Reasoning Inside The Box

Deduction in Herbrand Logics

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GCAI 2017

Herbrand Structures

- Herbrand structures: first order structures, domain=set of closed terms
- Used in: completeness, logic programming, Tweety, ...

Example

Language: $\langle 0, 1, f \rangle$ Domain: $\{0, 1, f(0,1), f(1, f(0,0)), \dots\}$

Herbrand logic: the logic that is induced by Herbrand structures

☺ Natural sub-class of structures

☺ Simpler than classical FOL

☺ $\mathbb{N}$ is axiomatizable

☺ TC is axiomatizable

☹ Not compact

☹ Inherently incomplete

☹ non-monotone

☹ No proof theory so far

The Herbrand Manifesto

Thinking Inside the Box

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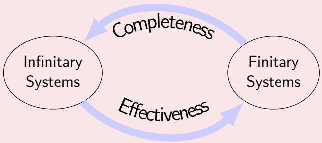
Main Contributions:

- Infinitary proof systems
- Finitary proof systems



Herbrand and semi-Herbrand Structures

First Order Structures

$M = \langle D, I \rangle$: D is the domain and I the interpretation

A first order structure is...

- semi-Herbrand if for every $d \in D$ there is $t \in cterms(\mathcal{L})$ s.t. $I(t) = d$
- Herbrand if the t is unique.
  - Alternatively: $D = cterms(\mathcal{L})$ and $I(t) = t$
- normal if $I(=)$ is $\{\langle a, a \rangle \mid a \in D\}$

Herbrand Logics

Consequence Relations

Let $\mathcal{C} \in \{cla, sHer, Her, cla_=, sHer_=, Her_=\}$

$\mathcal{T} \vdash_{\mathcal{C}} A$: every structure in $\mathcal{C}$ that satisfies $\mathcal{T}$ satisfies A

Proposition

$\vdash_{cla} \subsetneq \vdash_{sHer} = \vdash_{Her}$

Proposition

$\vdash_{cla_=} \subsetneq \vdash_{sHer_=} \subsetneq \vdash_{Her_=}$

Example

$$A\left\{\frac{t_1}{x}\right\}, A\left\{\frac{t_2}{x}\right\}, \ldots \underset{\vdash_{sHer}}{\overset{\nvdash_{cla}}{}} \forall x A$$

Example

$$\underset{\nvdash_{sHer_=}}{\overset{\vdash_{Her_=}}{}} \; c \neq d$$

Herbrand Logics are super-classical

Axiomatization of Equality

Equality Formulas

- *Equiv*: $x = x \qquad x = y \supset y = x \qquad (x = y \wedge y = z) \supset x = z$
- $Con(\mathcal{L})$: $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$
  $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (P(x_1, \ldots, x_n) \supset P(y_1, \ldots, y_n))$

semi-Herbrand Logic

$\mathcal{T} \vdash_{sHer_=} A$ iff $\mathcal{T}, Equiv, Con(\mathcal{L}) \vdash_{sHer} A$

Herbrand Logic

$\mathcal{T} \vdash_{cla_=} A$ ~~iff~~ $\mathcal{T}, Equiv, Con(\mathcal{L}) \vdash_{cla} A$

Axiomatization of Equality

(In)Equality Formulas

- *Equiv*: $x = x \qquad x = y \supset y = x \qquad (x = y \wedge y = z) \supset x = z$
- *Con*$(\mathcal{L})$: $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$
  $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (P(x_1, \ldots, x_n) \supset P(y_1, \ldots, y_n))$
- *inEq*$(\mathcal{L})$: $f(x_1, \ldots, y_n) \neq g(y_1, \ldots, y_n)$
  $x_i \neq y_i \supset f(\ldots, x_i, \ldots) \neq f(\ldots, y_i, \ldots)$

semi-Herbrand Logic

$\mathcal{T} \vdash_{sHer_=} A$ iff $\mathcal{T}, Equiv, Con(\mathcal{L}) \vdash_{sHer} A$

Herbrand Logic

$\mathcal{T} \vdash_{Her_=} A$ iff $\mathcal{T}, Equiv, Con(\mathcal{L}), inEq(\mathcal{L}) \vdash_{Her} A$

Axiomatization of Equality

(In)Equality Formulas

- *Equiv*: $x = x \qquad x = y \supset y = x \qquad (x = y \wedge y = z) \supset x = z$
- *Con*($\mathcal{L}$): $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$
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semi-Herbrand Logic

$\mathcal{T} \vdash_{sHer_=} A$ iff $\mathcal{T}, Equiv, Con(\mathcal{L}) \vdash_{sHer} A$

Herbrand Logic

$\mathcal{T} \vdash_{Her_=} A$ iff $\mathcal{T}, x = x, inEq(\mathcal{L}) \vdash_{Her} A$

