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Completeness and Cut-elimination for First-order Ideal Paraconsistent Four-valued Logic

Abstract. In this study, we prove the completeness and cut-elimination theorems for a first-order extension F4CC of Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic known as 4CC. These theorems are proved using Schütte's method, which can simultaneously prove completeness and cut-elimination.

Keywords: Ideal paraconsistent four-valued logic, 4CC, completeness theorem, cut-elimination theorem.

1. Introduction

In this study, we prove the completeness and cut-elimination theorems for a first-order extension F4CC of *Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic* known as 4CC [4, 5, 6]. These theorems are proved using Schütte's method [22, 19], which is used to simultaneously prove the completeness and cut-elimination theorems for Gentzen's sequent calculus LK [11] for first-order classical logic. To prove these theorems, we introduce an alternative Gentzen-type sequent calculus $\mathrm{S_{F4CC}}$ that is theorem-equivalent to a first-order extension $\mathrm{G_{F4CC}}$ of the original Gentzen-type sequent calculus $\mathrm{G_{4CC}}$ introduced by Arieli and Avron in [4, 5]. The proposed calculus $\mathrm{S_{F4CC}}$ is an extension of a Gentzen-type sequent calculus EPL, which was introduced by Kamide and Zohar in [12, 13].

The logic 4CC is an extension of *Belnap–Dunn logic* (also called *first-degree entailment logic* or *useful four-valued logic*) [7, 8, 10], and is also regarded as a variant of the *logic of logical bilattices* [2, 3]. It is also regarded as a specific type of *paraconsistent logic* [18], which has multiple names: it is called *paradefinite logic* by Arieli and Avron [4, 5], *non-alethic logic* by da Costa, and *paranormal logic* by Béziau [9]. Regardless of its name, paradefinite logic incorporates the properties of both *paraconsistency*, which

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rejects *the principle* $(\alpha \wedge {\sim}\alpha) \to \beta$ *of explosion*, and *paracompleteness*, which rejects *the law* $\alpha \vee {\sim}\alpha$ *of excluded middle.*

The logic 4CC is one of the most important ideal paradefinite logics that have natural many-valued semantics. 4CC is maximal relative to classical logic. This means, intuitively, that any attempt to add to it a tautology of classical logic, which is not provable in 4CC, should necessarily end-up with classical logic. For the exact definition and motivation of this property, see [6]. 4CC is also related to *connexive logics* [1, 16, 24], as it has a common characteristic Hilbert-style axiom scheme. For more information on the relationship between 4CC and connexive logics, see [13]. Below are the four-valued truth tables of 4CC, where t and f serve as the classical values *true* and false, while $\top$ and $\bot$ are two additional truth values:

$\wedge$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\vee$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\to$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	t	t	t	t
$\top$	t	f	$\top$	$\bot$
$\bot$	t	t	t	t

	${\sim}$
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

	$-$
t	t
f	f
$\top$	$\bot$
$\bot$	$\top$

As mentioned above, 4CC is an important ideal paradefinite logic. However, the first-order version F4CC of 4CC, which would be much more suitable for actual applications, has not yet been studied. A possible reason why F4CC was not considered is that elevating its propositional semantics to the first-order level is not trivial. The completeness theorem with respect to a valuation semantics for 4CC can be proved smoothly as shown in [12, 13] by using Lahav and Avron's unified semantic proof method [14], which was used to prove completeness and cut-elimination for a wide class of non-classical logics. However, the method by Lahav and Avron does not work for F4CC, as it is limited to propositional languages. The embedding-based proof method used in [12, 13] for proving the completeness theorem for other propositional paradefinite logics seems insufficient for F4CC, because it is not clear how a translation function can be defined for F4CC. To overcome these difficulties, in this study, we use Schütte's method for simultaneously proving the completeness and cut-elimination theorems for a Gentzen-type sequent calculus for F4CC. To use Schütte's method, we introduce a new Gentzen-type sequent calculus $\mathrm{S}_{\mathrm{F4CC}}$, as the original system $\mathrm{G}_{\mathrm{F4CC}}$ does not fit this method (see Remark 4.7). As a corollary of cut-elimination, we obtain a first-order variant of the *negative symmetry* property, introduced in [12].

The structure of this paper is summarized as follows. In Section 2, we introduce $\mathrm{S_{F4CC}}$ and $\mathrm{G_{F4CC}}$, and prove the equivalence between them. In Section 3, we introduce a valuation semantics for $\mathrm{S_{F4CC}}$, and show the soundness theorem with respect to this semantics. In Section 4, we prove the completeness and cut-elimination theorems for $\mathrm{S_{F4CC}}$ by using Schütte's method, and obtain the characteristic property of negative symmetry as a corollary of the cut-elimination theorem. In Section 5, we conclude and consider future work.

2. Sequent calculus

For the sake of simplicity of the discussion, a first-order language without individual constants and function symbols is considered. *Formulas* of the *first-order ideal paraconsistent four-valued logic* F4CC are constructed from countably many predicate symbols $p, q, ...$ and countably many individual variables $x, y, ...$ by the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\sim$ (paraconsistent negation), $-$ (conflation), $\forall$ (universal quantifier), and $\exists$ (existential quantifier). We use an expression $\alpha[y/x]$ to represent the formula which is obtained from the formula α by replacing all free occurrences of the individual variable x in α by the individual variable y, but avoiding a clash of variables by a suitable renaming of bound variables. A 0-ary predicate is regarded as a propositional variable. We use small letters $p, q, ...$ to denote not only predicate symbols but also atomic formulas, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to represent finite (possibly empty) sets of formulas. We use the expressions ${\sim}\Gamma$ and $-\Gamma$ to denote the sets $\{{\sim}\gamma \mid \gamma \in \Gamma\}$ and $\{-\gamma \mid \gamma \in \Gamma\}$, respectively. We use the symbol $=$ to denote the equality of symbols.

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. We use an expression $\alpha \Leftrightarrow \beta$ as the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. An expression $L \vdash S$ means that a sequent S is provable in a sequent calculus L. If L of $L \vdash S$ is clear from the context, we omit L in it. Two sequent calculi L_1 and L_2 are said to be *theorem-equivalent* if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: For any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Moreover, R is said to be *derivable* in L if there is a derivation from $S_1, \cdots, S_n$ to S in L. Note that a rule R of inference is admissible in a sequent calculus L if and only if the sequent calculi L and $L+R$ (i.e., the calculus obtained from L by the addition of R) are theorem-equivalent.

A Gentzen-type sequent calculus $\mathrm{S}_{\mathrm{F4CC}}$ for F4CC is defined as follows.

DEFINITION 2.1 ($\mathrm{S}_{\mathrm{F4CC}}$). *In the following definition, y represents an arbitrary individual variable, and z represents an individual variable which has the eigenvariable condition, i.e., z does not occur as a free individual variable in the lower sequent of the rule.*

The initial sequents of $\mathrm{S}_{\mathrm{F4CC}}$ are of the following form, for any atomic formula p,

$$p \Rightarrow p \qquad {\sim}p \Rightarrow {\sim}p \qquad -p \Rightarrow -p \qquad {\sim}p, -p \Rightarrow \qquad \Rightarrow {\sim}p, -p.$$

The structural inference rules of $\mathrm{S}_{\mathrm{F4CC}}$ are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The positive logical inference rules of $\mathrm{S}_{\mathrm{F4CC}}$ are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\rightarrow}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta}\ ({\rightarrow}\text{right})$$

$$\frac{\alpha[y/x], \Gamma \Rightarrow \Delta}{\forall x\alpha, \Gamma \Rightarrow \Delta}\ (\forall\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[z/x]}{\Gamma \Rightarrow \Delta, \forall x\alpha}\ (\forall\text{right})$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\exists x\alpha, \Gamma \Rightarrow \Delta}\ (\exists\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[y/x]}{\Gamma \Rightarrow \Delta, \exists x\alpha}\ (\exists\text{right}).$$

The negated logical inference rules of $\mathrm{S}_{\mathrm{F4CC}}$ are of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\alpha}\ ({\sim}{\sim}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{{\sim}{-}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{-}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}{-}\alpha}\ ({\sim}{-}\text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta \quad {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ ({\sim}\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta)}\ ({\sim}\wedge\text{right})$$

$$\frac{{\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim}\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta)}\ ({\sim}\vee\text{right})$$

$$\frac{\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{right})$$

$$\frac{{\sim}\alpha[z/x], \Gamma \Rightarrow \Delta}{{\sim}\forall x\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\forall\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha[y/x]}{\Gamma \Rightarrow \Delta, {\sim}\forall x\alpha}\ ({\sim}\forall\text{right})$$

$$\frac{{\sim}\alpha[y/x], \Gamma \Rightarrow \Delta}{{\sim}\exists x\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\exists\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha[z/x]}{\Gamma \Rightarrow \Delta, {\sim}\exists x\alpha}\ ({\sim}\exists\text{right}).$$

