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Gentzen-type Proof Systems for Non-classical Logics

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by

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Abstract

Sequent calculi constitute a prominent proof-theoretic framework, suitable for a variety of logics. These include classical and intuitionistic logic, modal logics, paraconsistent logics, and many-valued logics, including fuzzy logics. The most important property of useful sequent calculi is *analyticity*, whose most common instance is the *subformula property*. When a calculus is analytic, a certain limitation on the search space of derivations is achieved. In the case of propositional sequent calculi, this often leads to a finite bound on the search space, and decidability immediately follows. This limitation may also be useful for proving the consistency of a sequent calculus (the fact that the empty sequent is not derivable).

The most standard way to prove analyticity is via *cut-admissibility*, that is: the redundancy of the cut rule, which is usually the only rule whose premises may include a formula that is unrelated to the conclusion. Indeed, when cuts can be eliminated, and all other rules include in their premises only syntactic material from their conclusions, a simple induction on derivations entails that the end sequent can be derived using only its own syntactic material. Another major motivation for the elimination of the cut rule is the adequacy of cut-free calculi for efficient proof-search procedures.

Despite the usefulness of cut-admissibility, relying on it alone for proving analyticity leaves out various useful sequent calculi that are analytic, but do not enjoy cut-admissibility. Moreover, even when it is possible to obtain cut-admissibility, it might be easier to prove analyticity directly, rather than to go through complicated and error-prone proofs of cut-admissibility. In addition, a great deal of ingenuity is required for developing efficient proof-search algorithms for cut-free sequent calculi.

The main subject of this thesis is the notion of analyticity of sequent calculi discussed above. Our main contribution is a general analysis of it in several wide *families* of calculi. This analysis includes the following:

- We provide several sufficient criteria for analyticity, that are easy to check either "by hand" or in an automated way. These simple criteria can replace complex proofs of analyticity that go through cut-admissibility. In fact, many analyticity results from the literature are obtained as particular instances of this result. The value of these criteria is also demonstrated by several new useful calculi that we introduce in this work.

- We study the connection between analyticity and cut-admissibility, and prove that

in a wide variety of calculi, the two properties are equivalent. Besides theoretical interest, this result can be used to simplify proofs of cut-admissibility, whenever analyticity has already been established. Using this result, we show that some of the sufficient criteria for analyticity that we propose are also sufficient for cut-admissibility.

- We utilize analyticity to construct a uniform decision procedure for a wide family of sequent calculi. Our decision procedure relies only on analyticity, regardless of the admissibility of cut. Moreover, it is based on an efficient reduction to SAT, and thus all heuristic considerations are shifted to the mechanisms of off-the-shelf SAT-solvers. An implementation of this decision procedure is also described.

- Finally, we study the framework of non-deterministic matrices (Nmatrices) as a tool for constructing analytic sequent calculi for logics that are already given in some other form. A fundamental operation on Nmatrices is introduced, called *rexpansion*, and is shown to be a crucial (though so far implicit) ingredient in applying Nmatrices for the construction of analytic sequent calculi for non-classical logics.

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Chapter 1

Introduction

1.1 Background

Gentzen's seminal paper from 1934 [54] begins with the introduction of the first two natural deduction calculi: system NK for classical logic and system NJ for intuitionistic logic.[1] The calculus NJ admits the *subformula property*: whenever a hypothesis can be derived from a set of assumptions, there is a derivation of the same hypothesis that contains only subformulas of the hypothesis and the assumptions. However, this property fails for NK. For the purpose of providing a calculus with the subformula property for both intuitionistic and classical logic, Gentzen introduced two alternative systems: LK and LJ. The main formal entities that are manipulated in LK and LJ are called *sequents*. These have the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite lists of formulas[2] and " $\Rightarrow$ " is a meta-symbol. This more complex data structure provides a good mechanism for keeping track of the current premises in the derivation (the formulas that appear in Γ), as well as the current possible conclusions of these premises (the formulas that appear in Δ). And indeed, the intuitive reading of a sequent $\Gamma \Rightarrow \Delta$ is that the conjunction of the left side implies the disjunction of the right side. The manipulation of sequents instead of formulas allowed Gentzen to replace all elimination rules of natural deduction with rules that introduce formulas on the left side of the sequent. This way, the rules of the logical symbols admit the *local subformula property*: in each rule, the premises only include subformulas of the conclusion. In addition to the right-introduction and left-introduction rules (that correspond to introduction and elimination rules in natural deduction, respectively), LK and LJ also include several *structural rules*, that operate on the sequent level. Most of the structural rules also admit the local subformula property,

[1] The same year, Jaśkowski's similar notion of natural deduction was independently published [62].
[2] In this thesis we consider a variant in which Γ and Δ are finite *sets* of formulas.

with the single exception being the *cut* rule:

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

This is the only rule in LK and LJ that does not have the local subformula property. For this reason, a major portion of Gentzen's paper is devoted to the "Hauptsatz" (Main Theorem):

Theorem ([54])**.** *Every derivation of a sequent in LK (LJ) can be transformed into a derivation of the same sequent in LK (LJ), in which the cut rule is not used.*

As a corollary, Gentzen obtains the desired subformula property for the intuitionistic case (LJ), and unlike natural deduction, also for the classical case (LK). From this property, two additional characteristics of LK and LJ follow:

1. Consistency: the empty sequent is not derivable. This is true, as if it were derivable, the subformula property would have ensured a proof of it that includes no formulas at all. This is only possible if the empty sequent was one of the rules of the systems, which is not the case.

2. Decidability: when considering propositional classical and intuitionistic logics, each formula has finitely many subformulas. The subformula property of LK and LJ then provides a decision procedure for both logics, as it reduces the search space of possible derivations of a given sequent to a finite one.

Since then the framework of sequent calculi (as well as its extensions, like hypersequential calculi) is widely applied in proof theory and automated reasoning. In addition to classical and intuitionistic logics, many important non-classical logics were investigated and implemented through this framework and some of its variants. Examples include modal logics [85, 101], conditional logics [55], many-valued and fuzzy logics [25, 79] and paraconsistent logics [27, 28, 29].

For each of them, the process can be roughly described as follows:

1. A sequent calculus is designed for a particular logic.

2. The cut rule is shown to be admissible in this calculus.

3. The subformula property (or some natural generalization of it) is obtained as a corollary, and ensures that the calculus is consistent and decidable (in the propositional case).

