Rexpansions of Non-deterministic Matrices and Their Applications in Non-classical Logics

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Abstract

The operations of expansion and refinement on non-deterministic matrices (Nmatrices) are composed to form a new operation called rexpansion. Properties of this operation are investigated, together with their effects on the induced consequence relations. Using rexpansions, a semantic method for obtaining conservative extensions of (N)matrix-defined logics is introduced and applied to fragments of the classical two-valued matrix, as well as to other many-valued matrices and Nmatrices. The main application of this method is the construction and investigation of truth-preserving $\neg$-paraconsistent conservative extensions of Gödel fuzzy logic, in which $\neg$ has several desired properties. This is followed by some results regarding the relations between the constructed logics.

1 Introduction

Since its introduction in Avron and Lev (2005), the framework of *non-deterministic matrices*[1] (Nmatrices) has proven to be very useful, as it has almost all the advantages of the framework of ordinary matrices, while capturing logics that are practically left out by it. Accordingly, Nmatrices have been widely investigated and utilized in various areas, like many-valued logics (Kulicki and Trypuz 2012), paraconsistent logics (Avron 2007), and proof theory (Lahav 2013).

Nmatrices differ from (ordinary) matrices in that the truth value of a compound formula may not be uniquely determined by the truth values of its immediate subformulas, but only *constrained* by those truth values. This means that truth values of compound formulas are chosen non-deterministically from a set of options. The particular instance of ordinary matrices is obtained when all these sets are singletons. For some logics, this generalization provides an effective finite-valued semantics, where finite-valued matrices are beyond reach (see, e.g., Avron and Zamansky 2011).

[1] For a survey of Nmatrices, see Avron and Zamansky (2011).

In Arieli et al. (2011) and Avron (2007) two fundamental operations which are peculiar to Nmatrices were introduced: expansion and refinement. Both of them transform a given Nmatrix (that may be an ordinary matrix) to another one. The former amounts to a simple duplication of the truth values that are employed in the given Nmatrix, while the latter reduces the amount of non-determinism, by taking out possible values from the interpretations of the connectives. The two operations were shown useful for the modular construction of families of paraconsistent logics (Avron 2007, Avron et al. 2012), as well as for studying maximality properties in the constructed logics (Arieli et al. 2011).

In this paper we show that expansion and refinement are most useful when combined into one operation, which we call *rexpansion* (refined expansion). We investigate rexpansion as a powerful tool for generating new Nmatrices from existing ones. Properties of this combined operation are presented, along with its effects on the consequence relations which are induced by the operated Nmatrices. In particular, we identify a useful sufficient criterion for a rexpansion of an Nmatrix to result in an equivalent Nmatrix, that induces the same logic.

The main application of rexpansion in this paper is for the problem of conservatively extending a given logic **L** with new connectives which have some desirable properties. The method is to apply appropriate rexpansion to a matrix (or an Nmatrix) that is known to be characteristic for **L**, getting by this alternative semantics for it, for which the addition of the desired connectives is an easier task. The relations between the original logic and the extended one follow then from the general properties of rexpansions. We demonstrate this method with several examples, including matrices (and Nmatrices) for classical logic, paraconsistent logics, finite-valued logics and infinite-valued logics.

The most important demonstration of this technique provides a new (and as we show, significantly better) solution for the problem of constructing *paraconsistent fuzzy logics*. These are logics that are useful for modeling vague propositions, while avoiding the explosion principle, according to which any proposition follows from a contradiction. A first solution to this problem was given in Ertola et al. (2015), using a completely different approach. However, we show that this solution has some serious drawbacks, which are overcome in the solution proposed here. Our solution is obtained by performing different rexpansions on the Gödel matrix, and then augmenting the resulted Nmatrices with an involutive negation. We further investigate the connection between the various constructed logics.

The rest of this paper is organized as follows. We begin with a preliminaries section (Section 2), in which we review existing definitions and results in the theory of Nmatrices, including the operations of expansion and refinement. In Section 3 we combine these two operations into a single operation that is called *rexpansion*, and prove several results regarding this operation and its effects on consequence relations. Section 4 includes examples for applications of rexpansions in well-known logics from the literature. Section 5 focuses on paraconsistent conservative extensions of Gödel fuzzy logic that are obtained by performing various rexpansions on the Gödel matrix. We conclude with Section 6, in which several directions for further research are proposed.[2]

[2]A preliminary, concise version of this paper appears as Avron and Zohar (2017). Besides including full proofs, in this full version we have also expanded the demonstration of our approach in Section 4, and in particular included a detailed analysis of logics of formal inconsistency in Section 4.2. The investigation of paraconsistent conservative extensions of Gödel logic was expanded as well. In particular, the results in Section 5.3 are new.

2 Preliminaries

2.1 Propositional Logics and Matrices

A *propositional language* $\mathcal{L}$ consists of a countably infinite set of atomic variables *At* $= \{p_1, p_2, \ldots\}$ and a finite set $\diamondsuit_{\mathcal{L}}$ of propositional connectives. The set of all n-ary connectives of $\mathcal{L}$ is denoted by $\diamondsuit^n_{\mathcal{L}}$, and its set of well-formed formulas by $\mathcal{W}(\mathcal{L})$. We sometimes identify $\mathcal{L}$ with its set of connectives (e.g. when speaking about "the language $\{\wedge, \vee, \neg\}$"). A *propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ such that $\mathcal{L}$ is a propositional language and $\vdash_{\mathbf{L}}$ is a structural and non-trivial[3] (Tarskian) consequence relation for $\mathcal{L}$. In what follows, $\mathcal{L}$ denotes an arbitrary propositional language.

A notion that will be widely used in what follows, and especially when investigating relations between different logics, is that of *conservative extensions*:

Definition 2.1. A logic $\mathbf{L}_2 = \langle \mathcal{L}_2, \vdash_{\mathbf{L}_2} \rangle$ is *conservative over* a logic $\mathbf{L}_1 = \langle \mathcal{L}_1, \vdash_{\mathbf{L}_1} \rangle$ (or: $\mathbf{L}_2$ is a *conservative extension* of $\mathbf{L}_1$) if $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, and for every $\mathcal{T} \subseteq \mathcal{W}(\mathcal{L}_1)$ and $\varphi \in \mathcal{W}(\mathcal{L}_1)$ it holds that $\mathcal{T} \vdash_{\mathbf{L}_1} \varphi$ iff $\mathcal{T} \vdash_{\mathbf{L}_2} \varphi$.

The most standard way of defining logics semantically is by using *matrices* (Urquhart 2001):

Definition 2.2.

1. A *matrix* for $\mathcal{L}$ is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that:

   (a) $\mathcal{V}$ is a non-empty set (of *truth values*).

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (of *designated* truth values).

   (c) $\mathcal{O} : \diamondsuit_{\mathcal{L}} \to \bigcup_{i=0}^{\infty} (\mathcal{V}^i \to \mathcal{V})$ such that for every $i \in \mathbb{N}$ and $\diamond \in \diamondsuit^i_{\mathcal{L}}$, $\mathcal{O}(\diamond) : \mathcal{V}^i \to \mathcal{V}$. $\mathcal{O}(\diamond)$ is often regarded as the "truth table" of $\diamond$.

2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$. An $\mathcal{M}$-valuation is a function v from $\mathcal{W}(\mathcal{L})$ to $\mathcal{V}$ such that for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $v(\diamond(\psi_1, \ldots, \psi_n)) = \mathcal{O}(\diamond)(v(\psi_1), \ldots, v(\psi_n))$. An $\mathcal{M}$-valuation v is an $\mathcal{M}$-*model* of a formula ψ (in symbols: $v \vDash^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. It is an $\mathcal{M}$-*model* of a set $\mathcal{T}$ of formulas (in symbols: $v \vDash^{\mathcal{M}} \mathcal{T}$) if $v \vDash^{\mathcal{M}} \psi$ for every $\psi \in \mathcal{T}$. A formula ψ is an $\mathcal{M}$-*consequence* of a set $\mathcal{T}$ of formulas (in symbols: $\mathcal{T} \vdash_{\mathcal{M}} \psi$) if every $\mathcal{M}$-model of $\mathcal{T}$ is an $\mathcal{M}$-model of ψ. We say that $\mathcal{M}$ *induces a logic* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ (or that $\mathcal{M}$ is *characteristic for* $\mathbf{L}$) if $\vdash_{\mathcal{M}} = \vdash_{\mathbf{L}}$.

Many well-known non-classical logics are characterized using matrices:

Example 2.3. Asenjo–Priest's three-valued logic of paradox $\mathbf{LP}$ (Asenjo 1966, Priest 1979) and Kleene's three-valued logic $\mathbf{KL}$ (Kleene 1938) are both defined by matrices that differ only in the set of designated values. Consider the set $\mathcal{V}_3 = \{t, f, i\}$, and the interpretation function $\mathcal{O}_3$ that is defined by the following tables:

[3] This requirement is not always demanded in the literature, but we find it convenient (and natural) to include it here.

$\mathcal{O}_3(\wedge)$	t	f	i
t	t	f	i
f	f	f	f
i	i	f	i

$\mathcal{O}_3(\vee)$	t	f	i
t	t	t	t
f	t	f	i
i	t	i	i

	$\mathcal{O}_3(\neg)$
t	f
f	t
i	i

LP is characterized by the matrix $\mathcal{M}_{\mathbf{LP}} = \langle \mathcal{V}_3, \{t, i\}, \mathcal{O}_3 \rangle$, and **KL** by the matrix $\mathcal{M}_{\mathbf{KL}} = \langle \mathcal{V}_3, \{t\}, \mathcal{O}_3 \rangle$.

Example 2.4. Gödel fuzzy logic **G** (Dummett 1959) is characterized by the following matrix $\mathcal{M}_{\mathbf{G}} = \langle \mathcal{V}_{\mathbf{G}}, \mathcal{D}_{\mathbf{G}}, \mathcal{O}_{\mathbf{G}} \rangle$ for $\{\wedge, \vee, \supset, \perp\}$:

1. $\mathcal{V}_{\mathbf{G}} = [0, 1]$
2. $\mathcal{D}_{\mathbf{G}} = \{1\}$
3. $\mathcal{O}_{\mathbf{G}}(\perp) = 0$, $\mathcal{O}_{\mathbf{G}}(\vee)(a, b) = \max\{a, b\}$, $\mathcal{O}_{\mathbf{G}}(\wedge)(a, b) = \min\{a, b\}$, and $\mathcal{O}_{\mathbf{G}}(\supset)(a, b) = \begin{cases} 1 & a \leq b \\ b & a \not\leq b \end{cases}$.

Łukasiewicz fuzzy logic Ł (Łukasiewicz 1930) is characterized by a matrix that differs from $\mathcal{M}_{\mathbf{G}}$ solely in the interpretation of $\supset$, that is changed to: $\mathcal{O}(\supset)(a, b) \begin{cases} 1 & a \leq b \\ 1 - a + b & a \not\leq b \end{cases}$.

2.2 Non-deterministic Matrices

Matrices are *truth-functional*, that is, the truth value of a compound formula is uniquely determined by the truth values of its immediate subformulas. In Avron and Zamansky (2011), matrices are generalized to allow non-deterministic assignments of truth values to compound formulas.

Definition 2.5.

1. A *non-deterministic matrix (Nmatrix)* for $\mathcal{L}$ is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that:
   (a) $\mathcal{V}$ is a non-empty set (of *truth values*).
   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (of *designated* truth values).
   (c) $\mathcal{O} : \Diamond_{\mathcal{L}} \to \bigcup_{i=0}^{\infty} (\mathcal{V}^i \to P^+(\mathcal{V}))$ such that for every $i \in \mathbb{N}$ and $\diamond \in \Diamond_{\mathcal{L}}^i$, $\mathcal{O}(\diamond) : \mathcal{V}^i \to P^+(\mathcal{V})$ (where $P^+(\mathcal{V}) = P(\mathcal{V}) \setminus \{\emptyset\}$).
2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$-valuation is a function v from $\mathcal{W}(\mathcal{L})$ to $\mathcal{V}$ such that for every $\diamond \in \Diamond_{\mathcal{L}}^n$ and $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \mathcal{O}(\diamond)(v(\psi_1), \ldots, v(\psi_n))$. The definitions of $\mathcal{M}$-models and $\mathcal{M}$-consequences are as in Definition 2.2, using the non-deterministic notion of an $\mathcal{M}$-valuation.

To be considered as a particular instance of Nmatrices, we take matrices to be Nmatrices in which $\mathcal{O}(\diamond)(x_1, \dots, x_n)$ is a singleton for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $x_1, \dots, x_n \in \mathcal{V}$. In matrices mentioned below we freely interchange truth values with their singletons, whenever there is no danger of confusion.

Like matrices, Nmatrices provide an *analytic* semantic framework, in the sense that for every Nmatrix $\mathcal{M}$, every partial $\mathcal{M}$-valuation can be extended to a full $\mathcal{M}$-valuation.[4] A useful consequence of this property is the modular character that the framework of Nmatrices exhibits:

Definition 2.6. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, and $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. $\mathcal{M}_2$ is an *extension of* $\mathcal{M}_1$ *to* $\mathcal{L}_2$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$, and $\mathcal{O}_1(\diamond) = \mathcal{O}_2(\diamond)$ for every $\diamond \in \diamondsuit_{\mathcal{L}_1}$.

Proposition 2.7. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ be Nmatrices for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. If $\mathcal{M}_1$ is an extension of $\mathcal{M}_2$ to $\mathcal{L}_2$ then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$.*

2.3 Expansions and Refinements

Next we present two basic operations from Arieli et al. (2011) and Avron (2007), that can be performed on Nmatrices: *expansions* and *refinements*. Loosely speaking, an expansion of an Nmatrix is obtained by making several distinct copies of each truth value, so that the new designated values are the copies of the original ones, and each value in the interpretation of the connectives is replaced by all of its copies. This is formally defined as follows:

Definition 2.8.

1. A function F is called an *expansion function* if for every $x \in dom(F)$, $F(x)$ is a non-empty set, and $F(x) \cap F(y) = \emptyset$ whenever $x \neq y$. We say that F is an *expansion function for an Nmatrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\mathcal{L}$ if it is an expansion function and $dom(F) = \mathcal{V}$.

