# Cut-Admissibility as a Corollary of the Subformula Property

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# Background

## The Propositional Fragment of **LK** [Gentzen 1934]

$$(id)\ \frac{}{A \Rightarrow A} \qquad (cut)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (weak)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

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## The Cut-elimination Theorem ("Hauptsatz")

**Der Hauptsatz lautete:**

**Jede *LJ*- bzw. *LK*-Herleitung läßt sich in eine *LJ*- bzw. *LK*-Herleitung mit gleicher Endsequenz umwandeln, in welcher keine Schnitte vorkommen.**

# Background

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## The Cut-elimination Theorem ("Hauptsatz")

The *Hauptsatz* runs as follows:

Every *LJ*– or *LK*–derivation can be transformed into another *LJ*– or *LK*—derivation with the same endsequent, in which no cuts occur.

## The Propositional Fragment of **LK** [Gentzen 1934]

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## Analyticity ("The Subformula Property")

**2.513. Zusatz zum Hauptsatz (Teilformeln-Eigenschaft):** In einer *LJ*- bzw. *LK*-Herleitung ohne Schnitte sind alle vorkommenden *H*-*S*-Formeln T e i l f o r m e l n der in der Endsequenz auftretenden *S*-Formeln.

# Background

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## Analyticity ("The Subformula Property")

2.513. *Corollary of the Hauptsatz (subformula property)*: In an *LJ*– or *LK*–derivation without cuts, all occurring *D–S*–formulae are *subformulae* of the *S*–formula that occurs in the endsequent.

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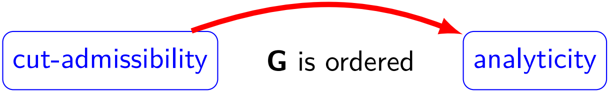
**LK** is ordered: premises are subformulas of **the** main formula

cut-admissibility → **LK** is ordered → analyticity

# Our Question



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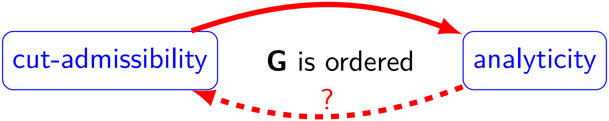

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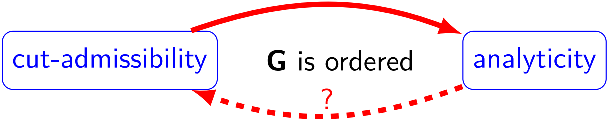
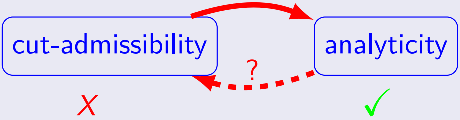



## "Axiomatic **LK**"

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \rightsquigarrow \frac{}{\Gamma, A \Rightarrow A \vee B, \Delta}$$




## More Counterexamples

- Modal logics B and S5
- Bi-intuitionistic logic

# Main Result

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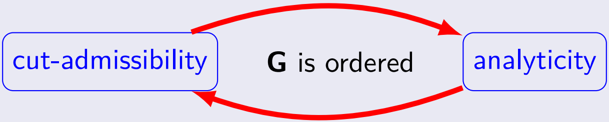
For two general families of calculi ("pure" and "intuitionistic"):




## Question

"But analyticity itself is usually proven using cut-admissibility!"

## Well...

- Theoretical interest
- Simpler proofs of cut-admissibility
  - Employ existing criteria for analyticity
  - From analytic cut-admissibility to full cut-admissibility

# Pure Sequent Calculi [Avron 1991]

- Propositional
- Sequents are pairs of finite sets
- Fully-structural (cut, weakening, id)
- No context-restriction

## Examples

- Prototype example: **LK**
- Primal infon iogic
- Paraconsistent logics
- Three and four valued logics

  ⋮

# Intuitionistic Sequent Calculi

- Propositional
- Sequents are pairs of finite sets
- Fully-structural (cut, weakening, id)
- Right context is empty for right-introduction rules whose premises have non-empty left sides

## Prototype Example: $\mathbf{LJ}'$ (Intuitionistic Logic) [Maehara 1954, Takeuti 1975]

$$(id)\ \frac{}{A \Rightarrow A} \qquad (cut)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (weak)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

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# Intuitionistic Sequent Calculi