The conflated logical inference rules of $\mathrm{S}_{\mathrm{F4CC}}$ *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{-{-}\alpha, \Gamma \Rightarrow \Delta}\ ({-}{-}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, -{-}\alpha}\ ({-}{-}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{-{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({-}{\sim}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -{\sim}\alpha}\ ({-}{\sim}\text{right})$$

$$\frac{-\alpha, -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ ({-}\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)}\ ({-}\wedge\text{right})$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta \quad -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({-}\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, -\alpha, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \vee \beta)}\ ({-}\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({-}{\rightarrow}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha{\rightarrow}\beta)}\ ({-}{\rightarrow}\text{right})$$

$$\frac{-\alpha[y/x], \Gamma \Rightarrow \Delta}{-\forall x\alpha, \Gamma \Rightarrow \Delta}\ ({-}\forall\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, -\alpha[z/x]}{\Gamma \Rightarrow \Delta, -\forall x\alpha}\ ({-}\forall\text{right})$$

$$\frac{-\alpha[z/x], \Gamma \Rightarrow \Delta}{-\exists x\alpha, \Gamma \Rightarrow \Delta}\ ({-}\exists\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, -\alpha[y/x]}{\Gamma \Rightarrow \Delta, -\exists x\alpha}\ ({-}\exists\text{right}).$$

REMARK 2.2. $({-}{\rightarrow}\text{left})$ *and* $({-}{\rightarrow}\text{right})$ *correspond to the Hilbert-style axiom scheme* $-(\alpha{\rightarrow}\beta) \leftrightarrow \alpha{\rightarrow}{-}\beta$*, which is a characteristic axiom scheme for some* connexive logics [1, 16, 23] *if* $-$ *is replaced by* ${\sim}$*. See* [17, 24] *for a comprehensive introduction to connexive logics.*

PROPOSITION 2.3. *The following sequents are provable in cut-free* $\mathrm{S}_{\mathrm{F4CC}}$*: For any formula* α*,*

1. $\alpha \Rightarrow \alpha$,
2. ${\sim}\alpha, -\alpha \Rightarrow$,
3. $\Rightarrow {\sim}\alpha, -\alpha$.

Proof. By induction on α. ■

Proposition 2.4. *The following sequents are provable in cut-free* $\mathrm{S_{F4CC}}$*: For any formulas* α *and* β*,*

1. $\sim\sim\alpha \Leftrightarrow \alpha$,
2. $\sim-\alpha \Leftrightarrow -\sim\alpha$,
3. $\sim(\alpha \wedge \beta) \Leftrightarrow \sim\alpha \vee \sim\beta$,
4. $\sim(\alpha \vee \beta) \Leftrightarrow \sim\alpha \wedge \sim\beta$,
5. $\sim(\alpha\to\beta) \Leftrightarrow \alpha \wedge \sim\beta$,
6. $\sim\forall x\alpha \Leftrightarrow \exists x\sim\alpha$,
7. $\sim\exists x\alpha \Leftrightarrow \forall x\sim\alpha$,
8. $--\alpha \Leftrightarrow \alpha$,
9. $-(\alpha \wedge \beta) \Leftrightarrow -\alpha \wedge -\beta$,
10. $-(\alpha \vee \beta) \Leftrightarrow -\alpha \vee -\beta$,
11. $-(\alpha\to\beta) \Leftrightarrow \alpha\to-\beta$,
12. $-\forall x\alpha \Leftrightarrow \forall x-\alpha$,
13. $-\exists x\alpha \Leftrightarrow \exists x-\alpha$.

Proof. Straightforward. ■

Proposition 2.5. *The following rules are derivable in* $\mathrm{S_{F4CC}}$*:*

$$\frac{\Gamma \Rightarrow \Delta, -\alpha}{\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\text{left}) \qquad \frac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\alpha}\ (\sim\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\alpha}{-\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}) \qquad \frac{\sim\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\alpha}\ (-\text{right}).$$

Proof. We show only the case for ($\sim$left) as follows.

$$\frac{\Gamma \Rightarrow \Delta, -\alpha \qquad \begin{array}{c}\vdots\ \text{Prop.2.3}\\ -\alpha, \sim\alpha \Rightarrow\end{array}}{\sim\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut}).$$

■

A first-order extension $\mathrm{G_{F4CC}}$ of Arieli-Avron's sequent calculus $\mathrm{G_{4CC}}$ [4, 5] for 4CC is defined as follows.

DEFINITION 2.6 ($\mathrm{G_{F4CC}}$). *A sequent calculus* $\mathrm{G_{F4CC}}$ *is obtained from* $\mathrm{S_{F4CC}}$ *by replacing* $\{(p \Rightarrow p)$, $({\sim}p \Rightarrow {\sim}p)$, $(-p \Rightarrow -p)$, $({\sim}p, -p \Rightarrow)$, $(\Rightarrow {\sim}p, -p)$, $(--$left$)$, $(--$right$)$, $(-{\sim}$left$)$, $(-{\sim}$right$)$, $(-\wedge$left$)$, $(-\wedge$right$)$, $(-\vee$left$)$, $(-\vee$right$)$, $(-\!\rightarrow$left$)$, $(-\!\rightarrow$right$)$, $(-\forall$left$)$, $(-\forall$right$)$, $(-\exists$left$)$, $(-\exists$right$)\}$ *with* $\{(\alpha \Rightarrow \alpha)$, $(-$left$)$, $(-$right$)\}$.

PROPOSITION 2.7. *The following sequents are provable in cut-free* $\mathrm{G_{F4CC}}$*: For any formula* α,

1. ${\sim}\alpha, -\alpha \Rightarrow$,
2. $\Rightarrow {\sim}\alpha, -\alpha$.

PROOF. By using $(-$left$)$ and $(-$right$)$. ■

THEOREM 2.8 (Equivalence between $\mathrm{S_{F4CC}}$ and $\mathrm{G_{F4CC}}$). *The systems* $\mathrm{S_{F4CC}}$ *and* $\mathrm{G_{F4CC}}$ *are theorem-equivalent.*

PROOF. • We show that for any sequent $\Gamma \Rightarrow \Delta$, if $\mathrm{G_{F4CC}} \vdash \Gamma \Rightarrow \Delta$, then $\mathrm{S_{F4CC}} \vdash \Gamma \Rightarrow \Delta$. This is shown by induction on the proofs P of $\Gamma \Rightarrow \Delta$ in $\mathrm{G_{F4CC}}$. We distinguish the cases according to the last inference of P. It is sufficient to consider the cases for ($\alpha \Rightarrow \alpha$ for any formula α), $(-$left$)$, and $(-$right$)$. The case for ($\alpha \Rightarrow \alpha$ for any formula α) holds by Proposition 2.3. The cases for $(-$left$)$ and $(-$right$)$ hold by Proposition 2.5.

• We show that for any sequent $\Gamma \Rightarrow \Delta$, if $\mathrm{S_{F4CC}} \vdash \Gamma \Rightarrow \Delta$, then $\mathrm{G_{F4CC}} \vdash \Gamma \Rightarrow \Delta$. This is shown by induction on the proofs Q of $\Gamma \Rightarrow \Delta$ in $\mathrm{S_{F4CC}}$. We distinguish the cases according to the last inference of P. We must consider the cases for $({\sim}p, -p \Rightarrow)$, $(\Rightarrow {\sim}p, -p)$, $({\sim}$left$)$, $({\sim}$right$)$, $(--$left$)$, $(--$right$)$, $(-{\sim}$left$)$, $(-{\sim}$right$)$, $(-\wedge$left$)$, $(-\wedge$right$)$, $(-\vee$left$)$, $(-\vee$right$)$, $(-\!\rightarrow$left$)$, $(-\!\rightarrow$right$)$, $(-\forall$left$)$, $(-\forall$right$)$, $(-\exists$left$)$, and $(-\exists$right$)$. We show some of the cases below. The rest are shown similarly.

1. Case $({\sim}$right$)$: The last inference of Q is of the form:

$$\dfrac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ ({\sim}\text{right}).$$

We obtain the required fact:

$$\dfrac{\dfrac{\vdots\ \text{Prop.2.7}}{\Rightarrow {\sim}\alpha, -\alpha} \qquad -\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ (\text{cut}).$$

2. Case ($\sim$left): The last inference of Q is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha}{\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\text{left}).$$

We obtain the required fact:

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha \quad \begin{array}{c} \vdots\ \text{Prop.2.7} \\ \sim\alpha, -\alpha \Rightarrow \end{array}}{\sim\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut}).$$

3. Case ($--$left): The last inference of Q is of the form:

$$\dfrac{\alpha, \Gamma \Rightarrow \Delta}{--\alpha, \Gamma \Rightarrow \Delta}\ (--\text{left}).$$

We obtain the required fact:

$$\dfrac{\dfrac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim-\alpha}\ (\sim-\text{right}) \quad \begin{array}{c} \vdots\ \text{Prop.2.7} \\ \sim-\alpha, --\alpha \Rightarrow \end{array}}{--\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut}).$$

4. Case ($-\!\rightarrow$left): The last inference of Q is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha\rightarrow\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\!\rightarrow\text{left}).$$

We obtain the required fact:

$$\dfrac{\dfrac{\begin{array}{c} \Gamma \Rightarrow \Delta, \alpha \\ \vdots\ (\text{we-left}), (\text{we-right}) \\ \Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \end{array} \quad \begin{array}{c} \dfrac{\begin{array}{c} \vdots\ \text{Prop.2.7} \\ \Rightarrow \sim\beta, -\beta \end{array} \quad -\beta, \Sigma \Rightarrow \Pi}{\Sigma \Rightarrow \Pi, \sim\beta}\ (\text{cut}) \\ \vdots\ (\text{we-left}), (\text{we-right}) \\ \Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim\beta \end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim(\alpha\rightarrow\beta)}\ (\sim\!\rightarrow\text{right})}{-(\alpha\rightarrow\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\text{left}).$$

5. Case ($-\forall$right): The last inference of Q is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha[z/x]}{\Gamma \Rightarrow \Delta, -\forall x\alpha}\ (-\forall\text{right}).$$

We obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, -\alpha[z/x] \quad \overset{\vdots\ \text{Prop.2.7}}{-\alpha[z/x], {\sim}\alpha[z/x] \Rightarrow}}{{\sim}\alpha[z/x], \Gamma \Rightarrow \Delta}\ (\text{cut})}{{\sim}\forall x\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\forall\text{left})}{\Gamma \Rightarrow \Delta, -\forall x\alpha}\ (-\text{right}).$$

■

3. Semantics

Prior to introduce a semantics for $\mathrm{S_{F4CC}}$, we introduce some notations. Let Γ be a non-empty finite set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of formulas. Then, we use the expressions Γ_* and Γ^* as abbreviations of $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$ and $\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$, respectively. If $\Gamma = \emptyset$, then we respectively use Γ_* and Γ^* as abbreviations of $q {\rightarrow} q$ and ${\sim}{-}(q{\rightarrow}q)$ where q is a fixed atomic formula.