2. For every expansion function F and $y \in \bigcup Im(F)$, we denote by $\widetilde{F}[y]$ the unique element $x \in dom(F)$ such that $y \in F(x)$.

3. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and F an expansion function for $\mathcal{M}$. The F*-expansion of* $\mathcal{M}$ is the Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$, where:

   (a) $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F(x)$.

   (b) $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F(x)$.

   (c) $\mathcal{O}_F(\diamond)(y_1, \dots, y_n) = \bigcup_{z \in \mathcal{O}(\diamond)(\widetilde{F}[y_1], \dots, \widetilde{F}[y_n])} F(z)$ for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $y_1, \dots, y_n \in \mathcal{V}_F$.

   $\mathcal{M}_2$ is an *expansion* of $\mathcal{M}_1$ if it is the F-expansion of it for some F.

Nothing but uniformly duplicating all truth values is done in expansions, and hence the consequence relation remains the same, as was shown in Arieli et al. (2011):

[4] Following Avron and Zamansky (2011), we use the term *analytic* for this property.

Proposition 2.9. *Let $\mathcal{M}_2$ be an expansion of $\mathcal{M}_1$. Then $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$.*

Example 2.10.

1. Two Nmatrices are isomorphic to one another if and only if one is the F-expansion of the other for some expansion function F (in which $F(x)$ is always a singleton).

2. Consider the usual matrix which is characteristic for classical logic, where the truth values are t and f. By assigning $\{t, \top\}$ to t and $\{f\}$ to f, we obtain an expansion function. The outcome of this expansion would be a non-deterministic matrix for classical logic, in which, for example, the interpretation of negation is $\mathcal{O}(\neg)(t) = \mathcal{O}(\neg)(\top) = \{f\}$ and $\mathcal{O}(\neg)(f) = \{t, \top\}$.

3. The classical matrix can be further expanded by assigning $[0, \frac{1}{2})$ to f and $\left[\frac{1}{2}, 1\right]$ to t. The outcome would be another non-deterministic matrix which is characteristic for classical logic. The interpretation of negation would then be $\mathcal{O}(\neg)(x) = [0, \frac{1}{2})$ whenever $x \geq \frac{1}{2}$ and $\mathcal{O}(\neg)(x) = \left[\frac{1}{2}, 1\right]$ whenever $x < \frac{1}{2}$.

Next, we define the *refinement* operation on Nmatrices. Loosely speaking, refining an Nmatrix means deleting some of its truth values, and then reducing the amount of non-determinism (each of these steps is optional). This is formally defined as follows:

Definition 2.11. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$. $\mathcal{M}_2$ is a *refinement* of $\mathcal{M}_1$ if:

1. $\mathcal{V}_2 \subseteq \mathcal{V}_1$.

2. $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$.

3. $\mathcal{O}_2(\diamond)(x_1, \ldots, x_n) \subseteq \mathcal{O}_1(\diamond)(x_1, \ldots, x_n)$ for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $x_1, \ldots, x_n \in \mathcal{V}_2$.

$\mathcal{M}_2$ is a *simple refinement* of $\mathcal{M}_1$ if in addition, $\mathcal{V}_2 = \mathcal{V}_1$.

Example 2.12. The infinite characteristic Nmatrix for classical logic from Example 2.10 can be (simply) refined by e.g. redefining $\mathcal{O}(\neg)$ in the following way: $\mathcal{O}(\neg)(x) = \{0\}$ whenever $x \geq \frac{1}{2}$ and $\mathcal{O}(\neg)(x) = \{1\}$ whenever $x < \frac{1}{2}$.

Clearly, refining an Nmatrix $\mathcal{M}$ can only reduce the set of $\mathcal{M}$-valuations. Consequently, we have the following proposition from Avron (2007):

Proposition 2.13. *Let $\mathcal{M}_2$ be a refinement of $\mathcal{M}_1$. Then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$.*

3 Refined Expansions

In this section we combine the two basic operations defined above and obtain *refined expansions* (in short: *rexpansions*). In what follows, $\mathcal{L}$ continues to denote a fixed propositional language, and by an Nmatrix we mean an Nmatrix for $\mathcal{L}$, unless stated otherwise.

3.1 Definition and Properties

We start by explicitly defining the combined operation and exploring its properties.

Definition 3.1. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices and F an expansion function for $\mathcal{M}_1$. We say that $\mathcal{M}_2$ is an *F-rexpansion of* $\mathcal{M}_1$ if it is a refinement of the F-expansion of $\mathcal{M}_1$. It is called:

1. *simple* if it is a simple refinement of the F-expansion of $\mathcal{M}_1$.
2. *preserving* if $F(x) \cap \mathcal{V}_2 \neq \emptyset$ for every $x \in \mathcal{V}_1$.
3. *strongly preserving* if it is preserving, and for every $x_1, \ldots, x_n \in \mathcal{V}_2$, $\diamond \in \diamondsuit^n_{\mathcal{L}}$, and $y \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1], \ldots, \widetilde{F}[x_n])$, it holds that the set $F(y) \cap \mathcal{O}_2(\diamond)(x_1, \ldots, x_n)$ is not empty.

$\mathcal{M}_2$ is called a *rexpansion of* $\mathcal{M}_1$ if it is an F-rexpansion of it for some expansion function F for $\mathcal{M}_1$. If $\mathcal{M}_2$ is a rexpansion of $\mathcal{M}_1$, then we may call $\mathcal{M}_2$ "preserving", "strongly preserving" or "simple" (without the suffix "rexpansion of $\mathcal{M}_1$") whenever that is clear from the context.

Loosely speaking, being a preserving rexpansion amounts to keeping at least one "copy" of every original truth value. Being strongly preserving means that this property holds not only for the set of truth values, but also for the interpretation of the connectives.

Example 3.2. The Nmatrix from Example 2.12 is a rexpansion of the classical matrix, which is simple and strongly preserving.

First, let us elaborate on the connections between the different properties of rexpansions:

Lemma 3.3. *Every simple rexpansion is preserving, every expansion is a strongly preserving rexpansion, and every preserving rexpansion of a matrix is strongly preserving.*

Proof. We prove that every preserving rexpansion of a matrix is strongly preserving. The other statements are trivial. Let $\mathcal{M}_1$ be a matrix, $\mathcal{M}_2$ an Nmatrix, and F an expansion function such that $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$. Let $x_1, \ldots, x_n \in \mathcal{V}_2$, $\diamond \in \diamondsuit^n_{\mathcal{L}}$, and $y \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1], \ldots, \widetilde{F}[x_1])$. We prove that $F(y) \cap \mathcal{O}_2(\diamond)(x_1, \ldots, x_n) \neq \emptyset$. $\mathcal{O}_2(\diamond)(x_1, \ldots, x_n) \subseteq \bigcup_{z \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1], \ldots, \widetilde{F}[x_1])} F(z)$ and $\mathcal{M}_1$ is a matrix. Therefore, $\mathcal{O}_2(\diamond)(x_1, \ldots, x_n) \subseteq F(y)$, which means that $F(y) \cap \mathcal{O}_2(\diamond)(x_1, \ldots, x_n) = \mathcal{O}_2(\diamond)(x_1, \ldots, x_n)$. This set cannot be empty, as $\mathcal{M}_2$ is an Nmatrix. $\square$

Next we provide a necessary and sufficient condition for an Nmatrix to be a rexpansion of another Nmatrix.

Proposition 3.4. *$\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is a rexpansion of $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ iff there is a function $f : \mathcal{V}_2 \to \mathcal{V}_1$ such that:*

1. *For every $x \in \mathcal{V}_2$, $x \in \mathcal{D}_2$ iff $f(x) \in \mathcal{D}_1$.*

2. *For every* $x_1, \ldots, x_n \in \mathcal{V}_2$ *and* $y \in \mathcal{O}_2(\diamond)(x_1, \ldots, x_n)$*, it holds that* $f(y) \in \mathcal{O}_1(\diamond)(f(x_1), \ldots, f(x_n))$.

Proof.
($\Leftarrow$): Suppose such a function f exists. For every subset Y of $\mathcal{V}_1$, denote the set $\{x \in \mathcal{V}_2 \mid f(x) \in Y\}$ by $f^{-1}[Y]$. Let $\mathcal{V}$ be some set such that $\mathcal{V} \cap \mathcal{V}_2 = \emptyset$ and $|\mathcal{V}| = |\mathcal{V}_1|$, and let $g : \mathcal{V}_1 \to \mathcal{V}$ be a bijection. We show that $\mathcal{M}_2$ is an F-rexpansion of $\mathcal{M}_1$ for

$$F = \lambda x \in \mathcal{V}_1. \begin{cases} f^{-1}[\{x\}] & x \in Im(f) \\ \{g(x)\} & otherwise \end{cases}$$

F is clearly an expansion function for $\mathcal{M}_1$. Now, $\mathcal{V}_2$ is the domain of f, and thus it is contained in $(\mathcal{V}_1)_F$. Next, by property (1) of f, $\mathcal{D}_2 = f^{-1}[\mathcal{D}_1]$, which, by the definition of F is equal to $(\mathcal{D}_1)_F \cap \mathcal{V}_2$. Finally, by property (2) of f,

$$\mathcal{O}_2(\diamond)(x_1, \ldots, x_n) \subseteq f^{-1}[\mathcal{O}_1(\diamond)(f(x_1), \ldots, f(x_n))] \subseteq$$

$$\bigcup_{z \in \mathcal{O}_1(\diamond)(f(x_1),\ldots,f(x_n))} F(z) = \bigcup_{z \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1],\ldots,\widetilde{F}[x_n])} F(z) = (\mathcal{O}_1)_F(\diamond)(x_1, \ldots, x_n)$$

($\Rightarrow$): If $\mathcal{M}_2$ is an F-rexpansion of $\mathcal{M}_1$ for some F, then the function $\lambda x \in \mathcal{V}_2.\widetilde{F}[x]$ satisfies the required conditions. □

Remark 3.5. In Avron and Zamansky (2011), the term ‘simple refinement’ was reserved for what is called here ‘refinement’, while the term ‘refinement’ was related to the functions from Proposition 3.4.

Another useful property of the rexpansion operation is that it induces some forms of transitivity:

Theorem 3.6.

1. *If* $\mathcal{M}_2$ *is a preserving rexpansion of* $\mathcal{M}_1$ *and* $\mathcal{M}_3$ *is a (preserving) rexpansion of* $\mathcal{M}_2$*, then* $\mathcal{M}_3$ *is a (preserving) rexpansion of* $\mathcal{M}_1$.

2. *If* $\mathcal{M}_2$ *is a strongly preserving rexpansion of* $\mathcal{M}_1$ *and* $\mathcal{M}_3$ *is a strongly preserving rexpansion of* $\mathcal{M}_2$*, then* $\mathcal{M}_3$ *is a strongly preserving rexpansion of* $\mathcal{M}_1$.

Proof. Let F and G be expansion functions such that $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a G-rexpansion of $\mathcal{M}_2$. For every $1 \leq i \leq 3$, assume that $\mathcal{M}_i = \langle \mathcal{V}_i, \mathcal{D}_i, \mathcal{O}_i \rangle$. Define $H = \lambda x \in \mathcal{V}_1. \bigcup_{y \in F(x) \cap \mathcal{V}_2} G(y)$. Using the fact that $\mathcal{M}_2$ is preserving, it can easily be shown that H is an expansion function for $\mathcal{M}_1$. We first prove that $\mathcal{M}_3$ is a H-rexpansion of $\mathcal{M}_1$:

$$\mathcal{V}_3 \subseteq (\mathcal{V}_2)_G = \bigcup_{y\in\mathcal{V}_2} G(y) = \bigcup_{y\in(\mathcal{V}_1)_F\cap\mathcal{V}_2} G(y) = \bigcup_{y\in\left(\bigcup_{x\in\mathcal{V}_1} F(x)\right)\cap\mathcal{V}_2} G(y) =$$

$$\bigcup_{y\in\bigcup_{x\in\mathcal{V}_1}(F(x)\cap\mathcal{V}_2)} G(y) = \bigcup_{x\in\mathcal{V}_1}\left(\bigcup_{y\in F(x)\cap\mathcal{V}_2} G(y)\right) = \bigcup_{x\in\mathcal{V}_1} H(x) = (\mathcal{V}_1)_H$$

and

$$(\mathcal{D}_2)_G = \left(\bigcup_{y\in\mathcal{D}_2} G(y)\right) = \left(\bigcup_{y\in(\mathcal{D}_1)_F\cap\mathcal{V}_2} G(y)\right) = \left(\bigcup_{y\in\left(\bigcup_{x\in\mathcal{D}_1} F(x)\right)\cap\mathcal{V}_2} G(y)\right) =$$

$$\left(\bigcup_{x\in\mathcal{D}_1}\bigcup_{y\in F(x)\cap\mathcal{V}_2} G(y)\right) = \left(\bigcup_{x\in\mathcal{D}_1} H(x)\right) = (\mathcal{D}_1)_H$$

which means that $\mathcal{D}_3 = (\mathcal{D}_2)_G \cap \mathcal{V}_3 = (\mathcal{D}_1)_H \cap \mathcal{V}_3$.

