

Non-deterministic Matrices in Action

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Matrices

Logical Matrices

A **matrix** \mathbf{M} consists of:

- a set \mathcal{V} of *truth values*
- a subset \mathcal{D} of *designated* truth values
- for each connective \diamond of arity n , a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \rightarrow \mathcal{V}$

M-valuations

$$v(\diamond(\varphi_1, \dots, \varphi_n)) = \mathcal{O}(\diamond)(v(\varphi_1), \dots, v(\varphi_n))$$

$$v(p) = a_1$$

$$v(q) = a_2$$

$$v(p \wedge q) = a_2$$

$$v(q \wedge p) = a_3$$

⋮

When Does $\mathcal{T} \vdash_{\mathbf{M}} \varphi$?

For every **M**-valuation v :

$$v(\psi) \in \mathcal{D} \text{ for every } \psi \in \mathcal{T} \implies v(\varphi) \in \mathcal{D}$$

Non-deterministic Matrices

Non-deterministic Logical Matrices [Avron et al.'05]

A **Nmatrix** \mathbf{M} consists of:

- a set \mathcal{V} of *truth values*
- a subset \mathcal{D} of *designated* truth values
- for each connective \diamond of arity n , a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$

M-valuations

$$v(\diamond(\varphi_1, \dots, \varphi_n)) \in \mathcal{O}(\diamond)(v(\varphi_1), \dots, v(\varphi_n))$$

$$v(p) = a_1$$

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Primal Infon Logic [Gurevich, Neeman'09]

- An extremely efficient logic
- The main logical engine in DKAL (Microsoft)

- $\mathcal{V} = \{0, 1\}$
- $\mathcal{D} = \{1\}$

$\mathcal{O}(\wedge)$	1	0
1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0,1}

$\mathcal{O}(\supset)$	1	0
1	{1}	{0}
0	{1}	{0,1}

- $v(p) = 0, v(q) = 0, v(p \vee q) = 0$ is an **M**-valuation
- $v(p) = 0, v(q) = 0, v(p \vee q) = 1$ is an **M**-valuation
- $p \vee p \not\vdash_{\mathbf{M}} p$
- No finite matrix

Rexpansions: Refined Expansions

$$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle, \mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$$

Expansion

\mathbf{M}_2 is an **expansion** of \mathbf{M}_1 if $\exists F$ s.t.:

- $\mathcal{V}_2 = \biguplus_{x \in \mathcal{V}_1} F(x)$
- $\mathcal{D}_2 = \biguplus_{x \in \mathcal{D}_1} F(x)$
- $\mathcal{O}_2(\diamond)(\bar{y}) = \biguplus_{z \in \mathcal{O}_1(\bar{x})} F(z)$,

Refinement

\mathbf{M}_2 is a **refinement** of \mathbf{M}_1 if:

- $\mathcal{V}_2 \subseteq \mathcal{V}_1$
- $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$
- $\mathcal{O}_2(\diamond)(\bar{x}) \subseteq \mathcal{O}_1(\diamond)(\bar{x})$

Proposition [Avron et al. '07,'11]

- If \mathbf{M}_2 is an expansion of \mathbf{M}_1 then $\vdash_{\mathbf{M}_1} = \vdash_{\mathbf{M}_2}$.
- If \mathbf{M}_2 is a refinement of \mathbf{M}_1 then $\vdash_{\mathbf{M}_1} \subseteq \vdash_{\mathbf{M}_2}$.

Rexpansions: Refined Expansions

$$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle, \mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$$

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\mathbf{M}_2 is a **rexpansion** of \mathbf{M}_1 if it is a refinement of some expansion of \mathbf{M}_1

Rexpansions: Refined Expansions

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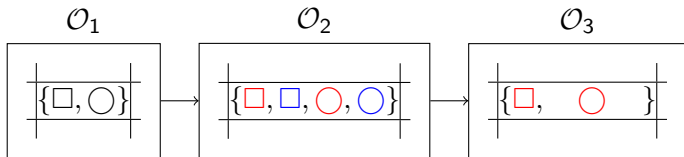
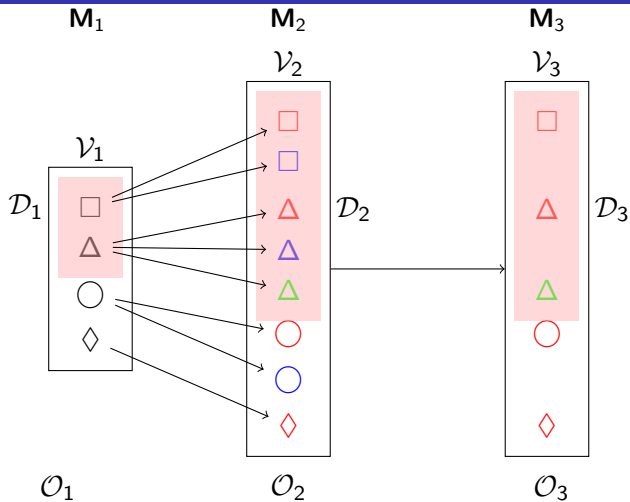
\mathbf{M}_2 is a **rexpansion** of \mathbf{M}_1 if it is a refinement of some expansion of \mathbf{M}_1