We follow the substitutional approach for first-order logic (see, e.g., [15, 20]).

Definition 3.1. *A structure $\mathcal{A} := \langle U, I^* \rangle$ is called a* paraconsistent model *if the following conditions hold:*

1. *U is a non-empty set,*
2. *I^* is a mapping such that*
   (a) *$p^{I^*}, ({\sim}p)^{I^*}, (-p)^{I^*} \subseteq U^n$ for an n-ary predicate symbol p,*
   (b) *for any predicate symbol p,*
      i. *$x \in ({\sim}p)^{I^*}$ iff $x \notin (-p)^{I^*}$,*
      ii. *$x \in (-p)^{I^*}$ iff $x \notin ({\sim}p)^{I^*}$.*

We introduce the notation $\underline{u}$ as the name of $u \in U$, and we denote $\mathcal{L}[\mathcal{A}]$ as the language obtained from the language $\mathcal{L}$ of F4CC *by adding the names of all the elements of U. A formula α is called a* closed formula *if α has no free individual variable. A formula of the form $\forall x_1 \cdots \forall x_m \alpha$ is called the* universal closure *of α if the free variables of α are $x_1, ..., x_m$. We write $cl(\alpha)$ for the universal closure of α.*

Definition 3.2 (Semantics for $\mathrm{S_{F4CC}}$). *Let $\mathcal{A} := \langle U, I^* \rangle$ be a paraconsistent model. The paraconsistent satisfaction relation $\mathcal{A} \models^* \alpha$ for any closed formula α of $\mathcal{L}[\mathcal{A}]$ are defined inductively by:*

1. $[\mathcal{A} \models^* p(\underline{u}_1, ..., \underline{u}_n)$ *iff* $(u_1, ..., u_n) \in p^{I^*}]$ *for any n-ary atomic formula* $p(\underline{u}_1, ..., \underline{u}_n)$,
2. $[\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$ *iff* $(u_1, ..., u_n) \in ({\sim}p)^{I^*}]$ *for any n-ary negated atomic formula* ${\sim}p(\underline{u}_1, ..., \underline{u}_n)$,
3. $[\mathcal{A} \models^* -p(\underline{u}_1, ..., \underline{u}_n)$ *iff* $(u_1, ..., u_n) \in (-p)^{I^*}]$ *for any n-ary conflated atomic formula* $-p(\underline{u}_1, ..., \underline{u}_n)$,
4. $\mathcal{A} \models^* \alpha \wedge \beta$ *iff* $\mathcal{A} \models^* \alpha$ *and* $\mathcal{A} \models^* \beta$,
5. $\mathcal{A} \models^* \alpha \vee \beta$ *iff* $\mathcal{A} \models^* \alpha$ *or* $\mathcal{A} \models^* \beta$,
6. $\mathcal{A} \models^* \alpha{\rightarrow}\beta$ *iff* $\mathcal{A} \not\models^* \alpha$ *or* $\mathcal{A} \models^* \beta$,
7. $\mathcal{A} \models^* \forall x\alpha$ *iff* $\mathcal{A} \models^* \alpha[\underline{u}/x]$ *for all* $u \in U$,
8. $\mathcal{A} \models^* \exists x\alpha$ *iff* $\mathcal{A} \models^* \alpha[\underline{u}/x]$ *for some* $u \in U$,
9. $\mathcal{A} \models^* {\sim}{\sim}\alpha$ *iff* $\mathcal{A} \models^* \alpha$,
10. $\mathcal{A} \models^* {\sim}{-}\alpha$ *iff* $\mathcal{A} \not\models^* \alpha$,
11. $\mathcal{A} \models^* {\sim}(\alpha \wedge \beta)$ *iff* $\mathcal{A} \models^* {\sim}\alpha$ *or* $\mathcal{A} \models^* {\sim}\beta$,
12. $\mathcal{A} \models^* {\sim}(\alpha \vee \beta)$ *iff* $\mathcal{A} \models^* {\sim}\alpha$ *and* $\mathcal{A} \models^* {\sim}\beta$,
13. $\mathcal{A} \models^* {\sim}(\alpha{\rightarrow}\beta)$ *iff* $\mathcal{A} \models^* \alpha$ *and* $\mathcal{A} \models^* {\sim}\beta$,
14. $\mathcal{A} \models^* {\sim}\forall x\alpha$ *iff* $\mathcal{A} \models^* {\sim}\alpha[\underline{u}/x]$ *for some* $u \in U$,
15. $\mathcal{A} \models^* {\sim}\exists x\alpha$ *iff* $\mathcal{A} \models^* {\sim}\alpha[\underline{u}/x]$ *for all* $u \in U$,
16. $\mathcal{A} \models^* {-}{-}\alpha$ *iff* $\mathcal{A} \models^* \alpha$,
17. $\mathcal{A} \models^* {-}{\sim}\alpha$ *iff* $\mathcal{A} \not\models^* \alpha$.
18. $\mathcal{A} \models^* -(\alpha \wedge \beta)$ *iff* $\mathcal{A} \models^* -\alpha$ *and* $\mathcal{A} \models^* -\beta$,
19. $\mathcal{A} \models^* -(\alpha \vee \beta)$ *iff* $\mathcal{A} \models^* -\alpha$ *or* $\mathcal{A} \models^* -\beta$,
20. $\mathcal{A} \models^* -(\alpha{\rightarrow}\beta)$ *iff* $\mathcal{A} \not\models^* \alpha$ *or* $\mathcal{A} \models^* -\beta$,
21. $\mathcal{A} \models^* -\forall x\alpha$ *iff* $\mathcal{A} \models^* -\alpha[\underline{u}/x]$ *for all* $u \in U$,
22. $\mathcal{A} \models^* -\exists x\alpha$ *iff* $\mathcal{A} \models^* -\alpha[\underline{u}/x]$ *for some* $u \in U$.

The paraconsistent satisfaction relation $\mathcal{A} \models^* \alpha$ *for any formula* α *of* $\mathcal{L}$ *are defined by* ($\mathcal{A} \models^* \alpha$ *iff* $\mathcal{A} \models^* cl(\alpha)$)*. A formula* α *of* $\mathcal{L}$ *is called F4CC-*valid *iff* $\mathcal{A} \models^* \alpha$ *holds for any paraconsistent model* $\mathcal{A}$*. A sequent* $\Gamma \Rightarrow \Delta$ *of* $\mathcal{L}$ *is called F4CC-valid (denoted by* $\mathrm{S_{F4CC}} \models \Gamma \Rightarrow \Delta$*) iff the formula* $\Gamma_* {\rightarrow} \Delta^*$ *of* $\mathcal{L}$ *is F4CC-*valid.

The following theorem provides an alternative definition of the above semantics:

THEOREM 3.3. *Requirement 2.b in Definition 3.1 and requirements 16–22 in Definition 3.2 can be replaced by the following requirement:*

$$\mathcal{A} \models^* -\alpha \ \textit{ iff } \ \mathcal{A} \not\models^* {\sim}\alpha \tag{I}$$

and the resulting class of structures stays the same.

PROOF.