As for $\mathcal{O}_3$, let $\diamond \in \diamondsuit^n_{\mathcal{L}}$, $x_1, \dots, x_n \in \mathcal{V}_3$, and $w \in \mathcal{O}_3(\diamond)(x_1, \dots, x_n)$. We show that $w \in (\mathcal{O}_1)_H(\diamond)(x_1, \dots, x_n)$. $\mathcal{M}_3$ is a refinement of $(\mathcal{M}_2)_G$, and hence $w \in (\mathcal{O}_2)_G(\diamond)(x_1, \dots, x_n)$. Now, there must exists $z \in \mathcal{O}_2(\diamond)(\widetilde{G}[x_1], \dots, \widetilde{G}[x_n])$ such that $w \in G(z)$, and since $\mathcal{M}_2$ is a refinement of $(\mathcal{M}_1)_F$, we have $z \in (\mathcal{O}_1)_F(\diamond)(\widetilde{G}[x_1], \dots, \widetilde{G}[x_n])$. Similarly, there exists $u \in \mathcal{O}_1(\diamond)(\widetilde{F}\left[\widetilde{G}[x_1]\right], \dots, \widetilde{F}\left[\widetilde{G}[x_1]\right])$ such that $z \in F(u)$. So we have that there exists $z \in F(u) \cap \mathcal{V}_2$ such that $w \in G(z)$. Hence $w \in H(u)$. To prove that $w \in (\mathcal{O}_1)_H(\diamond)(x_1, \dots, x_n)$, we show that $u \in \mathcal{O}_1(\diamond)(\widetilde{H}[x_1], \dots, \widetilde{H}[x_n])$. That is, we show that for every $1 \le i \le n$, $\widetilde{H}[x_i] = \widetilde{F}\left[\widetilde{G}[x_i]\right]$. For every $1 \le i \le n$ let $y_i = \widetilde{H}[x_i]$, $z_i = \widetilde{G}[x_i]$, and $w_i = \widetilde{F}[z_i]$. We prove that $y_i = w_i$: $x_i \in H(y_i)$, and hence there exists $y \in F(y_i) \cap \mathcal{V}_2$ such that $x_i \in G(y)$. Since G is an expansion function, $y = z_i$. Hence $y \in F(w_i)$. Similarly, F is an expansion function, and hence $y_i = w_i$.

Next, we show that if $\mathcal{M}_3$ is a preserving G-rexpansion of $\mathcal{M}_2$ then it is a preserving H-rexpansion of $\mathcal{M}_1$, that is, $H(x) \cap \mathcal{V}_3 \neq \emptyset$ for every $x \in \mathcal{V}_1$. Since $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$, there exists $y \in F(x) \cap \mathcal{V}_2$. And since $\mathcal{M}_3$ is a preserving G-rexpansion of $\mathcal{M}_2$, there exists $z \in G(y) \cap \mathcal{V}_3 \subseteq H(x) \cap \mathcal{V}_3$.

Finally, we show that if $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a strongly preserving G-rexpansion of $\mathcal{M}_2$, then $\mathcal{M}_3$ is a strongly preserving H-rexpansion of $\mathcal{M}_1$. Let $z_1, \dots, z_n \in \mathcal{V}_3$ and $\diamond \in \diamondsuit^n_{\mathcal{L}}$. We show that $H(x) \cap \mathcal{O}_3(\diamond)(z_1, \dots, z_n) \neq \emptyset$ for every $x \in \mathcal{O}_1(\diamond)(\widetilde{H}[z_1], \dots, \widetilde{H}[z_n])$. Let $x \in \mathcal{O}_1(\diamond)(\widetilde{H}[z_1], \dots, \widetilde{H}[z_n])$. For every $1 \le i \le n$ let $x_i = \widetilde{H}[z_i]$. Then there exists $y_i \in F(x_i) \cap \mathcal{V}_2$ such that $z_i \in G(y_i)$. Since $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$, there exists $y_0 \in F(x) \cap \mathcal{O}_2(\diamond)(y_1, \dots, y_n)$. Since $\mathcal{M}_3$ is a strongly preserving G-rexpansion of $\mathcal{M}_2$, there also exists $z_0 \in G(y_0) \cap \mathcal{O}_3(\diamond)(z_1, \dots, z_n) \subseteq H(x) \cap \mathcal{O}_3(\diamond)(z_1, \dots, z_n)$. $\square$

Corollary 3.7. *For every sequence $\mathcal{M}_1, \dots, \mathcal{M}_n$ of Nmatrices such that $\mathcal{M}_{i+1}$ is an expansion or a simple refinement of $\mathcal{M}_i$, we have that $\mathcal{M}_n$ is a preserving rexpansion of $\mathcal{M}_1$.*

3.2 Consequence Relations

In this section we investigate the effect rexpansions induce on semantically defined consequence relations. Our main theorem is the following:

Theorem 3.8. *If $\mathcal{M}_2$ is a rexpansion of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Moreover, if $\mathcal{M}_2$ is strongly preserving then $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$.*

Proof. The first part follows directly from Propositions 2.9 and 2.13 above. Suppose $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$. We prove that $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$. For this, it obviously suffices to prove that for every $\mathcal{M}_1$-valuation v there exists an $\mathcal{M}_2$-valuation v' such that $v \vDash^{\mathcal{M}_1} \psi$ iff $v' \vDash^{\mathcal{M}_2} \psi$ for every $\psi \in \mathcal{W}(\mathcal{L})$. Let $c : P(\mathcal{V}_2) \setminus \{\emptyset\} \rightarrow \mathcal{V}_2$ and suppose that for every $X \in P(\mathcal{V}_2) \setminus \{\emptyset\}$, $c(X) \in X$.[5] Let $\psi_1, \psi_2, \ldots$ be an enumeration of $\mathcal{W}(\mathcal{L})$ such that if ψ_i is a subformula of ψ_j then $i < j$. Now let v be an $\mathcal{M}_1$-valuation. For the construction of v', we first define a sequence $v_0, v_1, \ldots$ of *partial functions* from $\mathcal{W}(\mathcal{L})$ to $\mathcal{V}_2$: v_0 is the empty function, and for every $i > 0$, v_i is defined as follows. For every $\psi \in dom(v_{i-1})$, $v_i(\psi) = v_{i-1}(\psi)$. If $\psi_i \notin dom(v_{i-1})$, then:

1. If ψ_i is atomic and $F(v(\psi_i)) \cap \mathcal{V}_2$ is not empty, $v_i(\psi_i) = c(F(v(\psi_i)) \cap \mathcal{V}_2)$.
2. If ψ_i has the form $\diamond(\varphi_1, \ldots, \varphi_n)$ for $\varphi_1, \ldots, \varphi_n \in dom(v_{i-1})$ and $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \ldots, v_{i-1}(\varphi_n))$ is not empty, $v_i(\psi_i) = c(F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \ldots, v_{i-1}(\varphi_n)))$.

We prove by induction on i that:

1. $v_i(\psi) \in F(v(\psi))$ for every $\psi \in dom(v_i)$;
2. $dom(v_i) = \{\psi_1, \ldots, \psi_i\}$; and
3. v_i satisfies the conditions induced by $\mathcal{M}_2$, that is: $v_i(\diamond(\varphi_1, \ldots, \varphi_n)) \in \mathcal{O}_2(\diamond)(v_i(\varphi_1), \ldots, v_i(\varphi_n))$ whenever $\varphi_1, \ldots, \varphi_n$ and $\diamond(\varphi_1, \ldots, \varphi_n)$ are in $dom(v_i)$.

For $i = 0$, this trivially holds. Let $i > 0$.

1. Let $\psi \in dom(v_i)$. If $\psi \in dom(v_{i-1})$ then this holds by the induction hypothesis. Otherwise, $\psi = \psi_i$, and then this holds by definition.
2. By the induction hypothesis, $dom(v_{i-1}) = \{\psi_1, \ldots, \psi_{i-1}\}$, and therefore we have to prove that $\psi_i \in dom(v_i)$. If ψ_i is atomic, this amounts to showing that the set $F(v(\psi_i)) \cap \mathcal{V}_2$ is not empty, which holds as $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$. Otherwise, ψ_i has the form $\diamond(\varphi_1, \ldots, \varphi_n)$. By our enumeration and the induction hypothesis, $\varphi_1, \ldots, \varphi_n \in dom(v_{i-1})$, and therefore this amounts to showing that $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \ldots, v_{i-1}(\varphi_n))$ is not empty. By the induction hypothesis, we have that $v_{i-1}(\varphi_j) \in F(v(\varphi_j))$ for every $1 \leq j \leq n$. In other words, $v(\varphi_j) = \widetilde{F}[v_{i-1}(\varphi_j)]$ for every $1 \leq j \leq n$. By the fact that v is an $\mathcal{M}_1$-valuation, $v(\psi_i) \in \mathcal{O}_1(\diamond)(v(\varphi_1), \ldots, v(\varphi_n)) = \mathcal{O}_1(\diamond)(\widetilde{F}[v_{i-1}(\varphi_1)], \ldots, \widetilde{F}[v_{i-1}(\varphi_n)])$, and hence $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \ldots, v_{i-1}(\varphi_n)) \neq \emptyset$, as $\mathcal{M}_2$ is strongly preserving.

[5] The existence of such a function relies on the axiom of choice in case $\mathcal{V}_2$ is infinite.

3. Let $\diamond(\varphi_1,\ldots,\varphi_n),\varphi_1,\ldots,\varphi_n \in dom(v_i)$. We prove that $v_i(\diamond(\varphi_1,\ldots,\varphi_n)) \in \mathcal{O}_2(\diamond)(v_i(\varphi_1),\ldots,v_i(\varphi_n))$. If $\diamond(\varphi_1,\ldots,\varphi_n) \in dom(v_{i-1})$, then this holds by the induction hypothesis and our enumeration. Otherwise, $\psi_i = \diamond(\varphi_1,\ldots,\varphi_n)$, and then this holds by the induction hypothesis and the definition of v_i.

For every $\psi \in \mathcal{W}(\mathcal{L})$, let $i_\psi = \iota i \in \mathbb{N}.\psi = \psi_i$.[6] v' is defined by $v'(\psi) = v_{i_\psi}(\psi)$. First, we show that v' is an $\mathcal{M}_2$-valuation. Let $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $\varphi_1,\ldots,\varphi_n \in \mathcal{W}(\mathcal{L})$. Let $k = i_{\diamond(\varphi_1,\ldots,\varphi_n)}$. $v'(\diamond(\varphi_1,\ldots,\varphi_n)) = v_k(\diamond(\varphi_1,\ldots,\varphi_n))$, which belongs to $\mathcal{O}_2(\diamond)(v_k(\varphi_1),\ldots,v_k(\varphi_n)) = \mathcal{O}_2(\diamond)(v'(\varphi_1),\ldots,v'(\varphi_n))$. Second, we show that $v \vDash^{\mathcal{M}_1} \psi$ iff $v' \vDash^{\mathcal{M}_2} \psi$. Suppose $v \vDash^{\mathcal{M}_1} \psi$. Then $v(\psi) \in \mathcal{D}_1$. Now, by the construction of v', $v'(\psi) \in F(v(\psi)) \subseteq (\bigcup_{x\in\mathcal{D}_1} F(x)) \cap \mathcal{V}_2 = \mathcal{D}_2$, which means that $v' \vDash^{\mathcal{M}_2} \psi$. For the converse, suppose $v' \vDash^{\mathcal{M}_2} \psi$. Then $v'(\psi) \in \mathcal{D}_2 \subseteq (\bigcup_{x\in\mathcal{D}_1} F(x))$. Hence there exists $x \in \mathcal{D}_1$ such that $v'(\psi) \in F(x)$. Now, by the construction of v', $v'(\psi) \in F(v(\psi))$. Since F is an expansion function, $v(\psi) = x \in \mathcal{D}_1$, which means that $v \vDash^{\mathcal{M}_1} \psi$. □

The following corollary immediately follows as a consequence of Lemma 3.3 and Theorem 3.8:

Corollary 3.9. *Let $\mathcal{M}_2$ be a preserving rexpansion of $\mathcal{M}_1$. If $\mathcal{M}_1$ is a matrix then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.*

An important consequence of Corollary 3.9 and Proposition 2.7 (the usefulness of which is demonstrated in Sections 4 and 5) is a general method for providing a given logic with an alternative new semantics, and then use it for conservatively augmenting it with new connectives. This is established in the following corollary:

Corollary 3.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, $\mathcal{M}_1$ a matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some preserving rexpansion of $\mathcal{M}_1$. Then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2}\rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1}\rangle$.*

We conclude this section with a stronger instance of Corollary 3.10, that applies only for two-valued matrices:

Corollary 3.11. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, $\mathcal{M}_1 = \langle \{t,f\}, \{t\}, \mathcal{O}_1\rangle$ a matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some rexpansion of $\mathcal{M}_1$. Then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2}\rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1}\rangle$.*

Proof. By Definition 2.5, for every F-rexpansion $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}'\rangle$ of $\mathcal{M}_1$ we must have $F(t) \cap \mathcal{V}' \neq \emptyset$ (as otherwise $\mathcal{D}' = \emptyset$) and $F(f) \cap \mathcal{V}' \neq \emptyset$ (as otherwise $\mathcal{D}' = \mathcal{V}'$). The result then follows from Corollary 3.10. □

4 Some Basic Applications

In this section we present some examples of applications of rexpansion in non-classical logics. This is done by performing it on fragments of well-known matrices and Nmatrices, and thus obtaining conservative extensions of their induced logics.

[6] That is, i_ψ is the index of ψ in the enumeration.

Let $CL = \{\neg, \vee, \wedge, \supset, \bot\}$. Denote the (propositional) language whose set of connectives is CL by $\mathcal{CL}$, classical logic (over $\mathcal{CL}$) by $\mathbf{CL}$, and the classical two-valued matrix for $\mathcal{CL}$ by $\mathcal{M}_{\mathbf{CL}}$. For any $C \subseteq CL$, we denote the C-fragment of $\mathcal{CL}$ by $\mathcal{CL}^C$, the C-fragment of classical logic by $\mathbf{CL}^C$, and the C-fragment of the classical matrix by $\mathcal{M}^C_{\mathbf{CL}}$. We start with a direct consequence of Corollary 3.11:

Lemma 4.1. *Let $C \subseteq CL$. If $\mathcal{L}$ is a language such that $\mathcal{W}(\mathcal{CL}^C) \subseteq \mathcal{W}(\mathcal{L})$, and $\mathcal{M}$ is an extension to $\mathcal{L}$ of some rexpansion of $\mathcal{M}^C_{\mathbf{CL}}$, then $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is conservative over $\mathbf{CL}^C$.*

We shall use Lemma 4.1 to present conservative extensions of fragments of classical logic. Some of the resulting logics are *paraconsistent*, that is: unlike classical logic, they tolerate contradictions. Here is a formal definition, based on properties that were investigated in Carnielli et al. (2007), Marcos (2005), Arieli et al. (2011), Arieli and Avron (2015):

Definition 4.2. Let $\mathbf{L}$ be a logic in $\mathcal{CL}$. $\neg$ is a *weak negation in* $\mathbf{L}$ if $p \nvdash_{\mathbf{L}} \neg p$ and $\neg p \nvdash_{\mathbf{L}} p$. $\neg$ is a *negation in* $\mathbf{L}$ if $\mathbf{L}$ is *subclassical*, that is, $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ only if $\mathcal{T} \vdash_{\mathbf{CL}} \varphi$. $\mathbf{L}$ is *paraconsistent* if $\neg$ is a weak negation in $\mathbf{L}$ and $\neg p, p \nvdash_{\mathbf{L}} q$. It is *strictly paraconsistent* if it is paraconsistent and $\neg$ is a negation in $\mathbf{L}$. It is *boldly paraconsistent* if it is paraconsistent, and $\neg p, p \nvdash_{\mathbf{L}} \varphi$ whenever $\nvdash_{\mathbf{L}} \varphi$ and $p \notin At(\varphi)$.