\mathbf{M}_2 is a **preserving rexpansion** of \mathbf{M}_1 if $F(x) \cap \mathcal{V}_2 \neq \emptyset$ for every $x \in \mathcal{V}_1$

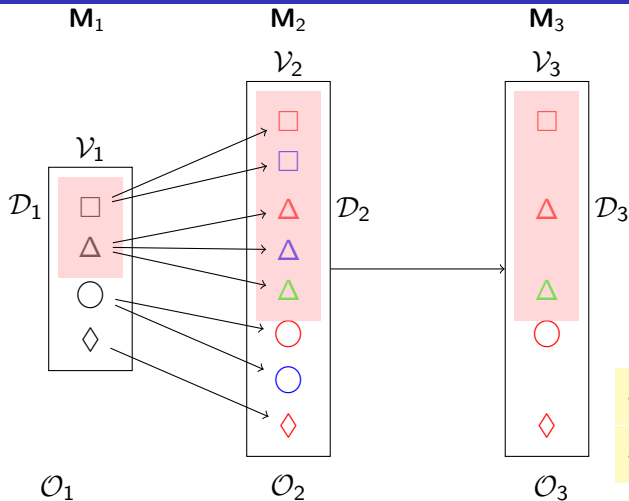
\mathbf{M}_2 is a **preserving rexpansion** of \mathbf{M}_1 , if in addition:

$$F(z) \cap \mathcal{O}_2(\diamond)(y_1, \dots, y_n) \neq \emptyset$$

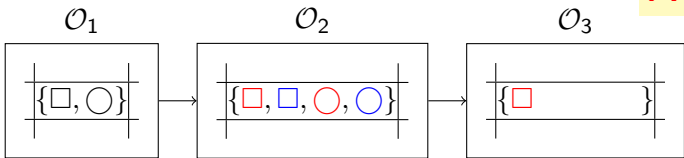
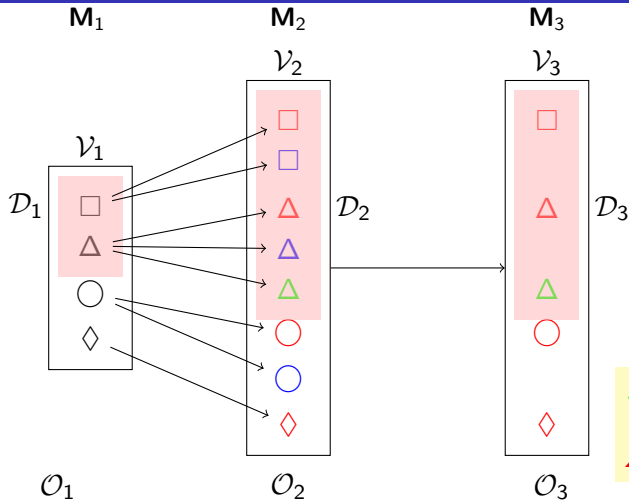
(Strongly) Preserving Rexpansions