4. In some cases, a proof-search procedure is designed for this particular calculus.

There are two main disadvantages in the process just described. First, the traditional route to the subformula property derives it as a corollary of cut-admissibility. However, there are several important logics for which a cut-free ordinary sequent calculus seems to be beyond reach, while a calculus that admits the subformula property was found. Examples for such logics include the modal logics B (of symmetric Kripke frames) and S5 (of universal Kripke frames) [85, 96, 101], and also bi-intuitionistic logic [84]. Moreover, even when a calculus admits both properties, it is sometimes much easier to prove the subformula property than cut-admissibility. Second, developing an efficient proof-search decision procedure for a given calculus requires a great deal of ingenuity (see, e.g., [45]). Efficiency is often gained by heuristics that are calculus-specific, which makes them hard to be generalized for other calculi.

Aiming for alternative routes than the one described above, this thesis focuses mainly on the subformula property and some of its natural generalizations, in most cases without relying on cut-admissibility, and utilizes some useful characteristics of it in order to design uniform decision procedures.

1.2 This Thesis

On Analyticity

Gentzen describes the subformula property concisely by: "No concepts enter into the proof other than those contained in its final result" [54]. Now, the obvious interpretation of this description identifies the notion of "containment" simply with the subformula relation. However, such a reading leaves out various calculi for non-classical logics, that, while not admitting the subformula property, do admit simple variants of it, that still ensure decidability and consistency. For example, in several calculi for many-valued and paraconsistent logics, every derivation of a sequent can be transformed into a derivation in which not only subformulas of the sequent occur, but also their negations (see, e.g., [13, 27]). Such variants of the subformula property are just as useful as the usual one to achieve decidability and consistency. Thus, in this thesis we consider several interpretations of what "containment" means, and introduce a parametrized notion of a subformula. We call the various variants of the subformula property that we propose *analyticity*, and say that a calculus is *analytic* if it admits one of them.

Main Results

Our main contribution is a systematic analysis of analyticity in three general families of propositional sequent calculi:

Pure Calculi These are propositional sequent calculi that include all the usual structural rules (id, cut, weakening, contraction and permutation), and whose logical derivation rules do not enforce any limitations on the context sequents.[3] While the most dominant example of a pure calculus is the propositional fragment of LK, pure calculi are suitable for various non-classical logics, including three and four-valued logics, and also many paraconsistent logics. For this family, we provide simple and decidable sufficient criteria for analyticity, that can be easily checked by either a logic researcher or practitioner, or in an automated way. Many analyticity results from the literature, as well as analyticity of new calculi that we introduce, are obtained as particular instances of this result. Further, we study the connection between analyticity and cut-admissibility in pure calculi, and show that for a wide sub-family, these properties are equivalent. We then provide a generic decision procedure for analytic pure calculi, that is based on a uniform reduction to SAT, thus leaving all logic-specific heuristics and optimizations that are common in proof-search algorithms to the realm of off-the-shelf SAT solvers. The final chapter of this thesis studies methods to construct pure calculi for logics that are defined by other means (e.g. by a Hilbert-type calculus or using a semantic definition).[4] The resulting calculi are subject to the reduction to SAT, as well as to the extension of pure calculi with modal operators, that is described next.

Pure Calculi with Modal Operators Usual sequent rules for modal logics are impure, as they operate on the context sequents and have several constraints on them. However, when a calculus can split into a pure part and a separate impure part that consists solely of modal rules, and these parts do not share any logical connectives, it is possible to lift some of the properties of the pure part to the whole calculus. This is the case, for example, for classical modal logics like K and S5, whose pure part is simply the propositional fragment of LK, and the modal part consists of a single rule for $\Box$. We study such calculi in a multimodal setting, and prove that the analyticity of a pure calculus is preserved when adding several well-known impure rules for modal operators. We then extend the decision procedure for analytic pure calculi to modal calculi of a certain form.

Intuitionistic Calculi Gentzen's calculus for intuitionistic logic LJ is not pure, as it employs only single-conclusion sequents. In particular, all right-introduction rules restrict the right side of the context sequent to be empty. In [97], Takeuti described

[3] We follow [11] and use the name *pure* for such calculi.

[4] In case the logic is given by a Hilbert calculus, we obtain proper sequent rules that are different from the original axioms.

Maehara's equivalent calculus for this logic [77], LJ', that relaxes this requirement. LJ' employs ordinary multiple conclusion sequents, and requires that the right side of the context sequent is empty only for the right introduction rules of implication and negation. Other calculi for constructive logics such as Nelson's N3 and N4 (see, e.g., [100]) employ a similar restriction. We identify a family of calculi, that we call *intuitionistic calculi*, and includes, among others, the aforementioned examples. We then prove that similarly to the case of pure calculi, cut-admissibility and analyticity are equivalent for a wide sub-family of intuitionistic calculi.

Main Tools

Gentzen's original method for proving analyticity (through cut-admissibility) is *syntactic*: it uses double induction on derivations, and constructively eliminates cuts, by replacing them with other rules that produce a derivation with a smaller inductive measurement. Since then, many analyticity proofs go through the same route, with the details changing according to the investigated calculus. While this approach is constructive – an algorithm can usually be extracted, that transforms a derivation with cut into a derivation without it – such syntactic proofs are very hard to verify, and are error-prone. The main issue is the vast number of (sub-)cases that need to be considered, and the fact that some cases that look similar to previously proven cases in the proof (and are therefore usually left to the reader), are actually different.

Another approach, that we adopt in this work, is the *semantic* approach. According to this approach, proving analyticity (and/or cut-admissibility) for a particular calculus amounts to a completeness proof of analytic (and/or cut-free) derivations in the calculus with respect to some semantics for which the full calculus is sound. Proofs in the semantic approach are easier to verify, and, more importantly in the context of this thesis, to generalize. Therefore, they are much more suitable for investigating *families* of calculi.