Remark 4.3. Inspired by Marcos (2005), the requirement for being a weak negation is regarded in Arieli et al. (2011) as a minimal condition that is expected from a unary connective to be called a negation. This is hardly enough, though, to characterize negation. Therefore, Arieli and Avron (2015) generalizes it to the requirement of subclassicality, leading to what is called here "strict paraconsistency". (Note that in practice, almost all non-classical logics in $\mathcal{CL}$ that have ever been studied are subclassical.) Finally, the requirement of bold paraconsistency connects paraconsistency to (and justifies it by) the broader principle of *relevance*: the inconsistency of p should not be a reason for inferring a formula that is completely irrelevant to p.

4.1 Rexpansion of Matrices

We start with finite-valued conservative extensions of classical logic.

Example 4.4. The $\{\wedge, \vee\}$-fragments of $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{KL}}$ (see Example 2.3) are simple rexpansions of $\mathcal{M}^{\{\wedge,\vee\}}_{\mathbf{CL}}$, as can be witnessed by the expansion functions $\lambda x \in \{t, f\} . \begin{cases} \{t, i\} & x = t \\ \{f\} & x = f \end{cases}$ and $\lambda x \in \{t, f\} . \begin{cases} \{t\} & x = t \\ \{f, i\} & x = f \end{cases}$, respectively. By Lemma 4.1, $\mathbf{LP}$ and $\mathbf{KL}$ are both conservative over $\mathbf{CL}^{\{\wedge,\vee\}}$. Note that neither of the matrices is a preserving expansion of the other: suppose for contradiction that $\mathcal{M}_{\mathbf{LP}}$ is a preserving F-expansion of $\mathcal{M}_{\mathbf{KL}}$. Then we must have that $\{t, i\} \subseteq F(t)$, and so $f \in F(i) \cup F(f)$. If $f \in F(i)$ then $F(f) \cap \{t, f, i\} = \emptyset$, and if $f \in F(f)$ then $F(i) \cap \{t, f, i\} = \emptyset$. Either way, $\mathcal{M}_{\mathbf{LP}}$ is not preserving. Clearly, $\mathcal{M}_{\mathbf{KL}}$ cannot be a preserving F-expansion of $\mathcal{M}_{\mathbf{LP}}$, as if this were the case, it would have two designated values.

Next, we consider the three-valued paraconsistent logics from Arieli and Avron (2015):

Example 4.5. Theorem 42 of Arieli and Avron (2015) characterizes all three-valued strictly paraconsistent logics in the language $\{\neg, \wedge, \vee, \supset\}$ that admit some natural properties. These logics coincide with the $\{\neg, \wedge, \vee, \supset\}$-fragments of the family of $8K$ conservative extensions of positive classical logic studied in Carnielli et al. (2007) and Carnielli and Marcos (2002). The three-valued matrices that induce these logics are all simple refinements of the following Nmatrix $\mathcal{M} = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O} \rangle$, where $\mathcal{O}$ is given by:

$\mathcal{O}(\wedge)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\mathcal{O}(\vee)$	t	f	$\top$
t	$\{t\}$	$\{t\}$	$\{t, \top\}$
f	$\{t\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\mathcal{O}(\supset)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t, \top\}$
f	$\{t\}$	$\{t\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

	$\mathcal{O}(\neg)$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t, \top\}$

Now, every simple refinement of $\mathcal{M}$ is an extension to $\{\wedge, \vee, \supset, \neg\}$ of some rexpansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$. Indeed, for $F(t) = \{t, \top\}$ and $F(f) = \{f\}$, it is easy to see that the $\{\wedge, \vee, \supset\}$-fragment of $\mathcal{M}$ is a simple refinement of $(\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}})_F = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O}' \rangle$, where $\mathcal{O}'$ is defined by:

$\mathcal{O}'(\wedge)$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\mathcal{O}'(\vee)$	t	f	$\top$
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\mathcal{O}'(\supset)$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

The fact that all these logics are conservative over positive classical logic is then obtained as a consequence of Lemma 4.1. Actually, by Corollary 3.11, all extensions of simple refinements of $(\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}})_F$ have this property. In addition, they have many of the natural properties demanded in Arieli and Avron (2015).

Example 4.6. The $\{\wedge, \vee, \supset, \neg\}$-fragment of the four-valued logic of bilattices from Arieli and Avron (1998) is characterized by the matrix $\mathcal{M}_4 = \langle \{t, f, \top, \bot\}, \{\top, \bot\}, \mathcal{O}_4 \rangle$, where $\mathcal{O}_4$ is given by:

$\mathcal{O}_4(\wedge)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\mathcal{O}_4(\vee)$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\mathcal{O}_4(\supset)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	t	t	t	t
$\top$	t	f	$\top$	$\bot$
$\bot$	t	t	t	t

	$\mathcal{O}_4(\neg)$
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

By dismissing $\supset$, we obtain a matrix for the logic of first-degree entailment (fde) from Anderson and Belnap (1975). Define an expansion function F for $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$ by $F(f) = \{f, \bot\}$ and $F(t) = \{t, \top\}$. It is easy to see that $\mathcal{M}_4$ is an extension to $\{\wedge, \vee, \supset, \neg\}$ of a simple refinement of $(\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}})_F = \langle \{t, \top, f, \bot\}, \{t, \top\}, \mathcal{O} \rangle$, where

$\mathcal{O}(\wedge)$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
f	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$
$\top$	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
$\bot$	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$

$\mathcal{O}(\vee)$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\bot$	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$

$\mathcal{O}(\supset)$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
$\bot$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

Arieli and Avron (1998) provided an analytic sequent calculus for $\vdash_{\mathcal{M}_4}$, and used it to prove that it is conservative over $\mathbf{CL}^{\{\wedge,\vee,\supset\}}$ (and that fde is conservative over $\mathbf{CL}^{\{\wedge,\vee\}}$). Here we obtain this result as a simple consequence of Lemma 4.1, by identifying the $\{\wedge,\vee,\supset\}$-fragment of $\mathcal{M}_4$ as a rexpansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$.

The next example concerns Gödel fuzzy logic $\mathbf{G}$ and its relation to classical logic.

Example 4.7. It is routine to verify that $\mathcal{M}_{\mathbf{G}}$ (Example 2.4) is an extension to $\{\wedge,\vee,\supset,\bot\}$ of a simple refinement of the F-expansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\bot\}}$, for $F(f)=[0,1)$ and $F(t)=\{1\}$. Consider $\wedge$, for example, and denote its classical interpretation by $\mathcal{O}(\wedge)$. Let $x,y\in[0,1]$ and $z=\min(x,y)$, and suppose $x'=\widetilde{F}[x]$ and $y'=\widetilde{F}[y]$. We show that $z\in F(\mathcal{O}(\wedge)(x',y'))$ (recall that in matrices we identify singletons with their unique elements). If $z<1$, then either $x<1$ or $y<1$, and so either $x'=f$ or $y'=f$, which means that $\mathcal{O}(\wedge)(x',y')=f$. In this case, we get $z\in[0,1)=F(f)=F(\mathcal{O}(\wedge)(x',y'))$. Otherwise, $z=1$, which means that $x=y=1$, and so $x'=y'=t$. In this case, $\mathcal{O}(x',y')=t$, and so we have $z\in\{1\}=F(t)=F(\mathcal{O}(\wedge)(x',y'))$. By Lemma 4.1, $\mathcal{M}_{\mathbf{G}}$ is conservative over $\mathbf{CL}^{\{\wedge,\vee,\bot\}}$.

Note that this argument does not survive the addition of implication: $\mathcal{O}_{\mathbf{G}}(\supset)(0.5,0.25)=0.25$, while $\mathcal{O}_{\mathbf{CL}}(\supset)(0,0)=1$ and $0.25\notin F(1)$. And indeed, $\mathbf{G}$ is not conservative over positive classical logic, as, for example, the classical tautology $((p\supset q)\supset p))\supset p$ is not valid in it (as can be seen e.g. by assigning 0.5 to p and 0.25 to q).

The process described in the above examples need not start with classical logic, as can be seen by the following example:

Example 4.8. Consider the following matrix $\mathcal{M}=\langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$, defined by $\mathcal{V}=\{t,f,\top,\bot\}$, $\mathcal{D}=\{t\}$, and $\mathcal{O}$ is given by:

$\mathcal{O}(\wedge)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	$\bot$
$\bot$	$\bot$	f	$\bot$	$\bot$

$\mathcal{O}(\vee)$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	$\top$
$\bot$	t	$\bot$	$\top$	$\bot$

	$\mathcal{O}_4(\neg)$
t	f
f	t
$\top$	t
$\bot$	t

Its conjunction and disjunction are interpreted as minimum and maximum (respectively) over the ordering $f\leq\bot\leq\top\leq t$. Its $\{\wedge,\vee\}$-fragment is a simple F-expansion of the

$\{\wedge,\vee\}$-fragment of $\mathcal{M}_{\mathbf{KL}}$ (Example 2.3), for $F(t)=\{t\}$, $F(f)=\{f,\bot\}$ and $F(i)=\{\top\}$. By Corollary 3.10, the logic it induces is conservative over the $\{\wedge,\vee\}$-fragment of $\mathbf{KL}$. It is a different logic than $\mathbf{KL}$, as it has tautologies (e.g. $p\vee\neg p$).

4.2 Rexpansion of Nmatrices

In all the examples above, rexpansions were performed on *matrices*. In this section, Nmatrices are taken as the starting points.

Avron et al. (2012) provided cut-free sequent calculi for many paraconsistent logics of the family called *Logics of Formal Inconsistency (LFIs)* (Carnielli et al. 2007, Carnielli and Marcos 2002). This is done uniformly, by first finding a characteristic Nmatrix, and then extracting a sequent calculus from it. The underlying language is $\mathcal{CL}^{\circ}=\mathcal{CL}\setminus\{\bot\}\cup\{\circ\}$, where $\circ$ is a unary connective which is intended to classify a given proposition as *consistent* (that is, $\circ\varphi$ should be read as "φ is consistent"). We show how rexpansions can be incorporated into this investigation in a useful way, that naturally uncovers relations between the different Nmatrices involved, as well as their induced logics.

The most basic logic that is investigated in Avron et al. (2012) is called BK. It is proven there to be characterized by the Nmatrix $\mathcal{M}_{BK}$, which is the extension to $\mathcal{CL}^{\circ}$ of the Nmatrix $(\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}})_F$ from Example 4.5, given by:

	$\mathcal{O}(\neg)$
t	$\{f\}$
f	$\{t,\top\}$
$\top$	$\{t,\top\}$

	$\mathcal{O}(\circ)$
t	$\{t,\top\}$
f	$\{t,\top\}$
$\top$	$\{f\}$

While BK serves as a basis for the modular construction of more powerful paraconsistent logics, its negation lacks some fundamental properties. For example, the following principles are not valid in it:

- Double negation: $(\mathbf{c})$ $\neg\neg\varphi\supset\varphi$ and $(\mathbf{e})$ $\varphi\supset\neg\neg\varphi$.
- De Morgan laws: e.g., $(\mathbf{n}^{\mathbf{r}}_{\wedge})$ $(\neg\varphi\vee\neg\psi)\supset\neg(\varphi\wedge\psi)$ and $(\mathbf{n}^{\mathbf{l}}_{\wedge})$ $\neg(\varphi\wedge\psi)\supset(\neg\varphi\vee\neg\psi)$.

Some basic properties that could be expected from the consistency operator are missing as well. Examples include:

- Inconsistency: $(\mathbf{i})$ $\neg\circ\varphi\supset(\varphi\wedge\neg\varphi)$
- Propagation laws: $(\mathbf{a})$ $(\circ\varphi\sharp\circ\psi)\supset\circ(\varphi\sharp\psi)$ for $\sharp\in\{\wedge,\vee,\supset\}$

Accordingly, a set A_0 of well-known axioms for LFIs is considered (that includes, among others, the aforementioned formulas), and is modularly incorporated into this Nmatrix: each subset of A_0 induces a simple refinement of $\mathcal{M}_{BK}$. For example, the addition of the axiom $(\mathbf{c})$ above amounts to setting $\mathcal{O}(\neg)(f)$ to $\{t\}$ (instead of $\{t,\top\}$). Further, the addition of $(\mathbf{a})$ amounts to ensuring that $\varphi\sharp\psi$ is given a value from $\{t,f\}$ whenever both φ and ψ are given values from $\{t,f\}$, for each $\sharp\in\{\wedge,\vee,\supset\}$. The Nmatrix that corresponds to the logic $BKca$, obtained by the addition of $(\mathbf{c})$ and $(\mathbf{a})$ to BK, turns out to be the extension to $\mathcal{CL}^{\circ}$ of the

Nmatrix $\mathcal{M}$ from Example 4.5, obtained by including the truth table for $\circ$ above. We denote the resulting Nmatrix by $\mathcal{M}_{BKca} = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O}_{BKca} \rangle$.

Things become more complicated when the following two well-known axioms for LFIs are added to A_0:

$$(\mathbf{l})\ \neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \qquad \text{and} \qquad (\mathbf{d})\ \neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$$

It was shown in Avron (2007) that most of the systems in the family induced by $A_0 \cup \{(\mathbf{l}), (\mathbf{d})\}$ that include at least one of $\{(\mathbf{l}), (\mathbf{d})\}$ cannot be characterized by a finite Nmatrix. This means that they go beyond the reach of $\mathcal{M}_{BK}$ and its refinements.