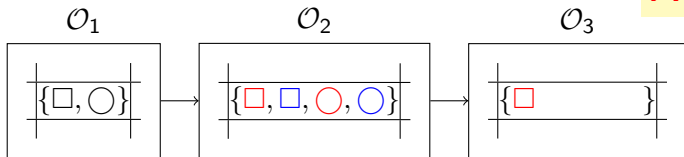
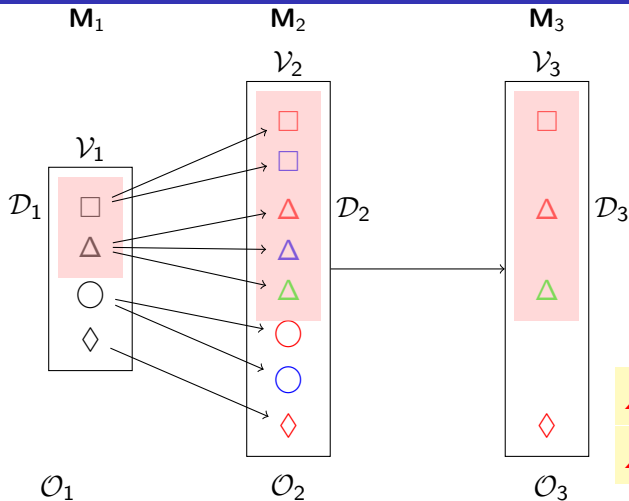
(Strongly) Preserving Rexpansions



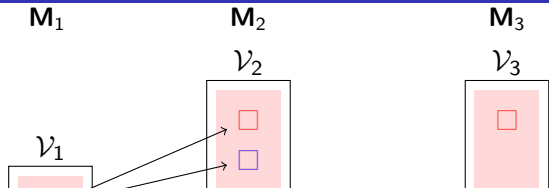
(Strongly) Preserving Rexpansions



(Strongly) Preserving Rexpansions



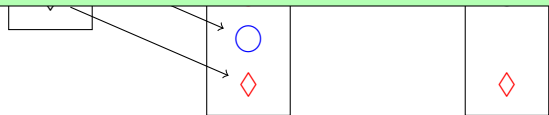
(Strongly) Preserving Rexpansions



Theorem:

If M_2 is a strongly preserving rexpansion of M_1 then $\vdash_{M_1} = \vdash_{M_2}$.

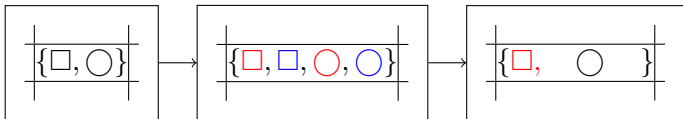
If M_2 is a preserving rexpansion of a **matrix** M_1 then $\vdash_{M_1} = \vdash_{M_2}$.



\mathcal{O}_1

\mathcal{O}_2

\mathcal{O}_3

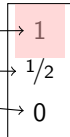


Three-valued Logics

Classical Logic



Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0}

$\mathcal{O}(\wedge)$	1	1/2	0
1	{1}	{0,1/2}	{0,1/2}
1/2	{0,1/2}	{0,1/2}	{0,1/2}
0	{0,1/2}	{0,1/2}	{0,1/2}

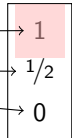
$\mathcal{O}(\vee)$	1	1/2	0
1	{1}	{1}	{1}
1/2	{1}	{0,1/2}	{0,1/2}
0	{1}	{0,1/2}	{0,1/2}

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$\mathcal{O}(\vee)$	1	0
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$\mathcal{O}(\wedge)$	1	1/2	0
1	{1}	{1/2}	{0}
1/2	{1/2}	{1/2}	{0}
0	{0}	{0}	{0}

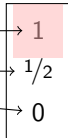
$\mathcal{O}(\vee)$	1	1/2	0
1	{1}	{1}	{1}
1/2	{1}	{1/2}	{1/2}
0	{1}	{1/2}	{0}

Three-valued Logics

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1/2	{1/2}	{1/2}	{0}
0	{0}	{0}	{0}

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$\mathcal{O}(\vee)$	1	1/2	0
1	{1}	{1}	{1}
1/2	{1}	{1/2}	{1/2}
0	{1}	{1/2}	{0}

Therefore, Kleene logic
is conservative over
 $\wedge\vee$ -classical logic

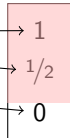
	$\mathcal{O}(\neg)$
1	{0}
1/2	{1/2}
0	{1}

Three-valued Logics

Classical Logic



Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\wedge)$	1	1/2	0
1	{1, 1/2}	{1, 1/2}	{0}
1/2	{1, 1/2}	{1, 1/2}	{0}
0	{0}	{0}	{0}

$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0}

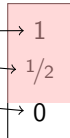
$\mathcal{O}(\vee)$	1	1/2	0
1	{1, 1/2}	{1, 1/2}	{1, 1/2}
1/2	{1, 1/2}	{1, 1/2}	{1, 1/2}
0	{1, 1/2}	{1, 1/2}	{0}

Three-valued Logics

Classical Logic



Priest Logic



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1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0}