While each of the aforementioned results is purely syntactic, almost all of their proofs are semantic, but in a way that is invisible to the "end-user", that does not need to know that the semantics even exists. The general recipe for proving our results goes as follows:

1. Formalize a syntactic hypothesis about a certain family of sequent calculi

2. Prove a general soundness and completeness theorem for this family

3. Using soundness and completeness, find a semantic hypothesis whose validity entails that of the syntactic one (often the two hypotheses will actually be equivalent)

4. Prove the semantic hypothesis using semantic tools

5. Conclude the validity of the syntactic hypothesis

But what semantic frameworks or principles should be used? An important desired feature of a semantic framework for the investigation of proof systems is *modularity*. The connection between the derivation rules and their semantics should be *local*, such that small changes in the derivation rules only have a local effect on the semantics, preferably only relating to the connectives that were changed in the proof system. For this reason, the main principle that is common to all the frameworks that we utilize in this thesis is the principle of *non-determinism*, that has proven useful in the analysis of families of sequent calculi (see, e.g., [21, 27, 70]). According to this principle, that is borrowed from automata and computability theory, the truth value of a compound formula is not uniquely determined by the truth values of its direct subformulas, but is only *constrained* by them. Further, more complex constraints may be incorporated, that do not only rely on the values assigned to immediate subformulas. The meaning of the connectives is thus not represented by functions (e.g. "$\neg(x) = 1 - x$", like in the case of classical negation). Instead, it is represented by relational constraints (e.g. "$x = 0 \to \neg x = 1$", like in the case of the CLuN negation [33] and other paraconsistent negations). This relaxed notion of a semantic framework is very flexible, and allows for a great amount of modularity. Such an approach is particularly useful for a semantical analysis of sequent calculi, by reading each derivation rule as a semantic constraint on the "legal" models. Unlike deterministic frameworks (such as many-valued matrices), each derivation rule has only a local effect, and thus its inclusion (or exclusion) is handled without the need to propagate any changes to the general semantics. The following is a list of the non-deterministic semantic frameworks that we use in this work:

Two-valued valuation functions These are functions that assign binary truth values to formulas, and generalize the bivaluation semantics of [35]. They are mainly used to study analyticity in pure calculi, and also to provide a uniform decision procedure for analytic pure calculi. By translating the rules of a given pure calculus into semantic constraints, one obtains a subset of such functions, for which the calculus is sound and complete. An important feature of these functions is that they may be partially defined, and in many cases, the ability to extend partial functions into full ones corresponds to analyticity.

Three-valued valuation functions This semantic framework is used to study cut-admissibility in pure calculi. It is obtained from two-valued valuation functions by the addition of a third truth value, whose purpose is to make the cut rule unsound (an idea that goes at least as far back as Schütte's work on cut-admissibility [90]). In such a framework, it is possible to obtain a semantics for which a pure calculus without cut is sound and complete. By relating the third truth value to the undefined elements in two-valued valuation functions, we are able to show the equivalence between cut-admissibility and analyticity in a wide sub-family of pure calculi.

Two-valued Kripke models This semantic framework is a particular instance of the Kripke valuations from [70], and it allows us to import the syntactic separation between pure rules and modal rules to the semantics: pure rules are represented by *local* constraints on the truth values in each world, while modal rules are represented by *global* constraints according to the usual meaning of $\Box$. Similarly to valuation functions, we consider Kripke models that are partially defined, and are thus able to generalize some of the results of pure calculi to their augmented version with modal operators. In particular, this semantic framework enables us to show that the analyticity of a pure calculus survives the addition of modal operators (by showing that the ability to extend partial valuations can be used to extend partial Kripke models). The similarities between Kripke models and valuation functions also allows us to extend the above decision procedure to certain modal operators.

Three-valued Kripke models These are obtained from two-valued Kripke models by the addition of a third truth value, and are a particular instance of the quasi-valuations from [70]. This framework is used to study the connection between cut-admissibility and analyticity in intuitionistic calculi. Unlike two-valued Kripke models, whose global constraints depend on modal operators, we utilize three-valued Kripke models for intuitionistic calculi, that do not employ modalities. Instead, the impure rules of intuitionistic calculi impose other (global) constraints on the valuations of each world. In the spirit of the semantics for intuitionistic logic, these models are *persistent*, which means that truth is hereditary in accessible worlds.

Non-deterministic matrices The above frameworks are very restrictive with respect to the number of truth values that are allowed to be used (either two or three), but are permissive with respect to the minimal semantic restrictions on the models that are employed. Non-deterministic matrices (Nmatrices) [21, 22], a natural generalization of ordinary many-valued matrices [99], are of a dual character: they are not restricted to any number of truth values; but, they are much more restrictive in the minimal requirements from the models of the framework. Nmatrices are obtained from ordinary matrices by allowing a non-deterministic choice of a truth value for a compound formula out of a set of possible values, determined by the values of its immediate subformulas. While the aforementioned semantic frameworks are mainly applied for studying existing sequent calculi, the framework of Nmatrices is useful for *constructing* sequent calculi for existing logics, and even for families of logics. And indeed, many sequent calculi for non-classical logics were constructed using the method of [25], by first finding a finite Nmatrix for the logic and then translating it to a sequent calculus (see, e.g., [26]). Using similar techniques, infinite Nmatrices are also useful for this task (see, e.g., [27, 29]). When studying families of logics, this method often requires some transformations on a certain basic Nmatrix, and then each logic of the family naturally induces some variant of the transformed Nmatrix. We introduce a fundamental operation on Nmatrices, that we call *rexpansion*, and show that many transformations on Nmatrices from the literature are actually, though implicitly, particular instances of this operation. We study the properties of rexpansion, and make explicit its usage in the construction of sequent calculi for non-classical logics. In addition, we introduce a method for conservatively extending a given logic or sequent calculus with new connectives that have several desired properties. The most important demonstration of this method that we present is a novel technique for the construction of paraconsistent fuzzy logics.

Outline

The structure of this thesis is as follows. Chapters 2 and 3 are devoted to the family of pure calculi. After explicitly defining this family and providing several examples in Section 2.2, their semantics, that is based on two-valued valuation functions, is introduced in Section 2.3. Then, a generalized analyticity property is defined in Section 2.5, where we also provide sufficient criteria for identifying and constructing analytic calculi. In Section 2.6 analyticity is proven to be equivalent to cut-admissibility in a wide subfamily of calculi. Chapter 3 presents a SAT-based decision procedure for analytic pure calculi. In particular, Section 3.3 describes an implementation of this procedure.

In Chapter 4 we investigate the extension of pure calculi with impure rules for well-known modal operators. Section 4.3 shows that analyticity survives the addition of such rules to a pure calculus, and Section 4.4 extends the reduction from Chapter 3 to a certain type of modal operators that we call *Next operators*. The results of this chapter are based on an extension of two-valued valuation functions to two-valued Kripke models.

Chapter 5 introduces the family of intuitionistic calculi, and extends the equivalence of Section 2.6 between cut-admissibility and analyticity to this family. This result is proved using a semantic view of intuitionistic calculi, that is based on three-valued Kripke models.