- Suppose the condition in (I) is met. We prove that requirements 2.b, and 16-22 are met. 2.b is implied by the combination of (I), 2 and 3. We show that 16, 17, 18 and 21 are met. The rest are shown similarly. For 16: By (I), $\mathcal{A} \models^* --\alpha$ iff $\mathcal{A} \not\models^* {\sim}-\alpha$, which by 10 holds iff $\mathcal{A} \models^* \alpha$. For 17: By (I), $\mathcal{A} \models^* -{\sim}\alpha$ iff $\mathcal{A} \not\models^* {\sim}{\sim}\alpha$, which by 9 holds iff $\mathcal{A} \not\models^* \alpha$. For 18: By (I), $\mathcal{A} \models^* -(\alpha \wedge \beta)$ iff $\mathcal{A} \not\models^* {\sim}(\alpha \wedge \beta)$, which by 11 holds iff $\mathcal{A} \not\models^* {\sim}\alpha$ and $\mathcal{A} \not\models^* {\sim}\beta$. This, again by (I) holds iff $\mathcal{A} \models^* -\alpha$ and $\mathcal{A} \models^* -\beta$ For 21: By (I), $\mathcal{A} \models^* -\forall x\alpha$ iff $\mathcal{A} \not\models^* {\sim}\forall x\alpha$, which by 14 holds iff $\mathcal{A} \not\models^* {\sim}\alpha[\underline{u}/x]$ for every $u \in U$. Again using (I), the latter holds iff $\mathcal{A} \models^* -\alpha[\underline{u}/x]$ for every $u \in U$.
- We prove that $\mathcal{A} \models^* -\alpha$ iff $\mathcal{A} \not\models^* {\sim}\alpha$ for every formula α and paraconsistent model $\mathcal{A}$. We do so by induction on α, explicitly covering most of the cases, leaving the rest for the reader. For the base case where α is atomic, this holds by 2.b of Definition 3.1. If $\alpha = {\sim}\beta$, then we have by 17 that $\mathcal{A} \models^* -\alpha$ iff $\mathcal{A} \not\models^* \beta$, which by 9 holds iff $\mathcal{A} \not\models^* {\sim}\alpha$. If $\alpha = -\beta$, then we have by 16 that $\mathcal{A} \models^* -\alpha$ iff $\mathcal{A} \models^* \beta$, which by 10 holds iff $\mathcal{A} \not\models^* {\sim}\alpha$. If $\alpha = \beta_1 \wedge \beta_2$, then we have by 18 that $\mathcal{A} \models^* -\alpha$ iff $\mathcal{A} \models^* -\beta_1$ and $\mathcal{A} \models^* -\beta_2$, which by the induction hypothesis holds iff $\mathcal{A} \not\models^* {\sim}\beta_1$ and $\mathcal{A} \not\models^* {\sim}\beta_2$. By 11, this holds iff $\mathcal{A} \not\models^* {\sim}\alpha$. If $\alpha = \forall x\beta$, then by 21 we have that $\mathcal{A} \models^* -\alpha$ iff $\mathcal{A} \models^* -\beta[\underline{u}/x]$ for all $u \in U$. By the induction hypothesis, this holds iff $\mathcal{A} \not\models^* {\sim}\beta[\underline{u}/x]$ for all $u \in U$, which by 14 holds iff $\mathcal{A} \not\models^* {\sim}\alpha$.

■

In particular, we have the following property, that will be used in the completeness proof below:

COROLLARY 3.4. *For any paraconsistent model $\mathcal{A}$ and any formula α, we have the following clauses:*

1. $\mathcal{A} \models^* {\sim}\alpha$ *iff* $\mathcal{A} \not\models^* -\alpha$,
2. $\mathcal{A} \models^* -\alpha$ *iff* $\mathcal{A} \not\models^* {\sim}\alpha$.

To conclude this section, we prove the soundness of the above semantics w.r.t. $\mathrm{S_{F4CC}}$.

THEOREM 3.5 (Soundness for $\mathrm{S_{F4CC}}$). *For any sequent S,*

if $\mathrm{S_{F4CC}} \vdash S$*, then* $\mathrm{S_{F4CC}} \models S$.

PROOF. By induction on the proof P of S. We distinguish the cases according to the last inference of P. We show some of the cases. The others are proved similarly.

1. Case $({\sim}p(\underline{u}_1, ..., \underline{u}_n), {-}p(\underline{u}_1, ..., \underline{u}_n) \Rightarrow)$: The last inference of P is of the form: ${\sim}p(\underline{u}_1, ..., \underline{u}_n), {-}p(\underline{u}_1, ..., \underline{u}_n) \Rightarrow$. We show that "${\sim}p(\underline{u}_1, ..., \underline{u}_n), {-}p(\underline{u}_1, ..., \underline{u}_n) \Rightarrow$ is F4CC-valid (i.e., ${\sim}p(\underline{u}_1, ..., \underline{u}_n) \wedge {-}p(\underline{u}_1, ..., \underline{u}_n) {\rightarrow} {\sim}{-}(q{\rightarrow}q)$ is F4CC-valid)." It is thus sufficient to show that for any paraconsistent model $\mathcal{A} = \langle U, I^* \rangle$, $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n) \wedge {-}p(\underline{u}_1, ..., \underline{u}_n) {\rightarrow} {\sim}{-}(\underline{q}{\rightarrow}\underline{q})$ where $\underline{q}$ is obtained from q by replacing all the free individual variables occurring in it with the names of some elements of U. We thus show $\mathcal{A} \not\models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n) \wedge {-}p(\underline{u}_1, ..., \underline{u}_n)$ or $\mathcal{A} \models^* {\sim}{-}(\underline{q}{\rightarrow}\underline{q})$. On the other hand, we have (*): $\mathcal{A} \not\models^* {\sim}{-}(\underline{q}{\rightarrow}\underline{q})$ (i.e., $\mathcal{A} \models^* \underline{q}{\rightarrow}\underline{q}$). Thus, we now show (**): $\mathcal{A} \not\models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n) \wedge {-}p(\underline{u}_1, ..., \underline{u}_n)$ (i.e., $\mathcal{A} \not\models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$ or $\mathcal{A} \not\models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$). By Corollary 3.4, if $\mathcal{A} \models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$, then $\mathcal{A} \not\models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$, and if $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$, then $\mathcal{A} \not\models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$. Thus, we have $\mathcal{A} \not\models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$ or $\mathcal{A} \not\models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$, and hence obtain (**). Thus, by (*) and (**), we obtain the required fact.
2. Case $(\Rightarrow {\sim}p(\underline{u}_1, ..., \underline{u}_n), {-}p(\underline{u}_1, ..., \underline{u}_n))$: The last inference of P is of the form: $\Rightarrow {\sim}p(\underline{u}_1, ..., \underline{u}_n), {-}p(\underline{u}_1, ..., \underline{u}_n)$. We show that "$\Rightarrow {\sim}p(\underline{u}_1, ..., \underline{u}_n), {-}p(\underline{u}_1, ..., \underline{u}_n)$ is F4CC-valid (i.e., $(q{\rightarrow}q){\rightarrow}{\sim}p(\underline{u}_1, ..., \underline{u}_n) \vee {-}p(\underline{u}_1, ..., \underline{u}_n)$ is F4CC-valid)." It is thus sufficient to show that for any paraconsistent model $\mathcal{A} = \langle U, I^* \rangle$, $\mathcal{A} \models^* (\underline{q}{\rightarrow}\underline{q}){\rightarrow}{\sim}p(\underline{u}_1, ..., \underline{u}_n) \vee {-}p(\underline{u}_1, ..., \underline{u}_n)$ where $\underline{q}$ is obtained from q by replacing all the free individual variables occurring in it with the names of some elements of U. We thus show $\mathcal{A} \not\models^* (\underline{q}{\rightarrow}\underline{q})$ or $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n) \vee {-}p(\underline{u}_1, ..., \underline{u}_n)$. On the other hand, we have (*): $\mathcal{A} \models^* \underline{q}{\rightarrow}\underline{q}$ (i.e., we have no $\mathcal{A} \not\models^* \underline{q}{\rightarrow}\underline{q}$). Thus, we now show (**): $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n) \vee {-}p(\underline{u}_1, ..., \underline{u}_n)$ (i.e., $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$ or $\mathcal{A} \models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$). By Corollary 3.4, if $\mathcal{A} \not\models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$, then $\mathcal{A} \models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$, and if $\mathcal{A} \not\models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$, then $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$. Thus, we have: $\mathcal{A} \models^* {-}p(\underline{u}_1, ..., \underline{u}_n)$ or $\mathcal{A} \models^* {\sim}p(\underline{u}_1, ..., \underline{u}_n)$, and hence obtain (**). Thus, by (*) and (**), we obtain the required fact.

3. Case ($\sim\exists$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, {\sim}\alpha[z/x]}{\Gamma \Rightarrow \Delta, {\sim}\exists x\alpha} \ ({\sim}\exists\text{right}).$$

We show that "$\Gamma \Rightarrow \Delta, {\sim}\alpha[z/x]$ is F4CC-valid" implies "$\Gamma \Rightarrow \Delta, {\sim}\exists x\alpha$ is F4CC-valid." By the hypothesis, (*): $\forall z_1 \cdots \forall z_n \forall z(\Gamma_* \to (\Delta^* \vee ({\sim}\alpha[z/x])))$ (where $z_1, ..., z_n$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, {\sim}\exists x\alpha$) is F4CC-valid. We show that $\mathcal{A} \models^* \forall z_1 \cdots \forall z_n(\Gamma_* \to (\Delta^* \vee ({\sim}\exists x\alpha)))$ for any paraconsistent model $\mathcal{A} := \langle U, I^* \rangle$, i.e., we show that for any $u_1, ..., u_n \in U$, $\mathcal{A} \models^* \underline{\Gamma}_* \to (\underline{\Delta}^* \vee ({\sim}\exists x\underline{\alpha}))$, where $\underline{\Gamma}_*, \underline{\Delta}^*$, and $\underline{\alpha}$ are respectively obtained from Γ_*, Δ^* and α by replacing $z_1, ..., z_n$ by $\underline{u}_1, ..., \underline{u}_n$. Here, we note that $({\sim}\exists x\alpha)[\underline{u}_1/z_1, ..., \underline{u}_n/z_n]$ (the result of the simultaneous substitution of z_i by $\underline{u}_i$ $(1 \leq i \leq n)$) is equivalent to ${\sim}\exists x(\alpha[\underline{u}_1/z_1, ..., \underline{u}_n/z_n])$, i.e., ${\sim}\exists x\underline{\alpha}$. By (*), we have $\mathcal{A} \models^* (\underline{\Gamma}_* \to (\underline{\Delta}^* \vee ({\sim}\underline{\alpha}[z/x])))[\underline{w}/z]$ for any $w \in U$. By the eigenvariable condition, z is not occurring freely in $\underline{\Gamma}_*, \underline{\Delta}^*$, and $\underline{\alpha}$. Thus, $\underline{\Gamma}_*[\underline{w}/z]$ and $\underline{\Delta}^*[\underline{w}/z]$ are equivalent to $\underline{\Gamma}_*$and $\underline{\Delta}^*$, respectively, and $\underline{\alpha}[z/x][\underline{w}/z]$ is equivalent to $\underline{\alpha}[\underline{w}/z][\underline{w}/x]$, i.e., $\underline{\alpha}[\underline{w}/x]$. Therefore, for any $w \in U$, we have that (a): $\mathcal{A} \models^* \underline{\Gamma}_* \to (\underline{\Delta}^* \vee {\sim}\underline{\alpha}[\underline{w}/x])$. Suppose that (b): $[\mathcal{A} \models^* \underline{\Gamma}_*$ and $\mathcal{A} \not\models^* \underline{\Delta}^*]$. Then, by (a), we have that for any $w \in U$, $\mathcal{A} \models^* {\sim}\underline{\alpha}[\underline{w}/x]$. Therefore, we obtain (c): $\mathcal{A} \models^* {\sim}\exists x\underline{\alpha}$. This means that (b) implies (c), i.e., $\mathcal{A} \models^* \underline{\Gamma}_*$ implies ($\mathcal{A} \models^* \underline{\Delta}^*$ or $\mathcal{A} \models^* {\sim}\exists x\underline{\alpha}$). Therefore, we have the required fact that $\mathcal{A} \models^* \underline{\Gamma}_* \to (\underline{\Delta}^* \vee ({\sim}\exists x\underline{\alpha}))$ for any $u_1, ...u_n \in U$.