- Propositional
- Sequents are pairs of finite sets
- Fully-structural (cut, weakening, id)
- Right context is empty for right-introduction rules whose premises have non-empty left sides

## Examples

- Prototype example: $\mathbf{LJ}'$
- Other constructive logics
- Paraconsistent constructive logics (e.g., Nelson's logic N4)

## Non-Examples

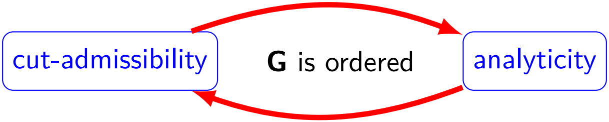
Calculi that include the following two rules:

$$(\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad (\Rightarrow \supset) \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

# Main Result

**Theorem**

1. Every ordered analytic pure calculus enjoys cut-admissibility
2. Every ordered analytic intuitionistic calculus enjoys cut-admissibility



# Can we do better?

**Theorem**

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**Calculus for FDE** [Arieli, Avron 1998]

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \neg A, \neg B, \Delta}{\Gamma \Rightarrow \neg(A \wedge B), \Delta}$$

- Not analytic, not ordered
- $\neg$-analytic, $\neg$-ordered
- Many calculi for non-classical only admit these weaker variants

# Can we do better?

**Theorem**

1. Every ¬-ordered ¬-analytic pure calculus enjoys cut-admissibility
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- Not analytic, not ordered
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# Can we do better?

## Theorem

1. Every $\prec$-ordered $\prec$-analytic pure calculus enjoys cut-admissibility
2. Every $\prec$-ordered $\prec$-analytic intuitionistic calculus enjoys cut-admissibility

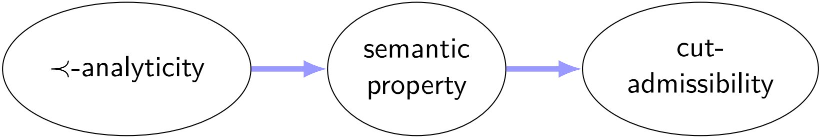
- $\prec$ must be:
  - irreflexive and transitive
  - safe: $\prec(A)$ is finite and computable.
  - structural: $A \prec B$ implies $\sigma(A) \prec \sigma(B)$
- $\prec$-ordered: in every rule $\frac{S}{s}$: $|s| = 1$ and $A \in S \wedge B \in s \Longrightarrow A \prec B$
- $\prec$-analytic: If $\vdash s$, then there is a proof that uses only $\prec(s)$

## Examples

- $\prec$="subformula of"
- $\prec$="subformula of + negations of subformula of"
- More combinations of unary connectives

# Proof

- We prove cut-admissibility, not cut-elimination.
- The proof is semantic:



# Semantics of Pure Calculi [Béziau 2001]

## Bivaluations

A bivaluation is a function $v : \mathcal{WFF} \to \{-1, 1\}$

## Satisfaction of sequents

$v \models \Gamma \Rightarrow \Delta$ if either $\exists A \in \Gamma . v(A) = -1$ or $\exists B \in \Delta . v(B) = 1$

## **G**-legal valuations

A valuation is **G**-legal if it respects the "semantic reading" of **G**.

## Example (Classical conjunction)

$$\frac{\Rightarrow A \quad \Rightarrow B}{\Rightarrow A \wedge B} \qquad \frac{A \Rightarrow}{A \wedge B \Rightarrow} \qquad \frac{B \Rightarrow}{A \wedge B \Rightarrow}$$

1. If $v(A) = 1$ and $v(B) = 1$ then $v(A \wedge B) = 1$
2. If $v(A) = -1$ then $v(A \wedge B) = -1$
3. If $v(B) = -1$ then $v(A \wedge B) = -1$

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## **G**-legal valuations

A valuation is **G**-legal if it respects the "semantic reading" of **G**.

## Example (Non-classical negations)

$$\frac{A \Rightarrow}{\Rightarrow \neg A} \qquad \frac{A \Rightarrow}{\neg\neg A \Rightarrow} \qquad \frac{\neg A \Rightarrow \quad \neg B \Rightarrow}{\neg(A \wedge B) \Rightarrow}$$

1. If $v(A) = -1$ then $v(\neg A) = 1$
2. If $v(A) = -1$ then $v(\neg\neg A) = -1$
3. If $v(\neg A) = -1$ and $v(\neg B) = -1$ then $v(\neg(A \wedge B)) = -1$

This semantics is non-deterministic.