Chapter 6 is dedicated to rexpansions of non-deterministic matrices and their applications in the construction of pure sequent calculi. Section 6.2 introduces the rexpansion operation and study its important properties, as well as its effects on semantically-defined consequence relations. In particular, it includes a method for obtaining conservative extensions of a given logic with new connectives that are expected to admit several properties. Section 6.4 demonstrates the usefulness of rexpansion in the construction of sequent calculi for non-classical logics. Section 6.5 uses rexpansions for the construction of paraconsistent fuzzy logics.

Finally, in Chapter 7 we conclude with a discussion of some directions for further research.

Related Work

Avron and Lev [21] introduced the family of canonical sequent calculi, a very restricted sub-family of pure calculi, and provided a sufficient syntactic criterion, called *coherence* for analyticity and cut-admissibility in them (see also [22]). Here we generalize this criterion in order to cover a much wider family of calculi. The general framework of [81] allows one to encode all pure calculi in linear logic, and use linear logic to reason about them. Among the pure calculi, it is again only the canonical ones for which a decidable criterion for analyticity is given in [81].

While most of the work in decision procedures for non-classical logics is done for specific logics, some tools focus on families of logics, that share the same syntax and basic structure of derivation rules. Examples of such tools include the Logic WorkBench [61], the Tableau Workbench [1], LoTREC [53], focusing on modal-like logics, and COOL [57], focusing on modal and hybrid logics. The decision procedure that we introduce here and its implementation are completely generic, and are applicable to all analytic pure calculi (and some of their extensions), without any restriction on the syntax itself. Another project with a similar goal is MetTeL [98], which incorporates a generic decision procedure for *tableau* calculi, and thus uses a different approach from the one taken here. We compare between the two approaches in Section 3.3.

Finally, some of the semantic frameworks that we employ here are either variants or particular instances of the general framework given in [70]. In particular, some of our completeness theorems, which we prove directly, can be also obtained using the method given there. However, in some cases, direct completeness proofs turn out to be simpler, and better targeted for the purposes of this thesis. While one of the main purposes of [70] is to provide semantic methods to prove syntactic properties, this thesis aims to take these methods to the next step, by building techniques and algorithms on top of it.

Publications Related to this Dissertation

Most of the contributions described in Chapters 2 to 5 are based on [71, 72, 73, 74, 103, 104]. The material of Chapter 6 is based on [23, 24]. The connection between each publication and its related chapters is described at the beginning of each chapter.

Chapter 2

Pure Sequent Calculi

In this chapter we define the family of *pure sequent calculi* [9], and provide a uniform semantic interpretation of such calculi, that will be the main tool for our investigation. Roughly speaking, pure sequent calculi are propositional calculi that include all the usual structural rules: exchange, contraction, cut, identity and weakening, and whose derivation rules do not enforce any limitations on the context formulas. As will be evident by our examples, this family is a prominent proof-theoretic framework, adequate for many propositional logics, including classical logic, many-valued logics, and various paraconsistent logics.

We start with Section 2.1, that is a section of preliminaries for this chapter, as well as for the chapters that follow. Section 2.2 explicitly defines the family of pure calculi and provides examples. In Section 2.3 we introduce a semantic framework for pure calculi, that will be the main tool for our investigation. Section 2.4 includes several useful transformations of sequent calculi that are used in later sections. In Section 2.5 we introduce a generalized notion of a subformula, that induces a parametrized notion of analyticity in sequent calculi, and provide method for identifying and constructing analytic calculi. Section 2.6 studies the connection between our general notion of analyticity and cut-admissibility. Finally, section Section 2.7 studies derivations in pure calculi that are restricted to single-conclusion sequents.

Publications Related to this Chapter

This chapter is mainly based on [71, 73, 74]. However, the results of [73] are strengthened here to apply on arbitrary languages and a more general notion of analyticity.

2.1 Preliminaries

A *propositional language* $\mathcal{L}$ consists of a countably infinite set of atomic variables $At = \{p_1, p_2, ...\}$ and a finite set $\diamondsuit_{\mathcal{L}}$ of propositional connectives. For every $n \geq 0$, the set of all n-ary connectives of $\mathcal{L}$ is denoted by $\diamondsuit^n_{\mathcal{L}}$. Well-formed formulas in a propositional language $\mathcal{L}$ are defined as usual. Given a set $\mathcal{F} \subseteq \mathcal{L}$, we say that a formula ψ is an $\mathcal{F}$*-formula* if $\psi \in \mathcal{F}$.

A *substitution* is a function from At to some propositional language. A substitution σ is extended to formulas of a propositional language $\mathcal{L}$ by $\sigma(\diamond(\psi_1, \dots, \psi_n)) = \diamond(\sigma(\psi_1), \dots, \sigma(\psi_n))$ for every connective $\diamond$, and to sets of formulas by $\sigma(\mathcal{F}) = \{\sigma(\psi) \mid \psi \in \mathcal{F}\}$.

A *propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ such that $\mathcal{L}$ is a propositional language and $\vdash_{\mathbf{L}}$ is a binary relation between subsets of $\mathcal{L}$ and formulas of $\mathcal{L}$, which is: (i) reflexive: if $\varphi \in \mathcal{T}$ then $\mathcal{T} \vdash_{\mathbf{L}} \varphi$; (ii) monotone: if $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T} \subseteq \mathcal{T}'$ then $\mathcal{T}' \vdash_{\mathbf{L}} \varphi$; (iii) transitive: if $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T}', \varphi \vdash_{\mathbf{L}} \psi$ then $\mathcal{T}, \mathcal{T}' \vdash_{\mathbf{L}} \psi$; (iv) structural: if $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ then $\sigma(\mathcal{T}) \vdash_{\mathbf{L}} \sigma(\varphi)$; (v) non-trivial: there exist $\emptyset \neq \mathcal{T} \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ such that $\mathcal{T} \not\vdash_{\mathbf{L}} \varphi$.[1]