For this reason, an *infinite* Nmatrix, that we denote by $\mathcal{M}^{\infty}_{BK} = \langle \mathcal{V}^{\infty}_{BK}, \mathcal{D}^{\infty}_{BK}, \mathcal{O}^{\infty}_{BK} \rangle$, is introduced, which utilizes the following three (disjoint) sets: $\mathcal{T} = \left\{ t^j_i \mid i \geq 0, j \geq 0 \right\}$, $\mathcal{I} = \left\{ \top^j_i \mid i \geq 0, j \geq 0 \right\}$, and $\mathcal{F} = \{f\}$. $\mathcal{M}^{\infty}_{BK}$ is then defined by $\mathcal{V}^{\infty}_{BK} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, $\mathcal{D}^{\infty}_{BK} = \mathcal{T} \cup \mathcal{I}$, and:

$$\mathcal{O}^{\infty}_{BK}(\wedge)(a, b) = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D}^{\infty}_{BK} & otherwise \end{cases} \qquad \mathcal{O}^{\infty}_{BK}(\vee)(a, b) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{D}^{\infty}_{BK} \text{ or } b \in \mathcal{D}^{\infty}_{BK} \\ \mathcal{F} & otherwise \end{cases}$$

$$\mathcal{O}^{\infty}_{BK}(\neg)(a) = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \\ \left\{ \top^{j+1}_i, t^{j+1}_i \right\} & a = \top^j_i \end{cases} \qquad \mathcal{O}^{\infty}_{BK}(\circ)(a) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & otherwise \end{cases}$$

$$\mathcal{O}^{\infty}_{BK}(\supset)(a, b) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \text{ or } b \in \mathcal{D}^{\infty}_{BK} \\ \mathcal{F} & otherwise \end{cases}$$

Now, $\mathcal{M}^{\infty}_{BK}$ is a strongly preserving F-rexpansion of M_{BK}, for

$$F = \lambda x \in \{t, f, \top\} . \begin{cases} \mathcal{T} & t \\ \mathcal{F} & f \\ \mathcal{I} & \top \end{cases}$$

Moreover, every subset A of A_0 induces a simple refinement of $\mathcal{M}^{\infty}_{BK}$, that is a strongly preserving F-rexpansion of the simple refinement of $\mathcal{M}_{BK}$ that is associated with A. In particular, without $(\mathbf{l})$ and $(\mathbf{d})$, we obtain an infinite characteristic Nmatrix for each system, equivalent to the three-valued one. For example, going back to $BKca$, we obtain the Nmatrix $\mathcal{M}^{\infty}_{BKca} = \langle \mathcal{V}^{\infty}_{BK}, \mathcal{D}^{\infty}_{BK}, \mathcal{O}^{\infty}_{BKca} \rangle$, where $\mathcal{O}^{\infty}_{BKca}$ is obtained from $\mathcal{O}^{\infty}_{BK}$ by setting $\mathcal{O}^{\infty}_{BKca}(\neg)(f) = \mathcal{T}$, and ensuring that $\varphi \sharp \psi$ is given a value from $\mathcal{T} \cup \mathcal{F}$, whenever both φ and ψ are given values from $\mathcal{T} \cup \mathcal{F}$, for every $\sharp \in \{\wedge, \vee, \supset\}$:

$$\mathcal{O}^{\infty}_{BKca}(\wedge)(a, b) = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & a, b \in \mathcal{T} \\ \mathcal{D}^{\infty}_{BK} & otherwise \end{cases} \qquad \mathcal{O}^{\infty}_{BKca}(\vee)(a, b) = \begin{cases} \mathcal{T} & a \in \mathcal{T} \text{ and } b \in \mathcal{T} \cup \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T} \cup \mathcal{F} \text{ and } b \in \mathcal{T} \\ \mathcal{F} & a, b \in \mathcal{F} \\ \mathcal{D}^{\infty}_{BK} & otherwise \end{cases}$$

$$\mathcal{O}^{\infty}_{BKca}(\neg)(a) = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{T} & a \in \mathcal{F} \\ \{\top_i^{j+1}, t_i^{j+1}\} & a = \top_i^j \end{cases} \quad \mathcal{O}^{\infty}_{BKca}(\circ)(a) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & otherwise \end{cases}$$

$$\mathcal{O}^{\infty}_{BKca}(\supset)(a, b) = \begin{cases} \mathcal{T} & a \in \mathcal{F} \text{ and } b \in \mathcal{T} \cup \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T} \cup \mathcal{F} \text{ and } b \in \mathcal{T} \\ \mathcal{F} & a \in \mathcal{D}^{\infty}_{BK} \text{ and } b \in \mathcal{F} \\ \mathcal{D}^{\infty}_{BK} & otherwise \end{cases}$$

It is routine to verify that $\mathcal{M}^{\infty}_{BKca}$ is a strongly preserving F-rexpansion $\mathcal{M}_{BKca}$. By Theorem 3.8, $\vdash_{\mathcal{M}_{BKca}} = \vdash_{\mathcal{M}^{\infty}_{BKca}}$.

When either $(\mathbf{l})$ or $(\mathbf{d})$ are included, however, what is obtained is again a preserving rexpansion of the corresponding three-valued Nmatrix, but not a strongly preserving one. For example, the refinement that is associated with $(\mathbf{l})$ amounts to the requirement that $\varphi \wedge \psi$ is assigned a value from $\mathcal{T}$ whenever φ is assigned $\top_i^j$ and ψ is assigned either $\top_i^{j+1}$ or t_i^{j+1}. Thus, the logic $BKcal$ (obtained from $BKca$ by the addition of $(\mathbf{l})$) is characterized by the Nmatrix $\mathcal{M}^{\infty}_{BKcal} = \langle \mathcal{V}^{\infty}_{BK}, \mathcal{D}^{\infty}_{BK}, \mathcal{O}^{\infty}_{BKcal} \rangle$, which is obtained from $\mathcal{M}^{\infty}_{BKca}$ by setting

$$\mathcal{O}^{\infty}_{BKcal}(\wedge) = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & a, b \in \mathcal{T} \\ \mathcal{T} & a = \top_i^j \text{ and } b \in \{\top_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D}^{\infty}_{BK} & otherwise \end{cases}$$

$\mathcal{M}^{\infty}_{BKcal}$ is indeed a preserving F-rexpansion of $\mathcal{M}_{BKca}$, but not a strongly preserving one. For example, $F(\top) \cap \mathcal{O}^{\infty}_{BKcal}(\wedge)(\top_i^j, \top_i^{j+1}) = \emptyset$, although $\top \in \mathcal{O}_{BKca}(\wedge)(\top, \top)$.

As was shown in Avron et al. (2012), the $\circ$-free fragment of $\mathcal{M}^{\infty}_{BKcal}$ is characteristic for da Costa's historical paraconsistent logic C_1 (da Costa 1974), thus providing this logic with an effective semantics.

5 Negations for Gödel Logic

The goal of this section is to develop reasonable logics in the language $\mathcal{CL}$ that simultaneously have two properties that were discussed in Section 4: paraconsistency and fuzziness.

The main problem we face in achieving the above goal is that the ordinary fuzzy logics (like the two described in Example 2.4) are defined via matrices with a single designated value. However, it is well known (Arieli and Avron 2015) that a logic which is induced by such a matrix cannot be paraconsistent. Therefore none of the standard fuzzy logics is paraconsistent. In order to develop logics that are both paraconsistent and fuzzy, it is necessary to replace the standard method of defining a fuzzy consequence relation by a weaker one. An additional step that can be made is to take $\neg$ as a primitive connective, and use new semantic interpretations for it. (In the standard fuzzy logics $\neg\psi$ is *defined* as $\psi \supset \bot$.)

The first attempt to achieve the goal of this section according to the above lines was made by Ertola et al. (2015). Their main idea was to follow a recent approach (Bou et al. 2009) to defining fuzzy consequence relations, that instead of preserving absolute truth (i.e. the truth value 1), preserves *degrees of truth*. Given a fuzzy matrix $\mathcal{M}$, which induces the ordinary (i.e. truth-preserving) fuzzy logic $\mathbf{L}$, this means that a formula ψ follows from a set of formulas $\mathcal{T}$ if there is a finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{T}$ such that the truth value which is assigned to ψ by some $\mathcal{M}$-valuation v is always greater than or equal to the minimal truth value that v assigns to $\varphi_1, \ldots, \varphi_n$. For the standard matrices used in fuzzy logics, the latter condition is equivalent to demanding the formula $(\varphi_1 \wedge \ldots \wedge \varphi_n) \supset \psi$ to be valid in the corresponding truth-preserving logic $\mathbf{L}$ (cf. Bou et al. 2009). This fact implies that $\mathbf{L}^{\leq}$, the degree-preservation logic induced by $\mathcal{M}$, has the same set of valid formulas as the truth-preserving logic $\mathbf{L}$ which is induced by $\mathcal{M}$. This makes $\mathbf{L}^{\leq}$ a natural variant of $\mathbf{L}$.

A good example of the method of Ertola et al. (2015) is provided by Łukasiewicz logic Ł (Example 2.4). The interpretation of $\neg$ there (where $\neg$ is taken as a defined connective, as explained above) is: $\mathcal{O}(\neg)(a) = 1 - a$. As said above, Ł itself cannot be paraconsistent. However, its degree-preserving variant $Ł^{\leq}$ *is* paraconsistent, as can be seen by any valuation v such that $v(p) = v(\neg p) = \frac{1}{2}$ and $v(q) = \frac{1}{4}$. $Ł^{\leq}$ is also subclassical (as it is contained in Ł, which is subclassical), and thus it is even strictly (though not boldly) paraconsistent. Moreover, it validates some basic classical equivalences connected with negation, like De Morgan's laws and the double negation laws. Unfortunately, $Ł^{\leq}$ has some very serious drawbacks as well. The main (but definitely not the only) one is that M.P. for $\supset$ is not valid in it. (This is exemplified by any valuation v in which $v(\varphi) = 0.5$, $v(\psi) = 0.4$ and $v(\varphi \supset \psi) = 0.9$.) Thus $\supset$ cannot be regarded in $Ł^{\leq}$ as an implication connective of any sort.

Is there a standard fuzzy logic $\mathbf{L}$ such that M.P. for $\supset$ is valid in $\mathbf{L}^{\leq}$? Of the three basic fuzzy logics (Łukasiewicz logic, Gödel logic and product logic), only in Gödel logic $\supset$ has this property (see, e.g., Hájek 1998). Hence it seems better to try to use $\mathbf{G}^{\leq}$ instead of $Ł^{\leq}$. However, in its original language (of $\{\wedge, \vee, \supset, \bot\}$) $\mathbf{G}^{\leq}$ is identical to $\mathbf{G}$. In particular, $\mathbf{G}^{\leq}$ is not paraconsistent with respect to the official negation of $\mathbf{G}$. To obtain a paraconsistent variant of $\mathbf{G}$, one should employ also the second idea mentioned above (and used in Ertola et al. 2015): to augment the language with a new negation connective. A particularly appealing choice is to augment $\mathbf{G}^{\leq}$ with the involutive Łukasiewicz negation. Denote the resulting logic by $\mathbf{G}^{\leq}_{\neg}$. As a fuzzy paraconsistent logic, $\mathbf{G}^{\leq}_{\neg}$ has all the nice properties of $Ł^{\leq}$ that were mentioned above. On the other hand it does not have its main shortcoming, because $\supset$ is in it a true implication connective: $\varphi \supset \psi$ follows in $\mathbf{G}^{\leq}_{\neg}$ from $\mathcal{T}$ iff ψ follows in it from $\mathcal{T} \cup \{\varphi\}$. What is more: $\mathbf{G}^{\leq}_{\neg}$ is a *conservative* extension to a richer language of the basic fuzzy logic $\mathbf{G}$. However, even $\mathbf{G}^{\leq}_{\neg}$ still has some serious drawbacks. Thus like $Ł^{\leq}$ it is not boldly paraconsistent.[7] Even more significant is the fact that (again like $Ł^{\leq}$) $\varphi \vee \neg\varphi$ is not valid in it. This is very important, since classical negation is determined by a combination of two principles: the law of contradiction (that implies that ψ follows from φ and $\neg\varphi$), and the law of excluded middle (validity of $\varphi \vee \neg\varphi$). Since we are seeking here paraconsistency, we are giving up the first of these two principles. So in order to justify viewing $\neg$ as a sort of *negation* (and in order to recover as much as possible from classical logic, while still being

[7] In both logics $q \vee \neg q$ is not valid, but it follows from $\{p, \neg p\}$, as the minimum value assigned to $\{p, \neg p\}$ is at most $\frac{1}{2}$, while the value assigned to $q \vee \neg q$ is at least $\frac{1}{2}$.

paraconsistent), we should keep intact at least the other basic principle of classical negation: the law of excluded middle.