Semantics

Semi-Herbrand Structures

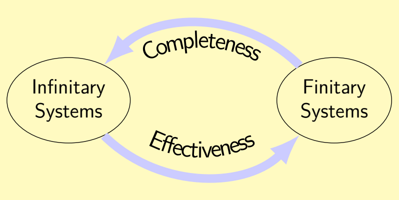
Herbrand Structures

Semi-Herbrand Structures with $=$

Herbrand Structures with $=$

Proof Systems

Infinitary Finitary



Herbrand Logics

Semantics

Semi-Herbrand Structures

⇕

Herbrand Structures

⇓

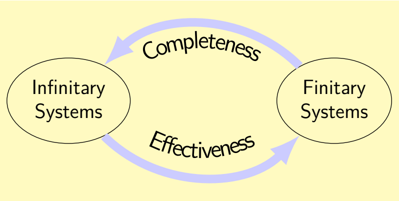
Semi-Herbrand Structures with $=$

⇓

Herbrand Structures with $=$

Proof Systems

Infinitary Finitary



Herbrand Logics

Semantics

Thinking inside the box

No nameless elements

Proof Theory

Reasoning inside the box

No free variables

What Are Sequents?

- *Sequents* have the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas.
- Intuition:

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \quad \leftrightsquigarrow \quad A_1 \wedge \ldots \wedge A_n \to B_1 \vee \ldots \vee B_m$$

- Special instance 1: Δ has one element: $\Gamma \Rightarrow A$
- Special instance 2: Γ is empty: $\Rightarrow A$

Example

- $A, B \Rightarrow A \wedge B$
- $A \Rightarrow A \vee B$
- $\Rightarrow A \vee \neg A$
- $A, \neg A \Rightarrow$
- $\Rightarrow A, \neg A$
- $A \Rightarrow A, B, C$

Sequent Calculus for Classical Logic **LK** [Gentzen 1934]

Γ and Δ are finite sets of formulas

Proof trees are finite

$$(id)\ \frac{}{\Gamma, A \Rightarrow A, \Delta} \qquad (cut)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(W \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$(\wedge \Rightarrow)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee \Rightarrow)\ \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

Sequent Calculus for Herbrand Logic $\mathbf{G_{Her}}$

Γ and Δ are ~~finite~~ sets of closed formulas

Proof trees are of finite height

$$(\forall \Rightarrow)\quad \frac{\Gamma, A\left\{\frac{t}{x}\right\} \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}\ \text{t is closed}$$

~~$$(\Rightarrow \forall)\quad \frac{\Gamma \Rightarrow A\left\{\frac{y}{x}\right\}, \Delta}{\Gamma \Rightarrow \forall x A, \Delta}\ \text{y fresh}$$~~

$$(\Rightarrow \forall)_H\quad \frac{\left\{\Gamma \Rightarrow A\left\{\frac{t}{x}\right\}, \Delta \mid t \in cterms(\mathcal{L})\right\}}{\Gamma \Rightarrow \forall x A, \Delta}$$

~~$$(\exists \Rightarrow)\quad \frac{\Gamma, A\left\{\frac{y}{x}\right\} \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta}\ \text{y fresh}$$~~

$$(\exists \Rightarrow)_H\quad \frac{\left\{\Gamma, A\left\{\frac{t}{x}\right\} \Rightarrow \Delta \mid t \in cterms(\mathcal{L})\right\}}{\Gamma, \exists x A \Rightarrow \Delta}$$

$$(\Rightarrow \exists)\quad \frac{\Gamma \Rightarrow A\left\{\frac{t}{x}\right\}, \Delta}{\Gamma \Rightarrow \exists x A, \Delta}\ \text{t is closed}$$

Intuition Behind $\mathbf{G_{Her}}$

$(\Rightarrow \forall)_H$

$$\frac{\Rightarrow A\left\{\frac{t_1}{x}\right\} \quad \Rightarrow A\left\{\frac{t_2}{x}\right\} \quad \Rightarrow A\left\{\frac{t_3}{x}\right\} \quad \ldots}{\Rightarrow \forall x A}$$

ω-rule [Schütte 1950]

$$\frac{\Rightarrow A(0) \quad \Rightarrow A(1) \quad \Rightarrow A(2) \quad \ldots}{\Rightarrow \forall x A}$$

Intuition Behind $\mathbf{G_{Her}}$

$(\Rightarrow \forall)_H$

$$\frac{\Rightarrow A\left\{\frac{t_1}{x}\right\} \quad \Rightarrow A\left\{\frac{t_2}{x}\right\} \quad \Rightarrow A\left\{\frac{t_3}{x}\right\} \quad \ldots}{\Rightarrow \forall x A}$$

ω-rule [Schütte 1950]

$$\frac{\Rightarrow A(0) \quad \Rightarrow A(1) \quad \Rightarrow A(2) \quad \ldots}{\Rightarrow \forall x A}$$