A *sequent* is a pair $\langle \Gamma, \Delta \rangle$, denoted $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. For a sequent $\Gamma \Rightarrow \Delta$, $\mathit{frm}(\Gamma \Rightarrow \Delta) = \Gamma \cup \Delta$. This notation is extended to sets of sequents by $\mathit{frm}(S) = \bigcup_{s \in S} \mathit{frm}(s)$. A sequent $\Gamma \Rightarrow \Delta$ is called an $\mathcal{F}$*-sequent* if $\mathit{frm}(\Gamma \Rightarrow \Delta) \subseteq \mathcal{F}$. We employ the standard sequent notations, e.g., when writing expressions like $\Gamma, \psi \Rightarrow \Delta$ or $\Rightarrow \psi$. The *union* of sequents is given by $(\Gamma_1 \Rightarrow \Delta_1) \cup (\Gamma_2 \Rightarrow \Delta_2) = (\Gamma_1 \cup \Gamma_2) \Rightarrow (\Delta_1 \cup \Delta_2)$. A sequent $\Gamma_1 \Rightarrow \Delta_1$ is a *subsequent* of a sequent $\Gamma_2 \Rightarrow \Delta_2$, denoted $(\Gamma_1 \Rightarrow \Delta_1) \subseteq (\Gamma_2 \Rightarrow \Delta_2)$, if $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$. Substitutions are also extended to sequents by $\sigma(\Gamma \Rightarrow \Delta) = \sigma(\Gamma) \Rightarrow \sigma(\Delta)$ and sets of sequents by $\sigma(S) = \{\sigma(s) \mid s \in S\}$.

Henceforth, $\mathcal{L}$ denotes an arbitrary propositional language. We sometimes identify $\mathcal{L}$ with its set of well-formed formulas (e.g., when writing "$\psi \in \mathcal{L}$"), or with its set of connectives (e.g., when writing "the language $\{\wedge, \vee, \neg\}$". The correct reading of such expressions will always be clear from the context. Whenever $\mathcal{L}$ can be inferred from the context, we may call $\mathcal{L}$-formulas *formulas.*

2.2 What Are Pure Calculi?

We start by defining *pure rules* and their applications, namely the steps that form derivations in pure calculi.

[1] This requirement is not always demanded in the literature, but we find it convenient (and natural) to include it here.

Definition 2.2.1. A *pure rule* is a pair $\langle S, s\rangle$, denoted $S \,/\, s$, where S is a finite set of sequents and s is a sequent. The elements of S are called the *premises* of the rule and s is called the *conclusion* of the rule.

An application of a pure rule is obtained by applying a substitution to the rule, and then adding a context-sequent.

Definition 2.2.2. An *application* of a pure rule $\{s_1, \dots, s_n\} \,/\, s$ is a pair of the form $\langle \{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \dots \cup c_n\rangle$ where σ is a substitution, s'_i is a subsequent of s_i for every $1 \leq i \leq n$ and $c_1, \dots, c_n$ are sequents (called *context sequents*). The sequents $\sigma(s'_i) \cup c_i$ are called the *premises* of the application and the sequent $\sigma(s) \cup c_1 \cup \dots \cup c_n$ is called the *conclusion* of the application. We often denote an application $\langle \{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \dots \cup c_n\rangle$ as a derivation step:

$$\frac{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n}{\sigma(s) \cup c_1 \cup \dots \cup c_n}$$

Example 2.2.3. The following is a pure rule:

$$p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$$

Applications of this rule have the following forms:

$$\frac{\Gamma, \psi_1 \Rightarrow \psi_2, \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta} \qquad \frac{\Gamma, \psi_1 \Rightarrow \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta} \qquad \frac{\Gamma \Rightarrow \psi_2, \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta}$$

Applications of the following rules

$$\Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow \qquad\qquad / \Rightarrow p_1 \supset p_1$$

have respectively the forms:

$$\frac{\Gamma_1 \Rightarrow \psi_1, \Delta_1 \qquad \Gamma_2, \psi_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi_1 \supset \psi_2 \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \frac{}{\Rightarrow \psi \supset \psi}$$

In contrast, the usual rule for introducing implication on the right side in intuitionistic logic is not a pure rule, since it allows only *left* context formulas.

We make three brief remarks regarding the above definition, that relate it to more common definitions of sequent rules from the literature:

1. Following [21], we use the object propositional language for specifying derivation rules, instead of meta-variables which are often used to present derivation schemes. Accordingly, applications of rules are obtained by applying a *substitution* on the premises and the conclusion of the rule, and freely adding context formulas.

2. We allow applications of pure rules to make use of subsequents of the premises, and not necessarily the full premises. While this is technically convenient (see, e.g.,

Section 2.4), it does not change the derivability relation induced by a given sequent calculus.

3. Applications of rules are *multiplicative*—allowing a different context sequent in each premise. Since all usual structural rules are assumed, one may equivalently consider *additive* applications, that require the same context sequent in all premises.

In turn, *pure sequent calculi* are finite sets of pure rules. To make them fully-structural (in addition to defining sequents as pairs of *sets*), the weakening rule, the identity axiom and the cut rule are allowed to be used in derivations. This is defined as follows:

Definition 2.2.4. A *pure calculus* is a finite set of pure rules. A *derivation* in a pure calculus $\mathbf{G}$ is defined as usual, where in addition to applications of the pure rules of $\mathbf{G}$, the following standard application schemes may be used:

$$(\textsc{weak})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \qquad (\textsc{id})\ \frac{}{\Gamma, \psi \Rightarrow \psi, \Delta} \qquad (\textsc{cut})\ \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

In ($\textsc{cut}$), ψ is called the *cut formula*.

Henceforth, unless stated otherwise, we consider only pure rules and pure calculi, and may refer to them simply as *rules* and *calculi*. By an $\mathcal{L}$-rule ($\mathcal{L}$-calculus) we mean a rule (calculus) that includes only connectives of $\mathcal{L}$. In what follows, unless stated otherwise, every calculus is an $\mathcal{L}$-calculus for some fixed propositional language $\mathcal{L}$.

Notation 2.2.5. For an $\mathcal{L}$-calculus $\mathbf{G}$, a set $\mathcal{F} \subseteq \mathcal{L}$ of formulas, a set S of $\mathcal{F}$-sequents and an $\mathcal{F}$-sequent s, we write $S \vdash_{\mathbf{G}}^{\mathcal{F}} s$ if there is a derivation of s from S in $\mathbf{G}$ consisting only of $\mathcal{F}$-sequents. For $S \vdash_{\mathbf{G}}^{\mathcal{L}} s$ (i.e., $\mathcal{F} = \mathcal{L}$), we may also write $S \vdash_{\mathbf{G}} s$. When S is finite, we usually omit the curly braces (writing, e.g., $\Rightarrow p_1, \ \Rightarrow p_2 \vdash_{\mathbf{G}}^{\mathcal{F}} \ \Rightarrow p_2$).