In this section we use rexpansions of the Gödel matrix $\mathcal{M}_{\mathbf{G}}$ for constructing even better paraconsistent fuzzy logics which are based on $\mathbf{G}$. Before describing our method, here is the list of properties that we would like a paraconsistent fuzzy logic $\mathbf{L}$ to have:

(i) $\mathbf{L}$ should be boldly paraconsistent;

(ii) $\mathbf{L}$ should be subclassical (and so, by (i), strictly paraconsistent);

(iii) $\mathbf{L}$ should be conservative over $\mathbf{G}$;

(iv) $\supset$, $\wedge$, and $\vee$ should respectively be an implication, a conjunction, and a disjunction for $\mathbf{L}$. This means that for every $\mathcal{T}$, φ, ψ, and σ we should have:

(iv).A $\mathcal{T} \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ iff $\mathcal{T} \vdash_{\mathbf{L}} \varphi \supset \psi$;
(iv).B $\mathcal{T} \vdash_{\mathbf{L}} \varphi \wedge \psi$ iff $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T} \vdash_{\mathbf{L}} \psi$;
(iv).C $\mathcal{T} \cup \{\varphi \vee \psi\} \vdash_{\mathbf{L}} \sigma$ iff $\mathcal{T} \cup \{\varphi\} \vdash_{\mathbf{L}} \sigma$ and $\mathcal{T} \cup \{\psi\} \vdash_{\mathbf{L}} \sigma$;

(v) $\mathbf{L}$ should validate $\varphi \vee \neg\varphi$;

(vi) $\mathbf{L}$ should validate the basic classical equivalence concerning $\neg$, $\vee$, and $\wedge$: $\varphi \equiv \neg\neg\varphi$, $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$, and $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$;

(vii) $\mathbf{L}$ should validate the following connections between negation and implication:

(vii).A $\varphi \supset (\neg\psi \supset \neg(\varphi \supset \psi))$
(vii).B $\neg(\varphi \supset \psi) \supset \neg\psi$
(vii).C $(\varphi \supset \psi) \supset (\neg(\varphi \supset \psi) \supset \varphi)$

A word of explanation is needed for the last item in this list. Ideally, we would have liked to add to item (vi) of the list above also the classical equivalence that connects $\neg$ and $\supset$: $\neg(\varphi \supset \psi) \equiv (\varphi \wedge \neg\psi)$. This, in turn, is equivalent to the validity of (vii).A, (vii).B and

$$\text{(vii).C'} \ \neg(\varphi \supset \psi) \supset \varphi$$

Unfortunately, we cannot include (vii).C' in our list, since together with items (iv) and (v), it immediately entails the validity of $\varphi \vee (\varphi \supset \psi)$, contradicting item (iii) of our list. So instead of (vii).C' we include in the list a weaker version, which is valid in $\mathbf{G}_{\neg}^{\leq}$, as well as in all the standard fuzzy logics (in which $\neg\psi$ is taken as $\psi \supset \bot$).

5.1 The Nmatrix $\mathcal{M}_{\mathbf{G}_{\neg}}^{t}$ and Its Refinements

The method of rexpansions allows us to present a better approach to the construction of paraconsistent conservative extensions of Gödel logic, which stays within the framework of truth-preservation. This is achieved by relaxing the principle of truth-functionality, and the preservation of absolute truth. The former is done by basing our construction on Nmatrices, and the latter by replacing "completely true" with "true enough", that is, taking a larger set of designated truth values. Formally:

Definition 5.1. Let $0 < t \leq 1$. ${\mathcal{M}_{\mathbf{G}}}^t$ is the Nmatrix for $\{\wedge, \vee, \supset, \perp\}$ obtained from $\mathcal{M}_{\mathbf{G}}$ by:

1. Taking $[t, 1]$ as the designated values.
2. Changing $\mathcal{O}(\supset)$ to $\mathcal{O}(\supset)(a, b) = \begin{cases} [t,1] & a \leq b \text{ or } b \geq t \\ \{b\} & a > b \text{ and } b < t \end{cases}$.

${\mathcal{M}_{\mathbf{G}}}^t_{\neg}$ is the extension of ${\mathcal{M}_{\mathbf{G}}}^t$ to $\{\wedge, \vee, \supset, \perp, \neg\}$, in which $\mathcal{O}(\neg)(a) = 1 - a$.

The next theorem shows that ${\mathcal{M}_{\mathbf{G}}}^t_{\neg}$ provides a satisfactory basis for constructing paraconsistent fuzzy logics.

Theorem 5.2. *Let $0 < t \leq 1$ and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a simple refinement of ${\mathcal{M}_{\mathbf{G}}}^t_{\neg}$. Then:*

1. *$\vdash_{\mathcal{M}}$ satisfies (iii), (iv).B, (iv).C, and (vi) (that is, it is conservative over $\mathbf{G}$, $\wedge$ is a conjunction, $\vee$ is a disjunction, and De Morgan and double negation laws are valid).*[8]
2. *If $1 \in \mathcal{O}(\supset)(x, y)$ whenever either $x = 0$ or $y = 1$ then $\vdash_{\mathcal{M}}$ satisfies (ii).*
3. *If $t > \frac{1}{2}$ then $\vdash_{\mathcal{M}}$ satisfies neither of (i), (iv), and (v).*
4. *If $t \leq \frac{1}{2}$ then $\vdash_{\mathcal{M}}$ satisfies (i) and (v).*

Proof. Suppose $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ and ${\mathcal{M}_{\mathbf{G}}}^t_{\neg} = \langle \mathcal{V}^t, \mathcal{D}^t, \mathcal{O}^t \rangle$.

1. It is straightforward to verify (iv).B, (iv).C and (vi). As for (iii), one verifies that ${\mathcal{M}_{\mathbf{G}}}^t$ is a simple F-expansion of $\mathcal{M}_{\mathbf{G}}$, for $F = \lambda x \in [0, 1] . \begin{cases} [t,1] & x = 1 \\ \{t \cdot x\} & x < 1 \end{cases}$. By Corollary 3.10, every simple refinement of ${\mathcal{M}_{\mathbf{G}}}^t_{\neg}$ induces a logic that is conservative over $\mathbf{G}$.
2. Suppose that $\mathcal{T} \not\vdash_{\mathbf{CL}} \varphi$. Then there exists a classical valuation v such that $v(\psi) = 1$ for every $\psi \in \mathcal{T}$ and $v(\varphi) = 0$. v is also an $\mathcal{M}$-valuation, and thus $\mathcal{T} \not\vdash_{\mathcal{M}} \varphi$.
3. If $v \models^{\mathcal{M}} p$ and $v \models^{\mathcal{M}} \neg p$, we must have $v(p), 1 - v(p) \geq t$, which is impossible for $t > \frac{1}{2}$. Therefore, $p, \neg p \vdash_{\mathcal{M}} q$ and (i) fails. Moreover, $\not\vdash_{\mathcal{M}} (p \wedge \neg p) \supset q$ (by assigning $\frac{1}{2}$ to p and 0 to q), and thus also (iv) fails. Finally, $v \not\models^{\mathcal{M}} p \vee \neg p$ for $v(p) = v(\neg p) = v(p \vee \neg p) = \frac{1}{2}$.
4. We start with (i): First we show that $\neg$ is a weak negation in $\vdash_{\mathcal{M}}$. Since $0 < t \leq \frac{1}{2}$ there exists $a < t$ such that $1 - a \geq t$. Any $\mathcal{M}$-valuation v in which $v(p) = 1 - a$ satisfies p but not $\neg p$, and any $\mathcal{M}$-valuation v in which $v(p) = a$ satisfies $\neg p$ but not p. Thus $p \not\vdash_{\mathcal{M}} \neg p$ and $\neg p \not\vdash_{\mathcal{M}} p$. Second, in any $\mathcal{M}$-valuation v in which $v(p) = v(\neg p) = \frac{1}{2}$ and $v(q) = 0$, we have $v \models^{\mathcal{M}} \{p, \neg p\}$ and $v \not\models^{\mathcal{M}} q$. Therefore $p, \neg p \not\vdash_{\mathcal{M}} q$. Next, we show that $\vdash_{\mathcal{M}}$ is boldly paraconsistent. Suppose $\not\vdash_{\mathcal{M}} \varphi$ and $p \notin At(\varphi)$. Then there exists an $\mathcal{M}$-valuation v such that $v(\varphi) < t$. Define a function v' as follows: $v'(\psi) = v(\psi)$ for every subformula ψ of φ (including φ itself), and $v'(p) = v'(\neg p) = \frac{1}{2}$. Now extend v' to an $\mathcal{M}$-valuation, and obtain that $p, \neg p \not\vdash_{\mathcal{M}} \varphi$. As for (v), for every $\mathcal{M}$-valuation v, if $v(\varphi) < t$ then $v(\neg\varphi) > t$ and vice versa, and hence $\vdash_{\mathcal{M}} \neg\varphi \vee \varphi$. □

[8] While the left-to-right direction of (iv).A may not hold, its right-to-left direction (namely M.P.) does hold.

The proof of Theorem 5.2 actually provides another interesting result regarding the Gödel matrix: the same logic would result if the designated values were taken to be any interval of the form $[t, 1]$ for any $0 < t < 1$.

Proposition 5.3. *Let* $0 < t < 1$ *and* $\mathcal{M}^t = \langle \mathcal{V}^t, \mathcal{D}^t, \mathcal{O}^t \rangle$, *where* $\mathcal{V}^t = [0, 1]$, $\mathcal{D}^t = [t, 1]$, *and* $\mathcal{O}^t = \mathcal{O}_{\mathbf{G}}$. *Then* $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathcal{M}^t}$.

Proof. $\mathcal{M}^t$ is a simple refinement of ${\mathcal{M}_{\mathbf{G}}}^t$, which is a simple rexpansion of $\mathcal{M}_{\mathbf{G}}$. By Corollary 3.9, we have $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathcal{M}^t}$. □

Other negations can be considered for $\mathbf{G}$, and rexpansions (and in particular Corollary 3.9) can be used in order to prove that the result is conservative over $\mathbf{G}$.

Lemma 5.4. *Let A be a set of axioms in $\mathcal{CL}$. If A is valid in ${\mathcal{M}_{\mathbf{G}}}^t_\neg$ then $\mathbf{G}^A$, the axiomatic extension of $\mathbf{G}$ with A, is conservative over $\mathbf{G}$.*

Proof. Clearly, $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}^A}$. Now let $\mathcal{T} \subseteq \mathcal{CL} \setminus \{\neg\}$ and $\varphi \in \mathcal{CL} \setminus \{\neg\}$. If $\mathcal{T} \vdash_{\mathbf{G}^A} \varphi$, then $\mathcal{T} \vdash_{{\mathcal{M}_{\mathbf{G}}}^t_\neg} \varphi$. And since $\vdash_{{\mathcal{M}_{\mathbf{G}}}^t_\neg}$ is conservative over $\mathbf{G}$, we must have $\mathcal{T} \vdash_{\mathbf{G}} \varphi$. □

Note that finding a new semantics for the augmented logic is not required, as only *soundness* is needed for the proof.

Example 5.5. Let A be a set consisting of the axioms from property (vi) above. Then G^A is an axiomatic extension of $\mathbf{G}$ with a negation that satisfies the usual double negation and De Morgan rules, and is conservative over $\mathbf{G}$.

5.2 Two Particular Refinements of ${\mathcal{M}_{\mathbf{G}}}^{\frac{1}{2}}_\neg$

Theorem 5.2 shows that simple refinements of ${\mathcal{M}_{\mathbf{G}}}^{\frac{1}{2}}_\neg$ enjoy many desirable properties one would expect from a paraconsistent fuzzy logic. However, they may lack some of the properties mentioned above. In particular, the formulas in (vii) are not valid in the logic that is induced by ${\mathcal{M}_{\mathbf{G}}}^{\frac{1}{2}}_\neg$ itself (for example, if $v(\varphi) = 0.7$, $v(\psi) = 0.8$, and $v(\varphi \supset \psi) = 0.7$, then v does not satisfy (vii).B). Moreover, (iv) does not hold in the simple refinement $\mathcal{M}^{0.7-0.8}$ of ${\mathcal{M}_{\mathbf{G}}}^{\frac{1}{2}}_\neg$, obtained by setting $\mathcal{O}(\supset)(a, b) = 0.7$ whenever $b \geq \frac{1}{2}$ or $a = b$, and $\mathcal{O}(\supset)(a, b) = 0.8$ whenever $b < \frac{1}{2}$ and $a < b$. Indeed, $v(\neg(\varphi \supset \varphi)) = 0.3 < \frac{1}{2}$ for every $\mathcal{M}^{0.7-0.8}$-valuation v, which means that $\neg(\varphi \supset \varphi) \vdash_{\mathcal{M}^{0.7-0.8}} \psi$. However, the $\mathcal{M}^{0.7-0.8}$-valuation u in which $u(\varphi) = u(\psi) = 0$ and $u(\varphi \supset \varphi) = 0.7$, shows that $\nvdash_{\mathcal{M}^{0.7-0.8}} \neg(\varphi \supset \varphi) \supset \psi$. Property (ii) also does not hold in $\vdash_{\mathcal{M}^{0.7-0.8}}$, as q follows from $\neg(p \supset p) \supset \neg(\bot \supset \neg(p \supset p))$ in it, but not in classical logic.

We present two particularly interesting simple refinements of ${\mathcal{M}_{\mathbf{G}}}^{\frac{1}{2}}_\neg$. The first is obtained by refining the interpretation of $\supset$ back to its original interpretation in $\mathcal{M}_{\mathbf{G}}$. The second is a reconstruction of a well-known semi-relevant logic (Anderson and Belnap 1975, Dunn and Restall 2002), in which all properties (i)–(vii) hold.

5.2.1 Closest to The Original: $det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})$

If we refine the interpretation of $\supset$ in $\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}$ to its original interpretation in $\mathcal{M}_{\mathbf{G}}$, we obtain a *matrix* for a paraconsistent fuzzy logic (denoted $det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})$), whose $\{\wedge, \vee, \supset, \bot\}$-fragment differs from $\mathcal{M}_{\mathbf{G}}$ solely in the choice of designated values. This seems as close as one can get to adding a paraconsistent involutive negation to $\mathcal{M}_{\mathbf{G}}$. Moreover, $\vdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})}$ strictly extends $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}}$, and satisfies all properties listed above, except for (vii).A. The only property whose verification is not routine is (iv).A, which we now prove.

Proposition 5.6 (Deduction Theorem for $\vdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})}$)**.** $\mathcal{T} \vdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_1 \supset \varphi_2$ *iff* $\mathcal{T}, \varphi_1 \vdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_2$.