■

4. Completeness and cut-elimination

In the following, we prove the (strong) completeness and cut-elimination theorems for $\mathrm{S}_{\mathrm{F4CC}}$ by using Schütte's method [22].

Definition 4.1. *A sequent* $\Gamma \Rightarrow \Delta$ *is called* saturated *if for any formulas* α *and* β*,*

1. $\alpha \wedge \beta \in \Gamma$ *implies* $(\alpha \in \Gamma$ *and* $\beta \in \Gamma)$*,*
2. $\alpha \wedge \beta \in \Delta$ *implies* $(\alpha \in \Delta$ *or* $\beta \in \Delta)$*,*
3. $\alpha \vee \beta \in \Gamma$ *implies* $(\alpha \in \Gamma$ *or* $\beta \in \Gamma)$*,*
4. $\alpha \vee \beta \in \Delta$ *implies* $(\alpha \in \Delta$ *and* $\beta \in \Delta)$*,*
5. $\alpha \to \beta \in \Gamma$ *implies* $(\alpha \in \Delta$ *or* $\beta \in \Gamma)$*,*

6. $\alpha{\rightarrow}\beta \in \Delta$ *implies* ($\alpha \in \Gamma$ *and* $\beta \in \Delta$),
7. $\forall x\alpha \in \Gamma$ *implies* ($\alpha[y/x] \in \Gamma$ *for any individual variable* y),
8. $\forall x\alpha \in \Delta$ *implies* ($\alpha[z/x] \in \Delta$ *for some individual variable* z),
9. $\exists x\alpha \in \Gamma$ *implies* ($\alpha[z/x] \in \Gamma$ *for some individual variable* z),
10. $\exists x\alpha \in \Delta$ *implies* ($\alpha[y/x] \in \Delta$ *for any individual variable* y),
11. ${\sim}{\sim}\alpha \in \Gamma$ *implies* $\alpha \in \Gamma$,
12. ${\sim}{\sim}\alpha \in \Delta$ *implies* $\alpha \in \Delta$,
13. ${\sim}{-}\alpha \in \Gamma$ *implies* $\alpha \in \Delta$,
14. ${\sim}{-}\alpha \in \Delta$ *implies* $\alpha \in \Gamma$,
15. ${\sim}(\alpha \wedge \beta) \in \Gamma$ *implies* (${\sim}\alpha \in \Gamma$ *or* ${\sim}\beta \in \Gamma$),
16. ${\sim}(\alpha \wedge \beta) \in \Delta$ *implies* (${\sim}\alpha \in \Delta$ *and* ${\sim}\beta \in \Delta$),
17. ${\sim}(\alpha \vee \beta) \in \Gamma$ *implies* (${\sim}\alpha \in \Gamma$ *and* ${\sim}\beta \in \Gamma$),
18. ${\sim}(\alpha \vee \beta) \in \Delta$ *implies* (${\sim}\alpha \in \Delta$ *or* ${\sim}\beta \in \Delta$),
19. ${\sim}(\alpha{\rightarrow}\beta) \in \Gamma$ *implies* ($\alpha \in \Gamma$ *and* ${\sim}\beta \in \Gamma$),
20. ${\sim}(\alpha{\rightarrow}\beta) \in \Delta$ *implies* ($\alpha \in \Delta$ *or* ${\sim}\beta \in \Delta$),
21. ${\sim}\forall x\alpha \in \Gamma$ *implies* (${\sim}\alpha[z/x] \in \Gamma$ *for some individual variable* z),
22. ${\sim}\forall x\alpha \in \Delta$ *implies* (${\sim}\alpha[y/x] \in \Delta$ *for any individual variable* y),
23. ${\sim}\exists x\alpha \in \Gamma$ *implies* (${\sim}\alpha[y/x] \in \Gamma$ *for any individual variable* y),
24. ${\sim}\exists x\alpha \in \Delta$ *implies* (${\sim}\alpha[z/x] \in \Delta$ *for some individual variable* z),
25. ${-}{-}\alpha \in \Gamma$ *implies* $\alpha \in \Gamma$,
26. ${-}{-}\alpha \in \Delta$ *implies* $\alpha \in \Delta$,
27. ${-}{\sim}\alpha \in \Gamma$ *implies* $\alpha \in \Delta$,
28. ${-}{\sim}\alpha \in \Delta$ *implies* $\alpha \in \Gamma$,
29. ${-}(\alpha \wedge \beta) \in \Gamma$ *implies* (${-}\alpha \in \Gamma$ *and* ${-}\beta \in \Gamma$),
30. ${-}(\alpha \wedge \beta) \in \Delta$ *implies* (${-}\alpha \in \Delta$ *or* ${-}\beta \in \Delta$),
31. ${-}(\alpha \vee \beta) \in \Gamma$ *implies* (${-}\alpha \in \Gamma$ *or* ${-}\beta \in \Gamma$),
32. ${-}(\alpha \vee \beta) \in \Delta$ *implies* (${-}\alpha \in \Delta$ *and* ${-}\beta \in \Delta$),
33. ${-}(\alpha{\rightarrow}\beta) \in \Gamma$ *implies* ($\alpha \in \Delta$ *or* ${-}\beta \in \Gamma$),
34. ${-}(\alpha{\rightarrow}\beta) \in \Delta$ *implies* ($\alpha \in \Gamma$ *and* ${-}\beta \in \Delta$),
35. ${-}\forall x\alpha \in \Gamma$ *implies* (${-}\alpha[y/x] \in \Gamma$ *for any individual variable* y),
36. ${-}\forall x\alpha \in \Delta$ *implies* (${-}\alpha[z/x] \in \Delta$ *for some individual variable* z),

37. $-\exists x\alpha \in \Gamma$ *implies* ($-\alpha[z/x] \in \Gamma$ *for some individual variable* z),

38. $-\exists x\alpha \in \Delta$ *implies* ($-\alpha[y/x] \in \Delta$ *for any individual variable* y).

We now introduce the notion of infinite sequent.

DEFINITION 4.2. *An expression* $\Gamma \Rightarrow \Delta$ *is called an* infinite sequent *if* Γ *and* Δ *are infinite (countable) sets of formulas. An infinite sequent* $\Gamma \Rightarrow \Delta$ *is called provable if a finite part* $\Gamma' \Rightarrow \Delta'$ *of the sequent is provable (i.e.,* Γ' *and* Δ' *are finite subsets of* Γ *and* Δ*, respectively).*

DEFINITION 4.3. *A* decomposition *of a sequent (or infinite sequent)* S *is defined as having the form* S' *or* $S'; S''$ *by*