Given a pure calculus $\mathbf{G}$, its associated derivability relation $\vdash_{\mathbf{G}}$ is defined between sets of sequents and sequents. This relation induces two consequence relations between sets of formulas and formulas.

1. $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ if there exists finite $\Gamma \subseteq \mathcal{T}$ such that $\vdash_{\mathbf{G}} \Gamma \Rightarrow \varphi$

2. $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ iff $\{ \Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} \ \Rightarrow \varphi$

In the case of pure calculi, these two definitions are easily seen to be equivalent.[2]

[2]The notation $\vdash_{\mathbf{G}}$ here is overloaded: it is used to denote both a relation between sequents and a relation between formulas. This overloading does not pose a problem, as the correct interpretation is always clear from the context.

Remark 2.2.6. Since pure sequent calculi manipulate multiple-conclusion sequents, they also naturally induce consequence relations in the sense of Scott [91] (see also [93]) between sets of formulas. For sets $\mathcal{T}$ and $\mathcal{S}$ of formulas, such a relation is defined by $\mathcal{T} \vdash_{\mathbf{G}} \mathcal{S}$ iff there are finite $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{T}$ such that $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$.

Next, we present several examples of pure sequent calculi. The most fundamental one is Gentzen's system for classical logic [54]:

Example 2.2.7 (Classical Logic)**.** The propositional language $\mathcal{CL}$ consists of three binary connectives $\wedge$, $\vee$, $\supset$, and one unary connective $\neg$. The propositional fragment of Gentzen's fundamental sequent calculus for classical logic [54] can be directly presented as a pure $\mathcal{CL}$-calculus, denoted **LK**, that consists of the following $\mathcal{CL}$-rules:

$(\neg \Rightarrow)$	$\Rightarrow p_1 \,/\, \neg p_1 \Rightarrow$	$(\Rightarrow \neg)$	$p_1 \Rightarrow \;/\; \Rightarrow \neg p_1$
$(\wedge \Rightarrow)$	$p_1, p_2 \Rightarrow \;/\, p_1 \wedge p_2 \Rightarrow$	$(\Rightarrow \wedge)$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow p_1 \wedge p_2$
$(\vee \Rightarrow)$	$p_1 \Rightarrow \;; p_2 \Rightarrow \;/\, p_1 \vee p_2 \Rightarrow$	$(\Rightarrow \vee)$	$\Rightarrow p_1, p_2 \,/\, \Rightarrow p_1 \vee p_2$
$(\supset \Rightarrow)$	$\Rightarrow p_1 \,; p_2 \Rightarrow \;/\, p_1 \supset p_2 \Rightarrow$	$(\Rightarrow \supset)$	$p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$

Note that there are several differences between Gentzen's original calculus from [54] and our presentation of it. First, instead of taking sequents to be lists of formulas, and including structural rules for manipulating lists, we consider sequents to be pairs of sets. Second, the rules $(\wedge \Rightarrow)$ and $(\Rightarrow \vee)$ were split in [54] to two rules each. For instance $(\Rightarrow \vee)$ was given by two separate derivation schemes:

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} \quad \text{and} \quad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

Nevertheless, our notion of an application of a rule, that allows for partial sequents to occur, considers these two application schemes as applications of $(\Rightarrow \vee)$.

Besides classical logic, pure calculi are useful for a wide variety of non-classical logics, as demonstrated by the following examples.

Example 2.2.8 (Many-valued Logics)**.** The paper [13] provides pure sequent calculi for well-known many-valued logics. For example, a calculus for Łukasiewicz three-valued logic $Ł_3$ [76], which we call $\mathbf{G}_{Ł_3}$, is obtained from **LK** by replacing the rules $(\supset \Rightarrow)$, $(\Rightarrow \supset)$ and $(\Rightarrow \neg)$ with the following rules:

$(\neg\neg \Rightarrow)$	$p_1 \Rightarrow \;/\, \neg\neg p_1 \Rightarrow$	$(\Rightarrow \neg\neg)$	$\Rightarrow p_1 \,/\, \Rightarrow \neg\neg p_1$
$(\neg\wedge \Rightarrow)$	$\neg p_1 \Rightarrow \;; \neg p_2 \Rightarrow \;/\, \neg(p_1 \wedge p_2) \Rightarrow$	$(\Rightarrow \neg\wedge)$	$\Rightarrow \neg p_1, \neg p_2 \,/\, \Rightarrow \neg(p_1 \wedge p_2)$
$(\neg\vee \Rightarrow)$	$\neg p_1, \neg p_2 \Rightarrow \;/\, \neg(p_1 \vee p_2) \Rightarrow$	$(\Rightarrow \neg\vee)$	$\Rightarrow \neg p_1 \,;\, \Rightarrow \neg p_2 \,/\, \Rightarrow \neg(p_1 \vee p_2)$
$(\neg \supset \Rightarrow)$	$p_1, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow$	$(\Rightarrow \neg \supset)$	$\Rightarrow p_1 \,;\, \Rightarrow \neg p_2 \,/\, \Rightarrow \neg(p_1 \supset p_2)$

$(\supset\Rightarrow)_{Ł_3}$ $\neg p_1 \Rightarrow\ ; p_2 \Rightarrow\ ;\ \Rightarrow p_1, \neg p_2 \,/\, p_1 \supset p_2 \Rightarrow$

$(\Rightarrow\supset)_{Ł_3}$ $p_1 \Rightarrow p_2 \,;\, \neg p_2 \Rightarrow \neg p_1 \,/\, \Rightarrow p_1 \supset p_2$

A pure calculus for the $\mathcal{CL}$-fragment of the logic of bilattices [3] (whose implication-free fragment coincides with the logic of first-degree entailments [2]), that we call $\mathbf{G}_4$, is obtained in a similar manner, by augmenting the positive fragment of **LK** with the above rules, excluding $(\supset\Rightarrow)_{Ł_3}$ and $(\Rightarrow\supset)_{Ł_3}$.