Proof. The fact that $\mathcal{T} \vdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_1 \supset \varphi_2$ implies $\mathcal{T}, \varphi_1 \vdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_2$ is easily verified using the interpretation of $\supset$. For the converse, suppose $\mathcal{T} \nvdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_1 \supset \varphi_2$. We prove that $\mathcal{T}, \varphi_1 \nvdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_2$. By our assumption, there exists a $det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})$-valuation v such that $v(\psi) \geq \frac{1}{2}$ for every $\psi \in \mathcal{T}$, and $v(\varphi_1 \supset \varphi_2) < \frac{1}{2}$. Let $r_1 = v(\varphi_1)$ and $r_2 = v(\varphi_2)$. Then: $v(\varphi_1 \supset \varphi_2) = r_2 < \frac{1}{2}$, $1 - r_2 > \frac{1}{2}$, and $r_1 > r_2$. If $r_1 \geq \frac{1}{2}$ then $v \models^{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_1$ and $v \not\models^{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_2$, and thus $\mathcal{T}, \varphi_1 \nvdash_{det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})} \varphi_2$. Hence we assume in addition that $r_1 < \frac{1}{2}$. We construct an appropriate countermodel by "fixing" v so that it satisfies $\mathcal{T}$ and φ_1, but still does not satisfy φ_2. This is done by replacing r_1 by $\frac{1}{2}$, and then making other necessary adjustments to keep the resulting valuation a $det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})$-valuation. Let

$$f = \lambda x \in [0,1] \,.\begin{cases} \frac{1}{2r_1}x & x < r_1 \\ \frac{1}{2} & r_1 \leq x \leq 1 - r_1 \\ \frac{x-1+2r_1}{2r_1} & x > 1 - r_1 \end{cases}$$

and let $v'(\varphi) = f(v(\varphi))$ for every φ. f is clearly an increasing function from $[0,1]$ to $[0,1]$. Now, $v'(\psi) \geq \frac{1}{2}$ for every $\psi \in \mathcal{T}$, as $v(\psi) \geq \frac{1}{2} > r_1$ for every such ψ. Also, $v'(\varphi_1) = \frac{1}{2}$, and $v'(\varphi_2) < \frac{1}{2}$, as $v(\varphi_2) = r_2 < r_1$. It is left to prove that v' is a $det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg})$-valuation. Suppose $det(\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}) = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$.

1. $v'(\varphi \wedge \psi) = f(v(\varphi \wedge \psi)) = f(\min\{v(\varphi), v(\psi)\}) = \min\{v'(\varphi), v'(\psi)\}$, as f is increasing.

2. Disjunction is shown similarly.

3. If $v'(\varphi) \leq v'(\psi)$ then $v(\varphi) \leq v(\psi)$, and then $v'(\varphi \supset \psi) = f(v(\varphi \supset \psi)) = f(1) = 1$. $v'(\varphi \supset \psi) = f(v(\varphi \supset \psi)) = f(v(\psi)) = v'(\psi)$.

4. We show that $v'(\neg\varphi) = 1 - v'(\varphi)$. If $v(\varphi) < r_1$, then $v(\neg\varphi) > 1 - r_1$. In such a case, $v'(\varphi) = \frac{1}{2r_1}v(\varphi)$ and $v'(\neg\varphi) = \frac{v(\neg\varphi)-1+2r_1}{2r_1} = \frac{2r_1 - v(\varphi)}{2r_1}$. In particular, $v'(\varphi) + v'(\neg\varphi) = \frac{v(\varphi)+2r_1-v(\varphi)}{2r_1} = 1$. If $r_1 \leq v(\varphi) \leq 1 - r_1$, then $v'(\varphi) = v'(\neg\varphi) = \frac{1}{2}$. And if $v(\varphi) > 1 - r_1$, then this case is symmetric to the first case.

5. $v'(\bot) = f(v(\bot)) = f(0) = 0$. □

5.2.2 A Semi-relevant Refinement: $\mathcal{M}_{RM^{\supset}}$

The matrix for the logic $RM^{\supset}$ (Avron 1986), that we denote by $\mathcal{M}_{RM^{\supset}}$ is a simple refinement of $\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}$ in which implication is interpreted by: $\mathcal{O}(\supset)(a,b) = \begin{cases} \{1-a\} & a \leq b \leq 1-a \\ \{b\} & otherwise \end{cases}$.

Note that $RM^{\supset}$ is shown in Avron (1986) to be equivalent to the famous Dunn-Meyer semi-relevant logic $\mathbf{RM}$ (Anderson and Belnap 1975). $RM^{\supset}$ satisfies all of the properties listed above. (All properties but (iv).A and (vii) follow from Theorem 5.2. (iv).A and (vii) were proved in Avron 1986.) In particular, it strictly extends $\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}}$.[9] Moreover, we show that $RM^{\supset}$ is unique with respect to the properties above:

Proposition 5.7. *$RM^{\supset}$ is the only finitary[10] logic that satisfies all properties (i)–(vii) above.*

Proof. Let $\mathbf{L}$ be such a logic. Denote by $\mathcal{H}$ the Hilbert calculus for $\mathbf{G}$ from Dummett (1959), and by $\mathcal{H}_{RM^{\supset}}$ the Hilbert calculus obtained from $\mathcal{H}$ by the addition of the axiom schemes of (v), (vi) and (vii). Note that the only rule of inference in $\mathcal{H}$ and $\mathcal{H}_{RM^{\supset}}$ is M.P. It was shown in Avron (1986) that the set of theorems of $\mathcal{H}_{RM^{\supset}}$ is the same as the set of formulas that are valid in $RM^{\supset}$. Since $RM^{\supset}$ is finitary[11] and admits the deduction theorem, it follows that $\mathcal{H}_{RM^{\supset}}$ is sound and complete for $RM^{\supset}$, that is, φ is derivable from $\mathcal{T}$ in $\mathcal{H}_{RM^{\supset}}$ iff $\mathcal{T} \vdash_{RM^{\supset}} \varphi$. Now, to satisfy (iii), all axiom schemes from $\mathcal{H}$ must be valid in $\mathbf{L}$, as otherwise, it would not be conservative over $\mathbf{G}$. To satisfy (v), (vi) and (vii), the axioms they include must be valid in $\mathbf{L}$ also. For (iv), M.P. must be valid in $\mathbf{L}$. Thus $\mathcal{H}_{RM^{\supset}}$ is sound for $\mathbf{L}$, and in particular, $RM^{\supset}$ is contained in $\mathbf{L}$. Now, if $\mathbf{L}$ strictly contains $RM^{\supset}$, then since M.P. is valid in $RM^{\supset}$ and $\mathbf{L}$ is both finitary and admits the deduction theorem (by (iv)), there exists a formula φ that is valid in $\mathbf{L}$ but not in $RM^{\supset}$. It is then a corollary of Avron (1986) that $\mathbf{L}$ has a finite characteristic matrix, and in particular, so does its $\neg$-free fragment. Since $\mathbf{G}$ cannot be finitely characterized by a matrix (Dummett 1959), $\mathbf{L}$ is not a conservative extension of $\mathbf{G}$, and thus (iii) fails. Therefore, we must have that $\mathbf{L}$ and $RM^{\supset}$ are identical. □

Table 1 summarizes the various logics and properties discussed in this section, and in particular, specifies the properties that hold in each logic. In the table, "*t*" means that the property holds, and "*f*" means that it does not. The column in the middle (titled "Simple Refinements of $\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}$") includes some cells with the symbol "$\top$". For the corresponding properties, the meaning is that some simple refinements of $\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}$ satisfy them, and some do

[9] The axiomatic extension of $RM^{\supset}$ with (vii).C', that we did not include in our list of requirements, is also considered in Avron (1986), and is proven to be equivalent to the 3-valued logic **PAC** (Arieli and Avron 2015), that is also known as RM_3.

[10] A logic $\mathbf{L}$ is called finitary if the compactness theorem holds for it, that is: $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ iff $\Gamma \vdash_{\mathbf{L}} \varphi$ for some finite $\Gamma \subseteq \mathcal{T}$.

[11] This follows from the equivalence between $RM^{\supset}$ and $\mathbf{RM}$ shown in Avron (1986), together with the fact that $\mathbf{RM}$ itself is finitary. The latter follows from the (strong) soundness and completeness theorem that was proven for $\mathbf{RM}$ in Avron (2016).

not. In contrast, property (iii) is not relevant for Ł$^{\leq}$, as it does not include any new connective. Thus it is marked with "$\perp$".

Table 1: Summary of Properties

Property	Ł$^{\leq}$	$\mathbf{G}^{\leq}$	Simple Refinements of $\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}$	$det(\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}})$	$RM^{\supset}$
(i)	f	f	t	t	t
(ii)	t	t	$\top$	t	t
(iii)	$\perp$	t	t	t	t
(iv)	f	t	$\top$	t	t
(v)	f	f	t	t	t
(vi)	t	t	t	t	t
(vii)	t	f	$\top$	f	t

5.3 What is the Cardinality of $\left\{\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t}}} \mid 0 < t \leq 1\right\}$?

We conclude by investigating the relation between the different logics that are induced by the Nmatrices $\mathcal{M}_{\mathbf{G}_{\neg}^{t}}$ (see Definition 5.1) themselves. These logics are *minimal* in the family of logics that are studied in Theorem 5.2, as different refinements of them may induce different extensions.

The main result of this concluding section can be summarized as follows:

1. All the Nmatrices $\mathcal{M}_{\mathbf{G}_{\neg}^{t}}$ for $0 < t \leq \frac{1}{2}$ induce the same logic;
2. There are exactly two logics that are induced for $t > \frac{1}{2}$.
3. All together, the answer to the above question is: three.[12]

The rest of this section is devoted to the proof of this result. We start by introducing the notion of $\langle t_1, t_2 \rangle$*-expanding functions* in Definition 5.8, and prove that they characterize all strongly preserving rexpansions between elements of $\left\{\mathcal{M}_{\mathbf{G}_{\neg}^{t}} \mid 0 < t \leq 1\right\}$ in Lemma 5.9. This fact is then used in Lemma 5.10, where the logics that are induced by these Nmatrices are identified, thus obtaining the aforementioned result in Corollary 5.11.

Definition 5.8. Let $0 < t_1 < t_2 \leq 1$. An expansion function $F : [0,1] \to P([0,1])$ is called $\langle t_1, t_2 \rangle$*-expanding* if:

(1) $\bigcup_{x \in [0,1]} F(x) = [0,1]$ and $\bigcup_{x \in [t_2,1]} F(x) = [t_1, 1]$.

(2) F is increasing: if $x < y$ then $x' < y'$ for every $x' \in F(x)$ and $y' \in F(y)$.

[12]We stress that the logics that are considered here are those that are induced by the Nmatrices $\mathcal{M}_{\mathbf{G}_{\neg}^{t}}$ themselves, not their refinements.

(3) $F(1-x) = \{1-y \mid y \in F(x)\}$ for every $x \in [0,1]$ (that is, $y \in F(x)$ iff $1-y \in F(1-x)$).

(4) $F(x)$ is a singleton whenever $x < t_2$ (and so because of (3), also when $x > 1-t_2$).

Lemma 5.9. *Let $F : [0,1] \to P([0,1])$ and $0 < t_1 < t_2 \leq 1$. Then the following statements are equivalent:*

1. *F is $\langle t_1, t_2\rangle$-expanding.*

2. *$\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$.*

3. *$\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is a strongly preserving F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$.*

Proof. Suppose $\mathcal{M}_{\mathbf{G}_\neg}^{t_1} = \langle [0,1], [t_1,1], \mathcal{O}_1\rangle$, $\mathcal{M}_{\mathbf{G}_\neg}^{t_2} = \langle [0,1], [t_2,1], \mathcal{O}_2\rangle$ and $(\mathcal{M}_{\mathbf{G}_\neg}^{t_2})_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F\rangle$.

$(\mathbf{1} \Rightarrow \mathbf{3})$: Let us calculate $(\mathcal{M}_{\mathbf{G}_\neg}^{t_2})_F$: $\mathcal{V}_F = [0,1]$ and $\mathcal{D}_F = \mathcal{D}_1 = [t_1,1]$, because of property (1). As for $\mathcal{O}_F$: Using property (2), we have that $\mathcal{O}_F(\wedge)(x,y) = F(\min\left\{\widetilde{F}[x], \widetilde{F}[y]\right\})$ and $\mathcal{O}_F(\vee)(x,y) = F(\max\left\{\widetilde{F}[x], \widetilde{F}[y]\right\})$. By property (3), we have $\mathcal{O}_F(\neg)(x) = F(1-\widetilde{F}[x])$. Combining properties (2) and (4) gives us $\mathcal{O}_F(\supset)(x,y) = \begin{cases} [t_1,1] & x \leq y \text{ or } y \geq t_1 \\ \{y\} & x > y \text{ and } y < t_1 \end{cases}$. Finally, $\mathcal{O}_F(\bot) = F(0)$.

We show that $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is a (simple) refinement of $(\mathcal{M}_{\mathbf{G}_\neg}^{t_2})_F$:

1. $\mathcal{O}_1(\wedge)(x,y) \subseteq \mathcal{O}_F(\wedge)(x,y)$: Assume w.l.g. that $x \leq y$. Since F is increasing, $\widetilde{F}[x] \leq \widetilde{F}[y]$, which means that $\mathcal{O}_1(\wedge)(x,y) = \{x\} \subseteq F(\widetilde{F}[x]) = F(\min\left\{\widetilde{F}[x], \widetilde{F}[y]\right\}) = \mathcal{O}_F(\wedge)(x,y)$.

2. $\mathcal{O}_1(\vee)(x,y) \subseteq \mathcal{O}_F(\vee)(x,y)$: this is shown similarly.

3. $\mathcal{O}_1(\neg)(x) \subseteq \mathcal{O}_F(\neg)(x)$: using property (3), we have that $\mathcal{O}_1(\neg)(x) = \{1-x\} \subseteq \left\{1-y \mid y \in F(\widetilde{F}[x])\right\} = F(1-\widetilde{F}[x]) = \mathcal{O}_F(\neg)(x)$.

4. $\mathcal{O}_1(\supset)(x,y) \subseteq \mathcal{O}_F(\supset)(x,y)$: If $x \leq y$ or $y \in [t_1,1]$ then $\mathcal{O}_1(\supset)(x,y) = [t_1,1] = \mathcal{O}_F(\supset)(x,y)$. Otherwise, $\mathcal{O}_1(\supset)(x,y) = \{y\}$, which conforms with the calculation of $O_F(\supset)$ above.