# Semantics of Pure Calculi without cut [Lahav, Avron 2011]

## Main Idea

The cut rule should not be sound.

$$\frac{\Rightarrow A \quad A \Rightarrow}{\Rightarrow}$$

## Bivaluations [Béziau 2001]

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## Example (Classical conjunction)

$$\frac{\Rightarrow A \quad \Rightarrow B}{\Rightarrow A \wedge B}$$

- If $v(A) \in \{0, 1\}$ and $v(B) \in \{0, 1\}$ then $v(A \wedge B) \in \{0, 1\}$

## Main Idea

The cut rule should not be sound.

$$\frac{\Rightarrow A \quad A \Rightarrow}{\Rightarrow}$$

## Trivaluations [Schütte 1958]

A trivaluation is a function $v : \mathcal{WFF} \to \{-1, 0, 1\}$

## Satisfaction of sequents

$v \models \Gamma \Rightarrow \Delta$ if either $\exists A \in \Gamma . v(A) \in \{-1, 0\}$ or $\exists B \in \Delta . v(B) \in \{0, 1\}$

## Example (Classical conjunction)

$$\frac{\Rightarrow A \quad \Rightarrow B}{\Rightarrow A \wedge B}$$

- If $v(A) \in \{0, 1\}$ and $v(B) \in \{0, 1\}$ then $v(A \wedge B) \in \{0, 1\}$

## Theorem (Soundness and Completeness)

| | | |
|---|---|---|
| Regular Derivations | ⟷ | Bivaluations |
| Cut-free Derivations | ⟷ | Trivaluations |

## Corollary

*Every* **G***-legal trivaluation can be transformed into a* **G***-legal bivaluation*

$\downarrow$

**G** *enjoys cut-admissibility*

## Proof

$\nvdash_{\mathbf{G}}^{cut-free} s \implies \exists$ trivaluation $v \not\models s \implies \exists$ bivaluation $v \not\models s \implies \nvdash_{\mathbf{G}} s$

# Proof for Pure Calculi

## Main Lemma

Every **G**-legal trivaluation can be transformed into a **G**-legal bivaluation.

## Proof

- Enumerate $\mathcal{WFF}$ according to $\prec$: $A_1, A_2, \ldots$
- "Characteristic sequent" $\Gamma_i \Rightarrow \Delta_i$ :

$$\{A_j \mid v(A_j) = 1\} \Rightarrow \{A_j \mid v(A_j) = -1\}$$

- Using analyticity, this sequent is not derivable (semantic proof)
- $v(A_{i+1}) = \begin{cases} 1 & \not\vdash_{\mathbf{G}} \Gamma_i, A_{i+1} \Rightarrow \Delta_i \\ -1 & \textit{otherwise} \end{cases}$
- Prove that the fixed trivaluation is still **G**-legal

$$A_1 = 1 \quad A_2 = -1 \quad \color{red}{A_3 = 0} \qquad A_4 = 1 \quad \color{red}{A_5 = 0}$$

$$\Uparrow$$

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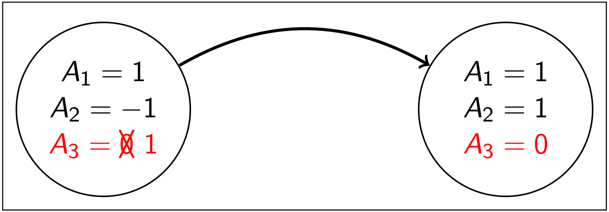
# Proof for Intuitionistic Calculi

- Similar structure
- Semantic framework: 3-valued valuations → 3-valued Kripke models
- Fixing 3-valued Kripke models into 2-valued ones is harder:

Persistent Models vs. **G**-legal Models



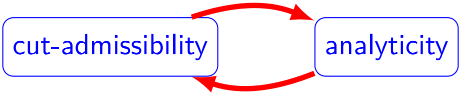
# Conclusion

We have seen:

- Two general cases in which cut-admissibility = analyticity:
  - Pure calculi
  - Intuitionistic calculi
- The proof is semantic

Future work:

- Modalities and quantifiers
- Single-conclusion calculi



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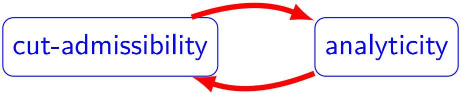


Thank You!