Example 2.2.9 (Paraconsistent Logics)**.** The paper [27] provides sequent calculi for many paraconsistent logics. For example, a pure calculus for da Costa's historical paraconsistent logic $\mathbf{C_1}$, which we call $\mathbf{G_{C_1}}$, consists of the rules of **LK** except for the left-introduction rule of negation, that is replaced by $(\neg\neg\Rightarrow)$, together with the following pure $\mathcal{CL}$-rules:

$$\Rightarrow p_1;\ \Rightarrow \neg p_1 \,/\, \neg(p_1 \wedge \neg p_1) \Rightarrow \qquad \neg p_1 \Rightarrow\,;\, \neg p_2 \Rightarrow\ /\, \neg(p_1 \wedge p_2) \Rightarrow$$

$$\neg p_1 \Rightarrow\,;\, p_2, \neg p_2 \Rightarrow\ /\, \neg(p_1 \vee p_2) \Rightarrow \qquad p_1, \neg p_1 \Rightarrow\,;\, \neg p_2 \Rightarrow\ /\, \neg(p_1 \vee p_2) \Rightarrow$$

$$p_1 \Rightarrow\,;\, p_2, \neg p_2 \Rightarrow\ /\, \neg(p_1 \supset p_2) \Rightarrow \qquad p_1, \neg p_1 \Rightarrow\,;\, \neg p_2 \Rightarrow\ /\, \neg(p_1 \supset p_2) \Rightarrow$$

Example 2.2.10 (Logic for efficient access control)**.** Primal infon logic [44], that we denote by **PIL**, was designed to efficiently reason about access control policies. While expressive enough for this purpose, it can be decided in linear time. The quotations-free fragment of its sequent calculus [34] can be presented as a pure calculus, which we denote by $\mathbf{G_{PIL}}$. It is obtained from the positive fragment of **LK** by adding the rules $\emptyset \,/\, \Rightarrow \top$ and $\emptyset \,/\, \bot \Rightarrow$, dismissing the left introduction rule of disjunction, and replacing the right introduction rule of implication with the following weaker rule:

$$\Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$$

To conclude, the following lemma shows that derivability in pure calculi admits closure under substitutions and context sequents.

Lemma 2.2.11. If $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$, then:

1. $\sigma(S) \vdash^{\sigma(\mathcal{F})}_{\mathbf{G}} \sigma(s)$ for every substitution σ.

2. $\{s' \cup c \mid s' \in S\} \vdash^{\mathcal{F} \cup frm(c)}_{\mathbf{G}} s \cup c$ for every sequent c.

2.3 Semantics

In this section we introduce a semantic interpretation of pure calculi, based on (possibly non-deterministic) two-valued valuation functions. This semantics will be the main tool that we use to characterize analyticity, and to provide a decision procedure for analytic pure calculi.

Our semantics follows [35] and uses *bivaluations*—functions assigning a binary truth value to each formula. The simple framework of bivaluations is applicable to a wide variety of propositional logics. The price for its simplicity and generality is the loss of truth-functionality: the truth value assigned to a compound formula is not always uniquely determined by the truth values assigned to its subformulas. Accordingly, it is insufficient to define bivaluations over atomic formulas, and hence they are defined in [35] over the entire language.

Here we extend the bivaluation framework by considering also *partial* bivaluations that assign truth values to *some* formulas. This allows us to have finite models which are essential in semantic decision procedures. Next, we precisely define (partial) bivaluations, and provide a general soundness and completeness theorem, relating each pure calculus **G** and set $\mathcal{F}$ of formulas to a set of partial bivaluations for which **G** is sound and complete, when only $\mathcal{F}$-formulas may appear in derivations.

Definition 2.3.1. A *bivaluation* is a function v from some set of propositional formulas, denoted $dom(v)$, to $\{0, 1\}$. A bivaluation v is extended to $dom(v)$-sequents by: $v(\Gamma \Rightarrow \Delta) = 1$ iff $v(\varphi) = 0$ for some $\varphi \in \Gamma$ or $v(\varphi) = 1$ for some $\varphi \in \Delta$. v is extended to sets (of $dom(v)$-formulas or sequents) by $v(X) = \min \{v(x) \mid x \in X\}$, where $\min \emptyset = 1$. Given a set $\mathcal{F}$ of formulas, by an *$\mathcal{F}$-bivaluation* we refer to a bivaluation v with $dom(v) = \mathcal{F}$.

To relate sequent calculi to bivaluations, we read pure rules as semantic constraints on bivaluations. This is formally defined as follows:

Definition 2.3.2. A bivaluation v *respects* a rule $s_1, \dots, s_n \,/\, s$ if $v(\{\sigma(s'_1), \dots, \sigma(s'_n)\}) \leq v(\sigma(s))$ for every subsequents $s'_1, \dots, s'_n$ of $s_1, \dots, s_n$ (respectively) and substitution σ such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(v)$. v is called **G**-*legal* for a calculus **G** if it respects all rules of **G**.

This definition captures many well-known semantic frameworks in a modular way.

Example 2.3.3 (Semantics of Classical Logic)**.** It is easy to see that a $\mathcal{CL}$-bivaluation v is **LK**-legal iff it respects the classical truth tables. For example, the first line of the truth table for conjunction is obtained as follows: Suppose $v(p_1) = v(p_2) = 1$. Then $v(\{\Rightarrow p_1, \ \Rightarrow p_2\}) = 1$. Since v is **LK**-legal, it respects $(\Rightarrow \wedge)$, and so $v(\Rightarrow p_1 \wedge p_2) = 1$. Therefore, $v(p_1 \wedge p_2) = 1$.

The generality of partial bivaluations allows them to go beyond classical logics, and thus to include some less standard examples:

Example 2.3.4. A bivaluation v respects the rule $p_1 \Rightarrow \ / \neg\neg p_1 \Rightarrow$ iff either $v(\varphi) = v(\neg\neg\varphi) = 0$ or $v(\varphi) = 1$, for every formula φ such that $\varphi, \neg\neg\varphi \in \mathit{dom}(v)$. Indeed, let σ be a substitution such that $\sigma(p_1) = \varphi$. If $v(\sigma(p_1) \Rightarrow\ ) = 1$, then we have $v(\varphi) = v(\sigma(p_1)) = 0$, and hence also $v(\sigma(\neg\neg p_1)) = v(\neg\neg\varphi) = 0$, which means that $v(\sigma(\neg\neg p_1) \Rightarrow\ ) = 1$. The converse is shown similarly. Note that every bivaluation v such that $\neg\neg\varphi \notin \mathit{dom}(v)$ whenever $\varphi \in \mathit{dom}(v)$ (trivially) respects this rule.

We prove a general soundness and completeness theorem, that ties the domain of bivaluations to the set of formulas that are allowed to appear in derivations.

Theorem 2.3.5 (Soundness and Completeness)**.** $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ iff $v(S) \leq v(s)$ for every **G**-legal $\mathcal{F}$-bivaluation v.