$\mathcal{O}(\wedge)$	1	1/2	0
1	{1}	{1/2}	{0}
1/2	{1/2}	{1/2}	{0}
0	{0}	{0}	{0}

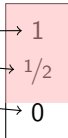
$\mathcal{O}(\vee)$	1	1/2	0
1	{1}	{1}	{1}
1/2	{1}	{1/2}	{1/2}
0	{1}	{1/2}	{0}

Three-valued Logics

Classical Logic



Priest Logic



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1/2	{1/2}	{1/2}	{0}
0	{0}	{0}	{0}

$\mathcal{O}(\vee)$	1	1/2	0
1	{1}	{1}	{1}
1/2	{1}	{1/2}	{1/2}
0	{1}	{1/2}	{0}

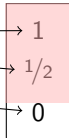
	$\mathcal{O}(\neg)$
1	{0}
1/2	{1/2}
0	{1}

Three-valued Logics

Classical Logic



Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0}

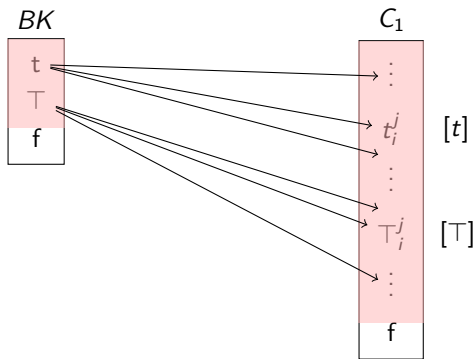
$\mathcal{O}(\wedge)$	1	1/2	0
1	{1}	{1/2}	{0}
1/2	{1/2}	{1/2}	{0}
0	{0}	{0}	{0}

$\mathcal{O}(\vee)$	1	1/2	0
1	{1}	{1}	{1}
1/2	{1}	{1/2}	{1/2}
0	{1}	{1/2}	{0}

Therefore, Priest logic
is conservative over
 $\wedge\vee$ -classical logic

	$\mathcal{O}(\neg)$
1	{0}
1/2	{1/2}
0	{1}

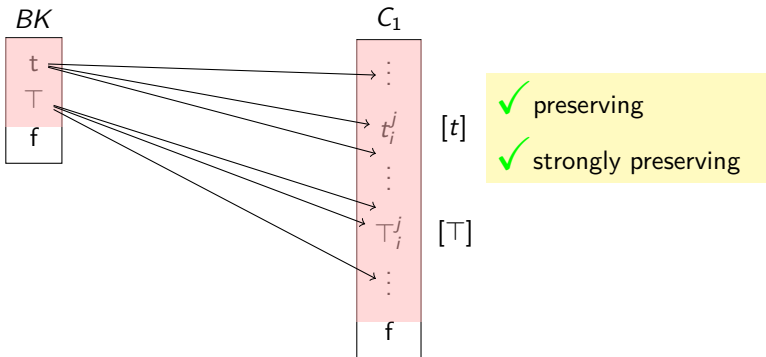
Paraconsistent Logics [Avron'07]



	$\mathcal{O}(\neg)$
t	$\{f\}$
T	$\{t, T\}$
f	$\{t, T\}$

$$\mathcal{O}(\neg)(x) = \begin{cases} \{f\} & x = t_i^j \\ [t] \cup [T] & x = T_i^j \\ [t] \cup [T] & x = f \end{cases}$$

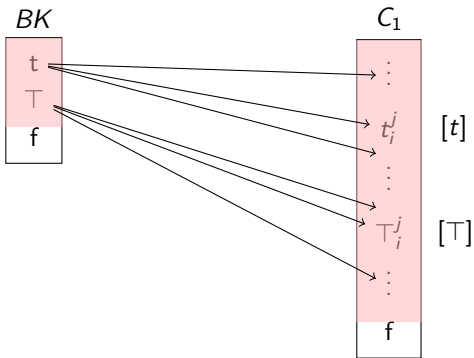
Paraconsistent Logics [Avron'07]



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Paraconsistent Logics [Avron'07]



✓ preserving

✗ strongly preserving

	$\mathcal{O}(\neg)$
t	$\{f\}$
T	$\{t, T\}$
f	$\{t, T\}$

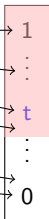
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Gödel Logic