5. $\mathcal{O}_1(\bot) \subseteq \mathcal{O}_F(\bot)$: We show that $0 \in F(0)$ $(= \mathcal{O}_F(\bot))$. Since $Im(F) = [0,1]$, $0 \in F(x)$ for some x. Assume for contradiction that $x > 0$. Since F is an expansion function, there exists some $y \in F(0)$. By property (2), $0 > y$, which is a contradiction.

Next, we prove that $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is a strongly preserving F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$. Clearly, it is preserving (as it is simple). The interpretations of all the connectives in $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ are deterministic, with the exception of $\supset$. Therefore, the only thing that needs to be verified is that $F(z) \cap \mathcal{O}_1(\supset)(x,y) \neq \emptyset$ whenever $z \in \mathcal{O}_2(\supset)(\widetilde{F}[x], \widetilde{F}[y])$ and either $x \leq y$ or $y \in [t_1,1]$. Let $z \in \mathcal{O}_2(\supset)(\widetilde{F}[x], \widetilde{F}[y])$. Since $x \leq y$ or $y \in [t_1,1]$, we have $\widetilde{F}[x] \leq \widetilde{F}[y]$

or $\widetilde{F}[y] \in [t_2, 1]$. Therefore, $z \in [t_2, 1]$, and so $F(z) \subseteq [t_1, 1]$. Since in this case, $\mathcal{O}_1(\supset)(x, y) = [t_1, 1]$, we have $F(z) \cap \mathcal{O}_1(\supset)(x, y) = F(z) \neq \emptyset$.

$(\mathbf{3} \Rightarrow \mathbf{2})$: Clearly, every strongly preserving F-rexpansion is an F-rexpansion.
$(\mathbf{2} \Rightarrow \mathbf{1})$: Suppose $\mathcal{M}_{\mathbf{G}_{\neg}^{t_1}}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_{\neg}^{t_2}}$. We prove that F is $\langle t_1, t_2 \rangle$-expanding, by verifying the four properties:

1. The correctness of property (1) is trivial.

2. If F is not increasing, then there exist $x, x', y, y' \in [0, 1]$ such that $x < y$, $x' \in F(x)$, $y' \in F(y)$ and $x' \geq y'$. Since F is an expansion function, $x' \neq y'$, thus $x' > y'$. Now, since $\mathcal{M}_{\mathbf{G}_{\neg}^{t_1}}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_{\neg}^{t_2}}$, $y' \in \mathcal{O}_1(\wedge)(x', y') \subseteq \mathcal{O}_F(\wedge)(x', y') = F(x)$. This is impossible, as $x \neq y$, $y' \in F(y)$, and F is a rexpansion function.

3. Let $x \in [0, 1]$. We prove that $F(1 - x) = \{1 - y \mid y \in F(x)\}$. For every $z \in F(1 - x)$, since $\mathcal{M}_{\mathbf{G}_{\neg}^{t_1}}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_{\neg}^{t_2}}$, $1 - z \in \mathcal{O}_1(\neg)(z) \subseteq \bigcup_{z' \in \mathcal{O}_2(\neg)(1-x)} F(z') = F(x)$, and therefore $z \in \{1 - y \mid y \in F(x)\}$. And for every $z \in \{1 - y \mid y \in F(x)\}$, $1 - z \in F(x)$, and therefore $z \in \mathcal{O}_1(\neg)(1 - z) \subseteq \bigcup_{z' \in \mathcal{O}_2(\neg)(x)} F(z') = F(1 - x)$.

4. If $F(x)$ is not a singleton for some $x < t_2$, then let $y_1, y_2 \in F(x)$ such that $y_1 < y_2$. In particular, $y_1, y_2 < t_1$. Therefore, since $\mathcal{M}_{\mathbf{G}_{\neg}^{t_1}}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_{\neg}^{t_2}}$, $y_1 \in \mathcal{O}_1(\supset)(y_2, y_1) \subseteq \bigcup_{z \in \mathcal{O}_2(\supset)(x,x)} F(z) = [t_1, 1]$, which is a contradiction. □

Now we apply Lemma 5.9 and Theorem 3.8 to the matrices $\mathcal{M}_{\mathbf{G}_{\neg}^{t}}$ for various values of t.

Lemma 5.10.

1. *$\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t}}} = \vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}}$ for every $0 < t < \frac{1}{2}$.*

2. *$\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t_1}}} = \vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t_2}}}$ for every $\frac{1}{2} < t_1 < t_2 < 1$.*

3. *$\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t}}} \subsetneq \vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{1}}}$ for every $\frac{1}{2} < t < 1$.*

4. *$\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}} \not\subseteq \vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t}}}$ and $\vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{t}}} \not\subseteq \vdash_{\mathcal{M}_{\mathbf{G}_{\neg}^{\frac{1}{2}}}}$ for every $\frac{1}{2} < t \leq 1$.*

Proof.

1. We construct an expansion function F that maps $[0, \frac{1}{2})$ to $[0, t)$ and $(\frac{1}{2}, 1]$ to $(1 - t, 1]$. The remaining value $\frac{1}{2}$ is duplicated to the remaining segment $[t, 1 - t]$. Namely:

$$F = \lambda x \in [0, 1] \,.\begin{cases} \{2tx\} & x < \frac{1}{2} \\ [t, 1 - t] & x = \frac{1}{2} \\ \{2tx + 1 - 2t\} & x > \frac{1}{2} \end{cases}$$

$F : [0, 1] \to P([0, 1])$ since $t < \frac{1}{2}$. By Lemma 5.9 and Theorem 3.8, it suffices to prove that F is $\left\langle t, \frac{1}{2} \right\rangle$-expanding. F is clearly an increasing expansion function with $\bigcup Im(F) = [0, 1]$ and $\bigcup_{x \in \left[\frac{1}{2}, 1\right]} F(x) = [t, 1]$. To see that property (3) is satisfied, we distinguish three cases: If $x < \frac{1}{2}$, $1 - x > \frac{1}{2}$, and then $F(1-x) = \{1-2tx\} = \{1-y \mid y \in F(x)\}$. If $x = \frac{1}{2}$, $1 - x = \frac{1}{2}$ and then $F(1-x) = [t, 1-t]$. Note that for every y, $y \in [t, 1-t]$ iff $1 - y \in [t, 1-t]$. Hence in this case, $F(1-x) = \{1-y \mid y \in [t, 1-t]\} = \{1-y \mid y \in F(x)\}$. If $x > \frac{1}{2}$, $1-x < \frac{1}{2}$, and then $F(1-x) = \{2t(1-x)\} = \{1-y \mid y \in \{2tx+1-2t\}\} = \{1-y \mid y \in F(x)\}$. Finally, property (4) clearly holds, as $F(x)$ is a singleton whenever $x < \frac{1}{2}$.

2. We construct a rexpansion function that maps $[t_2, 1]$ to $[t_1, 1]$, $[0, 1-t_2)$ to $[0, 1-t_1)$, and $[1-t_2, t_2)$ to $[1-t_1, t_1)$. Consider the following function:

$$f = \lambda x \in [0,1]\,.\begin{cases} \frac{1-t_1}{1-t_2} \cdot x & x < 1-t_2 \\ \frac{2t_1-1}{2t_2-1} \cdot x + \frac{t_2-t_1}{2t_2-1} & 1-t_2 \leq x < t_2 \\ \frac{1-t_1}{1-t_2} \cdot x + \frac{t_1-t_2}{1-t_2} & x \geq t_2 \end{cases}$$

Let $F = \lambda x \in [0,1]\,.\{f(x)\}$. By Lemma 5.9 and Theorem 3.8, it suffices to show that F is $\langle t_1, t_2 \rangle$-expanding. F is clearly an increasing expansion function, $\bigcup Im(F) = [0,1]$, and $F(x)$ is always a singleton. In addition, $\bigcup_{x \in [t_2, 1]} F(x) = f([t_2, 1]) = [t_1, 1]$. Finally, $F(1-x) = \{1-y \mid y \in F(x)\}$, as $f(1-x) = 1 - f(x)$ for every $x \in [0,1]$.

3. To show that $\vdash_{\mathcal{M}_{\mathbf{G}_\neg^t}} \subseteq \vdash_{\mathcal{M}_{\mathbf{G}_\neg^1}}$, we prove that for every $\mathcal{M}_{\mathbf{G}_\neg^1}$-valuation v there exists a $\mathcal{M}_{\mathbf{G}_\neg^t}$-valuation v' such that for every formula φ, $v \vDash^{\mathcal{M}_{\mathbf{G}_\neg^1}} \varphi$ iff $v' \vDash^{\mathcal{M}_{\mathbf{G}_\neg^t}} \varphi$.[13] Let v be a $\mathcal{M}_{\mathbf{G}_\neg^1}$-valuation. We construct v' by mapping the values that are strictly below 1 to being strictly below t, in a way that conforms with the interpretation of $\neg$. By making this mapping an increasing one, we conform with the interpretation of the other connectives. This is defined as follows: Let f be defined by:

$$f = \lambda x \in [0,1]\,.\begin{cases} 0 & x = 0 \\ (2t-1)x + 1 - t & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$

$f : [0,1] \to [0,1]$ is strongly increasing and $f(1-x) = 1 - f(x)$ for every $x \in [0,1]$.

Define $v'(\psi) = f(v(\psi))$ for every ψ. First, we prove that v' is a $\mathcal{M}_{\mathbf{G}_\neg^t}$-valuation: For $\wedge$, we have $v'(\varphi \wedge \psi) = f(v(\varphi \wedge \psi)) = f(\min\{v(\varphi), v(\psi)\}) = \min\{v'(\varphi), v'(\psi)\}$, as f is increasing. $\vee$ is shown similarly. In addition, $v'(\neg\psi) = f(v(\neg\psi)) = f(1 - v(\psi)) = 1 - f(v(\psi)) = 1 - v'(\psi)$ and $v'(\bot) = f(v(\bot)) = f(0) = 0$. Next, we show that the implication constraints are satisfied: If $v'(\varphi) \leq v'(\psi)$, then since f is increasing, $v(\varphi) \leq v(\psi)$. Since v is a $\mathcal{M}_{\mathbf{G}_\neg^1}$-valuation, $v(\varphi \supset \psi) = 1$, and hence $v'(\varphi \supset \psi) = 1 > t$. If $v'(\psi) \geq t$, then by the definition of f, $v'(\psi) = 1$, which means that $v(\psi) = 1$, and again, $v'(\varphi \supset \psi) = 1 > t$. Finally, if $v'(\varphi) > v'(\psi)$ and $v'(\psi) < t$ then we have $v(\varphi) > v(\psi)$

[13] The proof of this item does not use rexpansions: by Proposition 3.4, $\mathcal{M}_{\mathbf{G}_\neg^1}$ is not a rexpansion of $\mathcal{M}_{\mathbf{G}_\neg^t}$, as there is no function $f : [0,1] \to [0,1]$ satisfying $x \in \{1\}$ iff $f(x) \in [t,1]$.

and $v(\psi) < 1$. Since v is a $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation, $v(\varphi \supset \psi) = v(\psi) < 1$, and hence $v'(\varphi \supset \psi) = f(v(\psi)) = v'(\psi)$. Second, we prove that $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \psi$ iff $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \psi$, for every formula ψ. If $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \psi$ then $v(\psi) = 1$. In this case, $v'(\psi) = 1$ as well, and in particular, $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \psi$. In addition, if $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \psi$, then $v'(\psi) \geq t$, that is, $f(v(\psi)) \geq t$. By f's definition, we must have $v'(\psi) = f(v(\psi)) = 1$, which means that $v(\psi) = 1$. Therefore, $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \psi$.

To show that $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \neq \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}}$, note that $p \supset q \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \neg q \supset \neg p$, as for every $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation v such that $v(p \supset q) = 1$, we must have that $v(p) \leq v(q)$. In particular, $v(\neg q) \leq v(\neg p)$, and thus $v(\neg q \supset \neg p) = 1$. However, $p \supset q \nvdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \neg q \supset \neg p$, as can be seen by the following $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$-valuation: $v(p) = 1, v(q) = t, v(\neg p) = 0, v(\neg q) = 1 - t, v(p \supset q) = t, v(\neg q \supset \neg p) = 0$.

4. By Theorem 5.2, we have that $p, \neg p \vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} q$ but $p, \neg p \nvdash_{\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}} q$, and $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}} \neg\varphi \vee \varphi$ but $\nvdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \neg\varphi \vee \varphi$. □

Corollary 5.11. $\left\{\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \mid 0 < t \leq 1\right\} = \left\{\vdash_{\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}}, \vdash_{\mathcal{M}_{\mathbf{G}}{}^{\frac{3}{4}}_{\neg}}, \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}}\right\}$*, and its cardinality is* 3.

6 Conclusion and Further Research

We have investigated rexpansions – compositions of expansions and refinements on Nmatrices. Properties of this operation were proved, as well as their effects on consequence relations. Examples of applications of these results were also given, including the construction of conservative extensions for many logics from the literature, and in particular, paraconsistent conservative extensions of Gödel logic, that were investigated further.

Theorem 3.8 provides a sufficient condition for two Nmatrices to induce the same consequence relation. However, Example 4.4 shows that this condition is not necessary, since the $\{\wedge, \vee\}$-fragments of $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{KL}}$ induce the same logic, but neither is a strongly preserving expansion of the other. An interesting direction for further research is to characterize general cases in which the condition it suggests is also necessary. An intermediate goal in this direction is to further generalize the sufficient condition, so that it uniformly covers examples that are currently left out (like $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{KL}}$). Future work would also include more applications of rexpansions, in the spirit of Sections 4 and 5. Such applications can simplify known results from the literature of non-classical logics, and also the construction of new conservative extensions with certain properties. We have started to advance this line of work in Chapter 6 of Zohar (2018), where rexpansions are used to describe the modular construction of proof systems for non-classical logics, and plan to describe other modular constructions via rexpansions. Section 5 should be extended beyond Gödel logic, to provide a general method for the construction of paraconsistent fuzzy logics, based on rexpansions. Among the logics that are induced by refinements of $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$, only $RM^{\supset}$ has a known axiomatization. We leave it for future research to axiomatize other such logics, and in particular, the logic that is induced by $\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}$ itself. The decidability and complexity of these logics is also left for further research.

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