Proposition

Every closed sequent that is derivable in **LK** is derivable in $\mathbf{G_{Her}}$

Rules for Equality: Systems $\mathbf{G}_{\mathbf{sHer}_=}$ and $\mathbf{G}_{\mathbf{Her}_=}$

$$(paramodulation) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, s = t \Rightarrow, A', \Delta} s, t \in cterms(\mathcal{L})$$

(A' is obtained from A by replacing s by t)

$$(\Rightarrow =) \quad \frac{}{\Gamma \Rightarrow t = t, \Delta} t \in cterms(\mathcal{L})$$

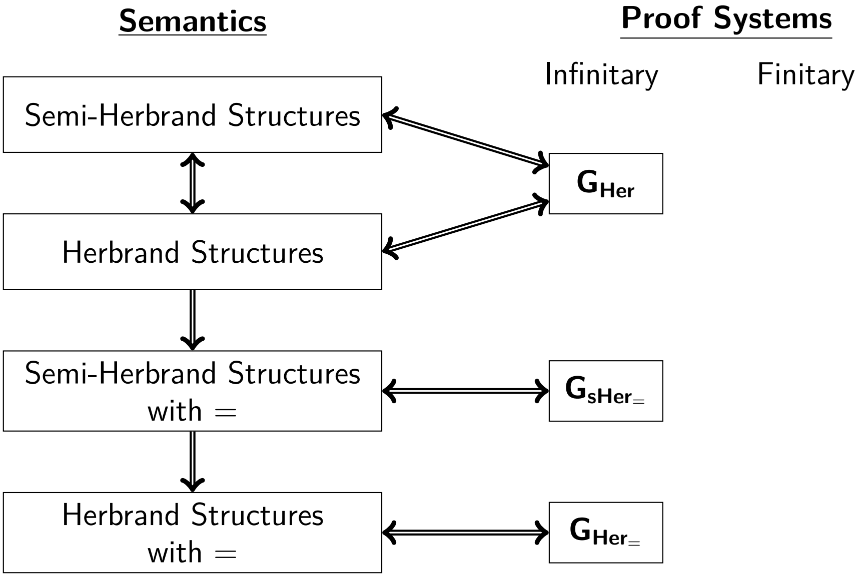
$$(= \Rightarrow) \quad \frac{}{\Gamma, s = t \Rightarrow \Delta} s \neq t \in cterms(\mathcal{L})$$

- $\mathbf{G}_{\mathbf{sHer}_=} = \mathbf{G}_{\mathbf{Her}} + (\Rightarrow =) + (paramodulation)$
- $\mathbf{G}_{\mathbf{Her}_=} = \mathbf{G}_{\mathbf{Her}} + (\Rightarrow =) + (= \Rightarrow)$

Theorem (soundness and completeness)

- $\mathbf{G}_{\mathbf{Her}}$ is sound and complete w.r.t. $\vdash_{Her}$ (and $\vdash_{sHer}$)
- $\mathbf{G}_{\mathbf{sHer}_=}$ is sound and complete w.r.t. $\vdash_{sHer_=}$
- $\mathbf{G}_{\mathbf{Her}_=}$ is sound and complete w.r.t. $\vdash_{Her_=}$

Herbrand Logics



Herbrand Logics

Semantics

Thinking inside the box

No nameless elements

Proof Theory

Peeking outside the box

free variables

Sequent Calculus for Herbrand Logic $\mathbf{G}^{\mathbf{IND}}_{\mathbf{Her}}$

Γ and Δ are finite sets of formulas

Proof trees are finite ($func(\mathcal{L})$ is finite)

$$(\forall\Rightarrow)\quad \frac{\Gamma, A\left\{\frac{t}{x}\right\}\Rightarrow\Delta}{\Gamma,\forall xA\Rightarrow\Delta}$$

~~$$(\Rightarrow\forall)\quad \frac{\Gamma\Rightarrow A\left\{\frac{y}{x}\right\},\Delta}{\Gamma\Rightarrow\forall xA,\Delta}\ \text{y fresh}$$~~

$$\frac{\left\{A\{\frac{x_1}{x}\},\ldots,A\{\frac{x_n}{x}\}\Rightarrow A\{\frac{f(x_1,\ldots,x_n)}{x}\}\mid f\in func(\mathcal{L})\right\}}{\Rightarrow\forall xA}$$

~~$$(\exists\Rightarrow)\quad \frac{\Gamma, A\left\{\frac{y}{x}\right\}\Rightarrow\Delta}{\Gamma,\exists xA\Rightarrow\Delta}\ \text{y fresh}$$~~