1. $\alpha, \beta, \alpha \wedge \beta, \Gamma \Rightarrow \Delta$ *is a decomposition of* $\alpha \wedge \beta, \Gamma \Rightarrow \Delta$,
2. $\Gamma \Rightarrow \Delta, \alpha \wedge \beta, \alpha$; $\Gamma \Rightarrow \Delta, \alpha \wedge \beta, \beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, \alpha \wedge \beta$,
3. $\alpha, \alpha \vee \beta, \Gamma \Rightarrow \Delta$; $\beta, \alpha \vee \beta, \Gamma \Rightarrow \Delta$ *is a decomposition of* $\alpha \vee \beta, \Gamma \Rightarrow \Delta$,
4. $\Gamma \Rightarrow \Delta, \alpha \vee \beta, \alpha, \beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, \alpha \vee \beta$,
5. $\alpha{\rightarrow}\beta, \Gamma \Rightarrow \Delta, \alpha$; $\beta, \alpha{\rightarrow}\beta, \Gamma \Rightarrow \Delta$ *is a decomposition of* $\alpha{\rightarrow}\beta, \Gamma \Rightarrow \Delta$,
6. $\alpha, \Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta, \beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta$,
7. $\alpha[y_1/x], ..., \alpha[y_m/x], \forall x\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* $\forall x\alpha, \Gamma \Rightarrow \Delta$ *where* $y_1, ..., y_m$ *are the free individual variables occurring in* $\forall x\alpha, \Gamma \Rightarrow \Delta$ *(if it has no free individual variable, then we adopt an arbitrary free variable in the language),*
8. $\Gamma \Rightarrow \Delta, \forall x\alpha, \alpha[z/x]$ *is a decomposition of* $\Gamma \Rightarrow \Delta, \forall x\alpha$ *where* z *is a* fresh *free individual variable (i.e.,* z *is not occurring in it),*
9. $\alpha[z/x], \exists x\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* $\exists x\alpha, \Gamma \Rightarrow \Delta$ *where* z *is a fresh free individual variable, (i.e.,* z *is not occurring in it),*
10. $\Gamma \Rightarrow \Delta, \exists x\alpha, \alpha[y_1/x], ..., \alpha[y_m/x]$ *is a decomposition of* $\Gamma \Rightarrow \Delta, \exists x\alpha$ *where* $y_1, ..., y_m$ *are the free individual variables occurring in* $\Gamma \Rightarrow \Delta, \exists x\alpha$*, (if it has no free individual variable, then we adopt an arbitrary free variable in the language),*
11. $\alpha, {\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* ${\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta$,
12. $\Gamma \Rightarrow \Delta, {\sim}{\sim}\alpha, \alpha$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}{\sim}\alpha$,
13. ${\sim}{-}\alpha, \Gamma \Rightarrow \Delta, \alpha$ *is a decomposition of* ${\sim}{-}\alpha, \Gamma \Rightarrow \Delta$,
14. $\alpha, \Gamma \Rightarrow \Delta, {\sim}{-}\alpha$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}{-}\alpha$,
15. ${\sim}\alpha, {\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$; ${\sim}\beta, {\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$ *is a decomposition of* ${\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$,

16. $\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta), {\sim}\alpha, {\sim}\beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta)$,
17. ${\sim}\alpha, {\sim}\beta, {\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta$ *is a decomposition of* ${\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta$,
18. $\Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta), {\sim}\alpha \ ; \ \Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta), {\sim}\beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta)$,
19. $\alpha, {\sim}\beta, {\sim}(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta$ *is a decomposition of* ${\sim}(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta$,
20. $\Gamma \Rightarrow \Delta, {\sim}(\alpha \rightarrow \beta), \alpha \ ; \ \Gamma \Rightarrow \Delta, {\sim}(\alpha \rightarrow \beta), {\sim}\beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}(\alpha \rightarrow \beta)$,
21. ${\sim}\alpha[z/x], {\sim}\forall x\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* ${\sim}\forall x\alpha, \Gamma \Rightarrow \Delta$ *where z is a fresh free individual variable, (i.e., z is not occurring in it),*
22. $\Gamma \Rightarrow \Delta, {\sim}\forall x\alpha, {\sim}\alpha[y_1/x], ..., {\sim}\alpha[y_m/x]$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}\forall x\alpha$ *where $y_1, ..., y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, {\sim}\forall x\alpha$, (if it has no free individual variable, then we adopt an arbitrary free variable in the language),*
23. ${\sim}\alpha[y_1/x], ..., {\sim}\alpha[y_m/x], {\sim}\exists x\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* ${\sim}\exists x\alpha, \Gamma \Rightarrow \Delta$ *where $y_1, ..., y_m$ are the free individual variables occurring in ${\sim}\exists x\alpha, \Gamma \Rightarrow \Delta$ (if it has no free individual variable, then we adopt an arbitrary free variable in the language),*
24. $\Gamma \Rightarrow \Delta, {\sim}\exists x\alpha, {\sim}\alpha[z/x]$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {\sim}\exists x\alpha$ *where z is a* fresh *free individual variable (i.e., z is not occurring in it),*
25. $\alpha, {-}{-}\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* ${-}{-}\alpha, \Gamma \Rightarrow \Delta$,
26. $\Gamma \Rightarrow \Delta, {-}{-}\alpha, \alpha$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {-}{-}\alpha$,
27. ${-}{\sim}\alpha, \Gamma \Rightarrow \Delta, \alpha$ *is a decomposition of* ${-}{\sim}\alpha, \Gamma \Rightarrow \Delta$,
28. $\alpha, \Gamma \Rightarrow \Delta, {-}{\sim}\alpha$ *is a decomposition of* $\Gamma \Rightarrow \Delta, {-}{\sim}\alpha$,
29. $-\alpha, -\beta, -(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$ *is a decomposition of* $-(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$,
30. $\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta), -\alpha \ ; \ \Gamma \Rightarrow \Delta, -(\alpha \wedge \beta), -\beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)$,
31. $-\alpha, -(\alpha \vee \beta), \Gamma \Rightarrow \Delta \ ; \ -\beta, -(\alpha \vee \beta), \Gamma \Rightarrow \Delta$ *is a decomposition of* $-(\alpha \vee \beta), \Gamma \Rightarrow \Delta$,
32. $\Gamma \Rightarrow \Delta, -(\alpha \vee \beta), -\alpha, -\beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, -(\alpha \vee \beta)$,
33. $-(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta, \alpha \ ; \ -\beta, -(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta$ *is a decomposition of* $-(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta$,
34. $\alpha, \Gamma \Rightarrow \Delta, -(\alpha \rightarrow \beta), -\beta$ *is a decomposition of* $\Gamma \Rightarrow \Delta, -(\alpha \rightarrow \beta)$,

35. $-\alpha[y_1/x], ..., -\alpha[y_m/x], -\forall x\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* $-\forall x\alpha, \Gamma \Rightarrow \Delta$ *where* $y_1, ..., y_m$ *are the free individual variables occurring in* $-\forall x\alpha, \Gamma \Rightarrow \Delta$ *(if it has no free individual variable, then we adopt an arbitrary free variable in the language),*
36. $\Gamma \Rightarrow \Delta, -\forall x\alpha, -\alpha[z/x]$ *is a decomposition of* $\Gamma \Rightarrow \Delta, -\forall x\alpha$ *where* z *is a* fresh *free individual variable (i.e.,* z *is not occurring in it),*
37. $-\alpha[z/x], -\exists x\alpha, \Gamma \Rightarrow \Delta$ *is a decomposition of* $-\exists x\alpha, \Gamma \Rightarrow \Delta$ *where* z *is a fresh free individual variable, (i.e.,* z *is not occurring in it),*
38. $\Gamma \Rightarrow \Delta, -\exists x\alpha, -\alpha[y_1/x], ..., -\alpha[y_m/x]$ *is a decomposition of* $\Gamma \Rightarrow \Delta, -\exists x\alpha$ *where* $y_1, ..., y_m$ *are the free individual variables occurring in* $\Gamma \Rightarrow \Delta, -\exists x\alpha$*, (if it has no free individual variable, then we adopt an arbitrary free variable in the language).*

DEFINITION 4.4. *A* decomposition tree of S *is a tree which expresses a process of some repeated decomposition of* S*.*

REMARK 4.5. *Roughly speaking, a decomposition rule corresponds to a (modified) inference rule of* $\mathrm{S_{F4CC}}$ − (cut) *(i.e., the right hand side of the decomposition rule corresponds to the premise(s) of an inference rule, and the left hand side corresponds to the conclusion of the inference rule). A decomposition tree corresponds to a bottom up proof search tree of* $\mathrm{S_{F4CC}}$ − (cut)*. In every decomposition of* S *(i.e.,* S' *or* $S'; S''$*), if* S *is unprovable in* $\mathrm{S_{F4CC}}$ − (cut)*, then so is* S' *or* S''*.*

LEMMA 4.6. *Let* $\Gamma \Rightarrow \Delta$ *be a given unprovable sequent in* $\mathrm{S_{F4CC}}$ − (cut)*. There exists an unprovable saturated (infinite) sequent* $\Gamma^\omega \Rightarrow \Delta^\omega$ *such that* $\Gamma \subseteq \Gamma^\omega$ *and* $\Delta \subseteq \Delta^\omega$*.*

PROOF. Let $\Gamma \Rightarrow \Delta$ be an unprovable sequent in $\mathrm{S_{F4CC}}$ − (cut). We construct $\Gamma^\omega \Rightarrow \Delta^\omega$ from $\Gamma \Rightarrow \Delta$ as follows.

1. We apply the decomposition instructions from Definition 4.3 to $\Gamma \Rightarrow \Delta$, in the following order, but without some decompositions, which are not related to the formulas in $\Gamma \Rightarrow \Delta$.

   $$(1) \longrightarrow (2) \longrightarrow (3) \longrightarrow \cdots \longrightarrow (38).$$

   In such a decomposition process, one of the decomposed elements S' and S'' of S is an unprovable sequent.
2. We repeat the same procedure as that of (1) infinitely often. Then, we obtain an infinite finitely branching decomposition tree.

3. By König's lemma, we have an infinite path on this decomposition tree as follows:

   $$\Gamma_0 \Rightarrow \Delta_0 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \infty,$$

   where $\Gamma_0 \Rightarrow \Delta_0$ is $\Gamma \Rightarrow \Delta$. In this sequence of the sequents on the infinite path, we have that $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \cdots$.
4. We put $\Gamma^\omega := \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^\omega := \bigcup_{i=0}^{\infty} \Delta_i$. We note that $\Gamma^\omega \cap \Delta^\omega = \emptyset$.

Then, we have that $\Gamma \subseteq \Gamma^\omega$ and $\Delta \subseteq \Delta^\omega$, and can verify that $\Gamma^\omega \Rightarrow \Delta^\omega$ is an unprovable saturated sequent.