Gödel Logic



G_t



$$0 < t \leq 1$$

$$\mathcal{O}(\wedge)(x, y) = \{\min(x, y)\}$$

$$\mathcal{O}(\wedge)(x, y) = \begin{cases} [t, 1] & \max(x, y) \geq t \\ \{\max(x, y)\} & \text{otherwise} \end{cases}$$

$$\mathcal{O}(\vee)(x, y) = \{\max(x, y)\}$$

$$\mathcal{O}(\vee)(x, y) = \begin{cases} [t, 1] & \min(x, y) \geq t \\ \{\min(x, y)\} & \text{otherwise} \end{cases}$$

$$\mathcal{O}(\supset)(x, y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & \text{otherwise} \end{cases}$$

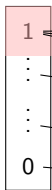
$$\mathcal{O}(\supset)(x, y) = \begin{cases} [t, 1] & x \leq y \text{ or } y \geq t \\ \{y\} & \text{otherwise} \end{cases}$$

$$\mathcal{O}(\perp)(x, y) = \{0\}$$

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Gödel Logic

Gödel Logic



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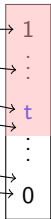
$$\mathcal{O}(\perp)(x, y) = \{0\}$$

Gödel Logic

Gödel Logic



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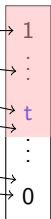
In Gödel logic, \mathcal{D} may be $[t, 1]$

Gödel Logic

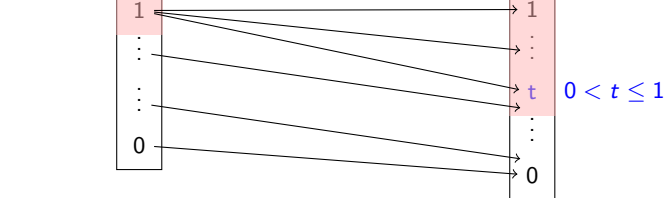
Gödel Logic



G_t



$$0 < t \leq 1$$



$$\mathcal{O}(\wedge)(x, y) = \{\min(x, y)\}$$

$$\mathcal{O}(\wedge)(x, y) = \{\max(x, y)\}$$

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$$\mathcal{O}(\supset)(x, y) = \begin{cases} [t, 1] & x \leq y \text{ or } y \geq t \\ \{y\} & \text{otherwise} \end{cases}$$

$$\mathcal{O}(\perp)(x, y) = \{0\}$$

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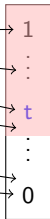
$$\mathcal{O}(\neg)(x) = 1 - x$$

Gödel Logic

Gödel Logic



G_t



$$0 < t \leq 1$$

$$\mathcal{O}(\wedge)(x, y) = \{\min(x, y)\}$$

$$\mathcal{O}(\wedge)(x, y) = \{\max(x, y)\}$$

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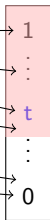
All refinements
are conservative!

Gödel Logic

Gödel Logic



G_t



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$$\mathcal{O}(\wedge)(x, y) = \{\min(x, y)\}$$

$$\mathcal{O}(\wedge)(x, y) = \{\max(x, y)\}$$

$$\mathcal{O}(\vee)(x, y) = \{\max(x, y)\}$$

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$$\mathcal{O}(\supset)(x, y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & \text{otherwise} \end{cases}$$

$$\mathcal{O}(\supset)(x, y) = \begin{cases} [t, 1] & x \leq y \text{ or } y \geq t \\ \{y\} & \text{otherwise} \end{cases}$$

$$\mathcal{O}(\perp)(x, y) = \{0\}$$

$$\mathcal{O}(\perp)(x, y) = \{0\}$$

$$\mathcal{O}(\neg)(x) = 1 - x$$

Exactly three:

$$\vdash_{G_{1/2}}, \vdash_{G_{3/4}}, \vdash_{G_1}$$

Gödel Logic

Gödel Logic



G_t



$$0 < t \leq 1$$

$$\mathcal{O}(\wedge)(x, y) = \{\min(x, y)\}$$

$$\mathcal{O}(\wedge)(x, y) = \{\max(x, y)\}$$

$$\mathcal{O}(\vee)(x, y) = \{\max(x, y)\}$$

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for $t \leq 1/2$: \vdash_{G_t}
is paraconsistent

Applications of Rexpansions:

- A **sufficient** criterion for conservative extensions
- Particular instances: paraconsistent (and) fuzzy logics

Future work:

- A **necessary** criterion for conservative extensions:
 - $\wedge\vee$ -Kleene = $\wedge\vee$ -Priest logic
 - None is (even) a rexpansion of the other
- Beyond Gödel logic
 - Łukasiewicz logic, product logic
 - Arbitrary t -norm based logics

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Thank you!