Example :

$$\frac{\Rightarrow A\left\{\frac{0}{x}\right\}\qquad A\Rightarrow A\left\{\frac{s(x)}{x}\right\}}{\Rightarrow\forall xA}$$

$$(\Rightarrow\exists)\quad \frac{\Gamma\Rightarrow A\left\{\frac{t}{x}\right\},\Delta}{\Gamma\Rightarrow\exists xA,\Delta}$$

Adding Equality

$$(paramodulation)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, s = t \Rightarrow, A', \Delta}$$

$$(\Rightarrow=)_{IND}\ \frac{}{\Gamma \Rightarrow x = x, \Delta}$$

$$(=\Rightarrow)_1\ \frac{}{\Gamma, f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m) \Rightarrow \Delta}$$

$$(=\Rightarrow)_2\ \frac{\Gamma, x_i = y_i \Rightarrow \Delta}{\Gamma, f(\ldots, x_i, \ldots) = f(\ldots, y_i, \ldots) \Rightarrow \Delta}$$

Example

$$(=\Rightarrow)_1 \frac{}{0 = s(x) \Rightarrow} \qquad (=\Rightarrow)_2 \frac{x = y \Rightarrow}{s(x) = s(y) \Rightarrow}$$

- $\mathbf{G}^{\mathbf{IND}}_{\mathbf{sHer}_=} = \mathbf{G}^{\mathbf{IND}}_{\mathbf{Her}} + (\Rightarrow=)_{IND} + (paramodulation)$
- $\mathbf{G}^{\mathbf{IND}}_{\mathbf{Her}_=} = \mathbf{G}^{\mathbf{IND}}_{\mathbf{sHer}_=} + (=\Rightarrow)_1 + (=\Rightarrow)_2$

Soundness

Theorem (soundness)

- $\mathbf{G}^{\mathbf{IND}}_{\mathbf{Her}}$ is sound w.r.t. $\vdash_{Her}$ (and $\vdash_{sHer}$)
- $\mathbf{G}^{\mathbf{IND}}_{\mathbf{sHer}_=}$ is sound w.r.t. $\vdash_{sHer_=}$
- $\mathbf{G}^{\mathbf{IND}}_{\mathbf{Her}_=}$ is sound w.r.t. $\vdash_{Her_=}$

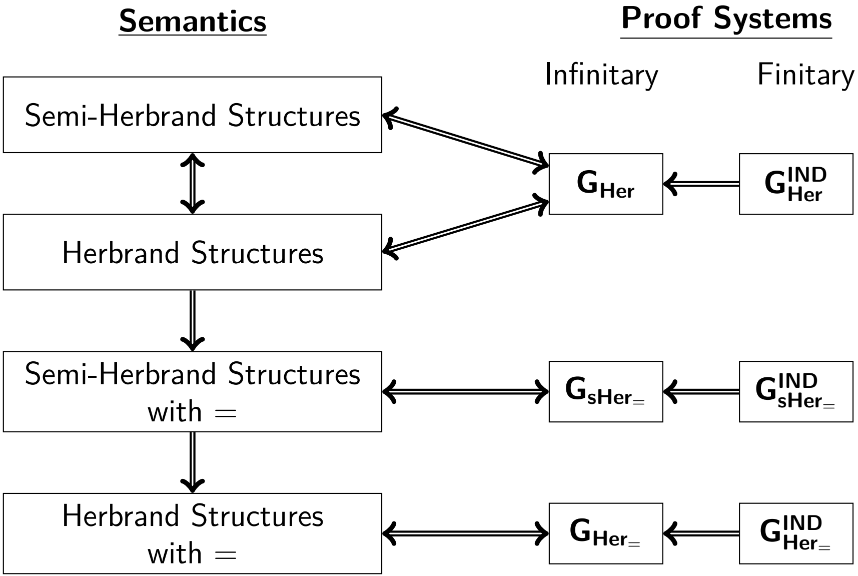
Corollary

- The induction principle is valid in (semi-)Herbrand structures.
  - $PA \equiv Q \equiv \Pi_2$ in Herbrand logics

More Properties

- All 3 systems are strictly stronger than **LK**
- $(\Longrightarrow)$ is provable from $(\Longrightarrow)_1, (\Longrightarrow)_2$

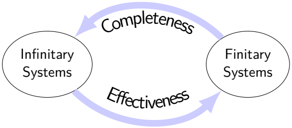
Herbrand Logics



Conclusions

We have seen:

- A modular definition of Herbrand semantics
- Infinitary proof systems
- Finitary approximations



Future work:

- Proof theoretical properties
- Semantics for the effective systems
- Relation to the Transitive Closure operator

Conclusions

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- A modular definition of Herbrand semantics
- Infinitary proof systems
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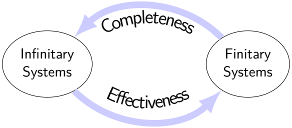

Future work:

- Proof theoretical properties
- Semantics for the effective systems
- Relation to the Transitive Closure operator

Thank you!