■

REMARK 4.7. *Schütte's method for proving completeness and cut-elimination theorems cannot apply to* $\mathrm{G_{F4CC}}$. *The reason is explained as follows. If the corresponding decomposition rules of* $\mathrm{G_{F4CC}}$ *including* $(-\mathrm{left})$ *and* $(-\mathrm{right})$ *are adopted, then it is not clear whether some repeated decomposition processes develop or not a saturated sequent which is defined naturally in Definition 4.1, because the decomposition rules for* $(-\mathrm{left})$ *and* $(-\mathrm{right})$ *are not related to the definition of saturated sequents. That is why we adopt* $\mathrm{S_{F4CC}}$ *as a base system for* F4CC.

LEMMA 4.8. *Let* $\Gamma \Rightarrow \Delta$ *be an unprovable sequent in* $\mathrm{S_{F4CC}} - (\mathrm{cut})$, *and* $\Gamma^\omega \Rightarrow \Delta^\omega$ *be an unprovable saturated sequent constructed from* $\Gamma \Rightarrow \Delta$ *by Lemma 4.6. We define a canonical paraconsistent model* $\mathcal{A} := \langle U, I^* \rangle$ *for* $\Gamma \Rightarrow \Delta$ *as follows:*

1. $U := \{z \mid z$ *is a free individual variable occurring in* $\Gamma^\omega \Rightarrow \Delta^\omega\}$,
2. I^* *is a mapping such that*
   (a) $p^{I^*} := \{(z_1, ..., z_m) \mid p(z_1, ..., z_m) \in \Gamma^\omega\}$,
   (b) $({\sim}p)^{I^*} := \{(z_1, ..., z_m) \mid {\sim}p(z_1, ..., z_m) \in \Gamma^\omega\}$,
   (c) $(-p)^{I^*} := \{(z_1, ..., z_m) \mid -p(z_1, ..., z_m) \in \Gamma^\omega\}$.

Then, we have: for any formula α,

1. $\alpha \in \Gamma^\omega$ *implies* $\mathcal{A} \models^* \underline{\alpha}$,
2. $\alpha \in \Delta^\omega$ *implies* $\mathcal{A} \not\models^* \underline{\alpha}$

where $\underline{\alpha}$ *is obtained form* α *by replacing every individual variable* x *occurring in* α *by the name* $\underline{x}$.

Proof. By induction on α.

• Base step: We show only the following case. The others can be shown similarly.

Case $\alpha = -p(\underline{u}_1, ..., \underline{u}_n)$: First we have: $-p(u_1, ..., u_n) \in \Gamma^\omega$ iff $(u_1, ..., u_n) \in (-p)^{I^*}$ (by the definitions of I^*) iff $\mathcal{A} \models^* -p(\underline{u}_1, ..., \underline{u}_n)$. Second, we have: $-p(u_1, ..., u_n) \in \Delta^\omega$ iff $-p(u_1, ..., u_n) \notin \Gamma^\omega$ (by the fact $\Gamma^\omega \cap \Delta^\omega = \emptyset$) iff $(u_1, ..., u_n) \notin (-p)^{I^*}$ (by the definitions of I^*) iff $\mathcal{A} \not\models^* -p(\underline{u}_1, ..., \underline{u}_n)$.

• Induction step: We show some of cases. The others can be shown similarly.

1. Case $\alpha = \beta \wedge \gamma$: First, we show that $\beta \wedge \gamma \in \Gamma^\omega$ implies $\mathcal{A} \models^* \underline{\beta} \wedge \underline{\gamma}$. Suppose $\beta \wedge \gamma \in \Gamma^\omega$. Then, we obtain $[\beta \in \Gamma^\omega$ and $\gamma \in \Gamma^\omega]$ by Definition 4.1. By the induction hypothesis, we obtain $[\mathcal{A} \models^* \underline{\beta}$ and $\mathcal{A} \models^* \underline{\gamma}]$. This means $\mathcal{A} \models^* \underline{\beta} \wedge \underline{\gamma}$. Second, we show that $\beta \wedge \gamma \in \Delta^\omega$ implies $\mathcal{A} \not\models^* \underline{\beta} \wedge \underline{\gamma}$. Suppose $\beta \wedge \gamma \in \Delta^\omega$. Then, we obtain $[\beta \in \Delta^\omega$ or $\gamma \in \Delta^\omega]$ by Definition 4.1. By the induction hypothesis, we obtain $[\mathcal{A} \not\models^* \underline{\beta}$ or $\mathcal{A} \not\models^* \underline{\gamma}]$. This means $\mathcal{A} \not\models^* \underline{\beta} \wedge \underline{\gamma}$.
2. Case $\alpha = \beta \to \gamma$: First, we show that $\beta \to \gamma \in \Gamma^\omega$ implies $\mathcal{A} \models^* \underline{\beta} \to \underline{\gamma}$. Suppose $\beta \to \gamma \in \Gamma^\omega$. Then, we obtain $[\beta \in \Delta^\omega$ or $\gamma \in \Gamma^\omega]$ by Definition 4.1. By the induction hypothesis, we obtain $[\mathcal{A} \not\models^* \underline{\beta}$ or $\mathcal{A} \models^* \underline{\gamma}]$. This means $\mathcal{A} \models^* \underline{\beta} \to \underline{\gamma}$. Second, we show that $\beta \to \gamma \in \Delta^\omega$ implies $\mathcal{A} \not\models^* \underline{\beta} \to \underline{\gamma}$. Suppose $\beta \to \gamma \in \Delta^\omega$. Then, we obtain $[\beta \in \Gamma^\omega$ and $\gamma \in \Delta^\omega]$ by Definition 4.1. By the induction hypothesis, we obtain $[\mathcal{A} \models^* \underline{\beta}$ and $\mathcal{A} \not\models^* \underline{\gamma}]$. This means $\mathcal{A} \not\models^* \underline{\beta} \to \underline{\gamma}$.
3. Case $\alpha = {\sim}{\sim}\beta$: First, we show that ${\sim}{\sim}\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^* {\sim}{\sim}\underline{\beta}$. Suppose ${\sim}{\sim}\beta \in \Gamma^\omega$. Then, we obtain $\beta \in \Gamma^\omega$ by Definition 4.1. By the induction hypothesis, we obtain $\mathcal{A} \models^* \underline{\beta}$, and hence obtain $\mathcal{A} \models^* {\sim}{\sim}\underline{\beta}$. Second, we show that ${\sim}{\sim}\beta \in \Delta^\omega$ implies $\mathcal{A} \not\models^* {\sim}{\sim}\underline{\beta}$. Suppose ${\sim}{\sim}\beta \in \Delta^\omega$. Then, we obtain $\beta \in \Delta^\omega$ by Definition 4.1. By the induction hypothesis, we obtain $\mathcal{A} \not\models^* \underline{\beta}$, and hence obtain $\mathcal{A} \not\models^* {\sim}{\sim}\underline{\beta}$).
4. Case $\alpha = {\sim}{-}\beta$: First, we show that ${\sim}{-}\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^* {\sim}{-}\underline{\beta}$. Suppose ${\sim}{-}\beta \in \Gamma^\omega$. Then, we obtain $\beta \in \Delta^\omega$ by Definition 4.1. By the induction hypothesis, we obtain $\mathcal{A} \not\models^* \underline{\beta}$, and hence obtain $\mathcal{A} \models^* {\sim}{-}\underline{\beta}$. Second, we show that ${\sim}{-}\beta \in \Delta^\omega$ implies $\mathcal{A} \not\models^* {\sim}{-}\underline{\beta}$. Suppose ${\sim}{-}\beta \in \Delta^\omega$. Then, we obtain $\beta \in \Gamma^\omega$ by Definition 4.1. By the induction hypothesis, we obtain $\mathcal{A} \models^* \underline{\beta}$, and hence obtain $\mathcal{A} \not\models^* {\sim}{-}\underline{\beta}$).

5. Case $\alpha = {\sim}(\beta{\to}\gamma)$: First, we show that ${\sim}(\beta{\to}\gamma) \in \Gamma^\omega$ implies $\mathcal{A} \models^* {\sim}(\underline{\beta}{\to}\underline{\gamma})$. Suppose ${\sim}(\beta{\to}\gamma) \in \Gamma^\omega$. Then, we obtain [$\beta \in \Gamma^\omega$ and ${\sim}\gamma \in \Gamma^\omega$] by Definition 4.1. By the induction hypothesis, we obtain [$\mathcal{A} \models^* \underline{\beta}$ and $\mathcal{A} \models^* {\sim}\underline{\gamma}$]. This means $\mathcal{A} \models^* {\sim}(\underline{\beta}{\to}\underline{\gamma})$. Second, we show that ${\sim}(\beta{\to}\gamma) \in \Delta^\omega$ implies $\mathcal{A} \not\models^* \underline{\beta}{\to}\underline{\gamma}$. Suppose ${\sim}(\beta{\to}\gamma) \in \Delta^\omega$. Then, we obtain [$\beta \in \Delta^\omega$ or ${\sim}\gamma \in \Delta^\omega$] by Definition 4.1. By the induction hypothesis, we obtain [$\mathcal{A} \not\models^* \underline{\beta}$) or $\mathcal{A} \not\models^* {\sim}\underline{\gamma}$)]. This means $\mathcal{A} \not\models^* {\sim}(\underline{\beta}{\to}\underline{\gamma})$).
6. Case $\alpha = {\sim}\forall x\beta$: First, we show that ${\sim}\forall x\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^* {\sim}\forall x\underline{\beta}$. Suppose ${\sim}\forall x\beta \in \Gamma^\omega$. Then, we obtain ${\sim}\beta[z/x] \in \Gamma^\omega$ for some $z \in U$, by Definition 4.1. By the induction hypothesis, we obtain that $\mathcal{A} \models^* {\sim}\underline{\beta}[\underline{z}/x]$ for some $z \in U$. This means $\mathcal{A} \models^* {\sim}\forall x\underline{\beta}$. Second, we show that ${\sim}\forall x\beta \in \Delta^\omega$ implies $\mathcal{A} \not\models^* {\sim}\forall x\underline{\beta}$. Suppose ${\sim}\forall x\beta \in \Delta^\omega$. Then, we obtain ${\sim}\beta[y_i/x] \in \Delta^\omega$ for any $y_i \in U$, by Definition 4.1. By the induction hypothesis, we obtain $\mathcal{A} \not\models^* {\sim}\underline{\beta}[\underline{y_i}/x]$) for any $y_i \in U$. This means $\mathcal{A} \not\models^* {\sim}\forall x\underline{\beta}$.
7. Case $\alpha = {\sim}\exists x\beta$: First, we show that ${\sim}\exists x\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^* {\sim}\exists x\underline{\beta}$. Suppose ${\sim}\exists x\beta \in \Gamma^\omega$. Then we obtain ${\sim}\beta[y_i/x] \in \Gamma^\omega$ for any $y_i \in U$, by Definition 4.1. By the induction hypothesis, we obtain that $\mathcal{A} \models^* {\sim}\underline{\beta}[\underline{y_i}/x]$ for any $y_i \in U$. This means $\mathcal{A} \models^* {\sim}\exists x\underline{\beta}$. Second, we show that ${\sim}\exists x\beta \in \Delta^\omega$ implies $\mathcal{A} \not\models^* {\sim}\exists x\underline{\beta}$. Suppose ${\sim}\exists x\beta \in \Delta^\omega$. Then, we obtain ${\sim}\beta[z/x] \in \Delta^\omega$ for some $z \in U$ by Definition 4.1. By the induction hypothesis, we obtain $\mathcal{A} \not\models^* {\sim}\underline{\beta}[\underline{z}/x]$ for some $z \in U$. This means $\mathcal{A} \not\models^* {\sim}\exists x\underline{\beta}$.
8. Case $\alpha = -(\beta{\to}\gamma)$: First, we show that $-(\beta{\to}\gamma) \in \Gamma^\omega$ implies $\mathcal{A} \models^* -(\underline{\beta}{\to}\underline{\gamma})$. Suppose $-(\beta{\to}\gamma) \in \Gamma^\omega$. Then, we obtain [$\beta \in \Delta^\omega$ or $-\gamma \in \Gamma^\omega$] by Definition 4.1. By the induction hypothesis, we obtain [$\mathcal{A} \not\models^* \underline{\beta}$ or $\mathcal{A} \models^* -\underline{\gamma}$]. This means $\mathcal{A} \models^* -(\underline{\beta}{\to}\underline{\gamma})$. Second, we show that $-(\beta{\to}\gamma) \in \Delta^\omega$ implies $\mathcal{A} \not\models^* -\underline{\beta}{\to}\underline{\gamma}$. Suppose $-(\beta{\to}\gamma) \in \Delta^\omega$. Then, we obtain [$\beta \in \Gamma^\omega$ and ${\sim}\gamma \in \Delta^\omega$] by Definition 4.1. By the induction hypothesis, we obtain [$\mathcal{A} \models^* \underline{\beta}$) and $\mathcal{A} \not\models^* -\underline{\gamma}$)]. This means $\mathcal{A} \not\models^* -(\underline{\beta}{\to}\underline{\gamma})$).

■

Theorem 4.9 (Strong completeness for $\mathrm{S_{F4CC}}$). *For any sequent S, if S is valid, then* $\mathrm{S_{F4CC}} - (\mathrm{cut}) \vdash S$.

Proof. Let S be $\Gamma \Rightarrow \Delta$. Then, we prove the following: if $\Gamma \Rightarrow \Delta$ is unprovable in $\mathrm{S_{F4CC}} - (\mathrm{cut})$, then there exists a model $\mathcal{A}$ such that $\Gamma \Rightarrow \Delta$ is not F4CC-valid in $\mathcal{A}$. Suppose that $\Gamma \Rightarrow \Delta$ is unprovable in $\mathrm{S_{F4CC}} - (\mathrm{cut})$.

Then, by Lemma 4.8, we can construct a canonical model $\mathcal{A}$ satisfying the conditions in this lemma. Thus, we have $\mathcal{A} \models^* \underline{\gamma}$ and $\mathcal{A} \not\models^* \underline{\delta}$ for any $\gamma \in \Gamma \subseteq \Gamma^\omega$ and any $\delta \in \Delta \subseteq \Delta^\omega$. Hence, we obtain "$\mathcal{A} \not\models^* \underline{\Gamma}_* \rightarrow \underline{\Delta}^*$," and hence "$\mathcal{A} \not\models^* cl(\Gamma_* \rightarrow \Delta^*)$." Therefore, $\Gamma \Rightarrow \Delta$ is not F4CC-valid in $\mathcal{A}$. ■

THEOREM 4.10 (Cut-elimination for $\mathrm{S}_{\mathrm{F4CC}}$). *The rule* (cut) *is admissible in cut-free* $\mathrm{S}_{\mathrm{F4CC}}$.

PROOF. By combining Theorems 4.9 and 3.5. ■

By using Theorem 4.10, we can strengthen Proposition 2.5.

PROPOSITION 4.11. *The rules* ($\sim$left)*,* ($\sim$right)*,* ($-$left)*, and* ($-$right) *are admissible in cut-free* $\mathrm{S}_{\mathrm{F4CC}}$*:*

PROOF. We show only the case for ($\sim$left). Suppose $\mathrm{S}_{\mathrm{F4CC}} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta, -\alpha$. Then, we can derive $\sim\alpha, \Gamma \Rightarrow \Delta$ from $\Gamma \Rightarrow \Delta, -\alpha$ by Proposition 2.5 using (cut). We thus obtain $\mathrm{S}_{\mathrm{F4CC}} - (\mathrm{cut}) \vdash \sim\alpha, \Gamma \Rightarrow \Delta$ by Theorem 4.10. ■

By using Proposition 4.11, we can obtain the following characteristic property of $\mathrm{S}_{\mathrm{F4CC}}$

THEOREM 4.12 (Admissibilities of negative symmetry rules in $\mathrm{S}_{\mathrm{F4CC}}$). *The following negative symmetry rules are admissible in cut-free* $\mathrm{S}_{\mathrm{F4CC}}$*:*

$$\frac{-\Delta \Rightarrow -\Gamma}{\sim\Gamma \Rightarrow \sim\Delta}\ (\sim\Uparrow-) \qquad \frac{\sim\Delta \Rightarrow \sim\Gamma}{-\Gamma \Rightarrow -\Delta}\ (-\Uparrow\sim).$$

PROOF. By Proposition 4.11. ■

5. Conclusion

In this paper, we have studied a first-order extension of the paradefinite logic 4CC. This extension, which relies on the Gentzen-type sequent calculus EPL introduced in [12, 13] for 4CC, was obtained by augmenting the latter system with natural rules for the quantifiers, as well as for their interaction with the connectives $\sim$ and $-$. A cut-free proof system was presented, as well as a two-valued first-order semantics, that served as the main tool for proving cut-elimination.

In what follows, we clarify the connection between the results of this paper and our previous papers [12, 13]. The paper [13] is an extension of the conference paper [12], and hence the results of [12] are included in [13]. The propositional fragment of the sequent calculus $\mathrm{S}_{\mathrm{F4CC}}$ which was introduced in this paper is equivalent to the sequent calculus EPL which was introduced in [13] for 4CC. The sequent calculus EPL is theorem-equivalent to the original sequent calculus $\mathrm{G}_{\mathrm{4CC}}$ [4, 5] for 4CC. Moreover, the sequent calculus $\mathrm{G}_{\mathrm{F4CC}}$ which was introduced in this paper for F4CC is a straightforward first-order extension of $\mathrm{G}_{\mathrm{4CC}}$. Thus, as shown in this paper, $\mathrm{G}_{\mathrm{F4CC}}$ is theorem-equivalent to $\mathrm{S}_{\mathrm{F4CC}}$. Although the results of [13] for EPL were shown only for the propositional case (i.e., the case for 4CC), the results of this paper were shown for the first-order case (i.e., the case for F4CC). Thus, the proofs of the main theorems including cut-elimination and completeness are completely different in these papers (i.e., the present paper used Schütte's method for proving these theorems, but the paper [13] used Lahav and Avron's semantic method proposed in [14]). On the other hand, the proofs of the negative symmetry property for $\mathrm{S}_{\mathrm{F4CC}}$ is the same as that for EPL. In addition, a subsystem of EPL called PL, which is obtained from EPL by deleting the initial sequents $\sim\alpha, -\alpha \Rightarrow$ and $\Rightarrow \sim\alpha, -\alpha$, was introduced in [13] wherein the cut-elimination and completeness theorems for PL were proved using an embedding-based method. We can also show the same results for such a first-order subsystem of $\mathrm{S}_{\mathrm{F4CC}}$ in a similar way as those for PL.

As Schütte's method was also used for second-order logic [21], it would be interesting to see whether the current work can be generalized for a second-order extension of 4CC. A second question that is left for future research is whether the methods used here and other methods can be applied to prove completeness theorems with respect to natural many-valued and algebraic semantics for F4CC.

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