Proof.
Soundness: Let v be a **G**-legal $\mathcal{F}$-bivaluation. Suppose $v(S) = 1$. We prove that $v(s) = 1$ by induction on the length of the derivation of s from S in **G** (which consists only of $\mathcal{F}$-sequents). If $s \in S$, or s is the conclusion of an application of (ID), (CUT), or (WEAK) then this is straightforward.

Suppose now that s is the conclusion of an application of some rule $s_1, \dots, s_n \,/\, s_{n+1} \in \mathbf{G}$. Then there are subsequents $s'_1, \dots, s'_n$ of $s_1, \dots, s_n$ (respectively), a substitution σ and $\mathcal{F}$-sequents $c_1, \dots, c_n$ such that $s = \sigma(s_{n+1}) \cup c_1 \cup \dots c_n$, $\sigma(\mathit{frm}(\{s'_1, \dots, s'_n, s_{n+1}\})) \subseteq \mathcal{F}$, $c_1, \dots, c_n \subseteq \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}} \sigma(s'_i) \cup c_i$ for every $1 \leq i \leq n$ with a shorter derivation. By the induction hypothesis, $v(\sigma(s'_i) \cup c_i) = 1$ for every $1 \leq i \leq n$. If $v(c_i) = 1$ for some $1 \leq i \leq n$ then $v(\sigma(s_{n+1}) \cup c_1, \dots, c_n) = 1$. Otherwise, for every $1 \leq i \leq n$, $v(\sigma(s'_i)) = 1$. Since v is **G**-legal, $v(\sigma(s_{n+1})) = 1$ and hence $v(\sigma(s_{n+1}) \cup c_1, \dots, c_n) = 1$.
Completeness: Assume $S \nvdash^{\mathcal{F}}_{\mathbf{G}} s$. We construct a **G**-legal $\mathcal{F}$-bivaluation v such that $v(S) > v(s)$. Since $\mathcal{F}$ may be infinite, this construction requires the following generalization of sequents: An *ω-sequent* is a pair $\langle L, R\rangle$, denoted $L \Rightarrow R$, where L and R are (possibly infinite) subsets of $\mathcal{L}$. We write $S \vdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow R$ if there exist finite $\Gamma \subseteq L$ and $\Delta \subseteq R$ such that $S \vdash^{\mathcal{F}}_{\mathbf{G}} \Gamma \Rightarrow \Delta$. Other definitions and notations involving sequents are adopted to ω-sequents in the obvious way. Call an ω-sequent $L \Rightarrow R$ *maximal unprovable* if the followings hold:

- $L \cup R \subseteq \mathcal{F}$
- $S \nvdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow R$
- $S \vdash^{\mathcal{F}}_{\mathbf{G}} L, \varphi \Rightarrow R$ for every $\varphi \in \mathcal{F} \setminus L$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow \varphi, R$ for every $\varphi \in \mathcal{F} \setminus R$.

It is routine to extend s to a maximal unprovable ω-sequent $L \Rightarrow R$. Then, a countermodel v is defined as follows: $v = \lambda\varphi \in \mathcal{F}. \begin{cases} 1 & \varphi \in L \\ 0 & \varphi \in R \end{cases}$. From the fact that $L \Rightarrow R$ is maximal unprovable, using (CUT) and (ID), we have that $L \uplus R = \mathcal{F}$, and so v is well-defined. Clearly, $v(s) = 0$. In addition, for every $\Gamma \Rightarrow \Delta \in S$, we have that $S \vdash^{\mathcal{F}}_{\mathbf{G}} \Gamma \Rightarrow \Delta$, which means that either $\Gamma \not\subseteq L$ or $\Delta \not\subseteq R$. Either way, $v(\Gamma \Rightarrow \Delta) = 1$.

It is left to show that v is **G**-legal. Let $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a rule of **G**, $\Gamma'_1 \Rightarrow \Delta'_1, \dots, \Gamma'_n \Rightarrow \Delta'_n$ respective subsequents of $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$, and σ a substitution, such that $\sigma(\mathit{frm}(\{\Gamma'_1 \Rightarrow \Delta'_1, \dots, \Gamma'_n \Rightarrow \Delta'_n, \Gamma_0 \Rightarrow \Delta_0\})) \subseteq \mathcal{F}$ and $v(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 1$ for every $1 \leq i \leq n$. We prove that $v(\sigma(\Gamma_0 \Rightarrow \Delta_0)) = 1$. By our assumption, for every $1 \leq i \leq n$ there exists either $\varphi \in \Gamma'_i$ such that $v(\sigma(\varphi)) = 0$ (and then $\sigma(\varphi) \in R$) or $\varphi \in \Delta'_i$ such that $v(\sigma(\varphi)) = 1$ (and then $\sigma(\varphi) \in L$). We construct a sequent $\Gamma \Rightarrow \Delta$ as follows. For every $1 \leq i \leq n$, we include in Γ a formula $\sigma(\varphi)$ for some $\varphi \in \Delta'_i$ such that $v(\sigma(\varphi)) = 1$, or, if such φ does not exist, we include in Δ a formula $\sigma(\varphi)$ for some $\varphi \in \Gamma'_i$ such that $v(\sigma(\varphi)) = 0$. Then, we have $(\Gamma \Rightarrow \Delta) \subseteq (L \Rightarrow R)$. In addition, using (ID), we have $S \vdash^{\mathcal{F}}_{\mathbf{G}} \sigma(\Gamma'_i), \Gamma \Rightarrow \sigma(\Delta'_i), \Delta$ for every $1 \leq i \leq n$. By applying $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ with $\Gamma \Rightarrow \Delta$ as a context sequent, we obtain that $S \vdash^{\mathcal{F}}_{\mathbf{G}} \sigma(\Gamma_0), \Gamma \Rightarrow \sigma(\Delta_0), \Delta$, and therefore, $S \vdash^{\mathcal{F}}_{\mathbf{G}} \sigma(\Gamma_0), L \Rightarrow \sigma(\Delta_0), R$. Since $S \not\vdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow R$, we have $\sigma(\Gamma_0 \Rightarrow \Delta_0) \not\subseteq L \Rightarrow R$, and so either $v(\psi) = 0$ for some $\psi \in \sigma(\Gamma_0)$ or $v(\psi) = 1$ for some $\psi \in \sigma(\Delta_0)$. Either way, we have $v(\sigma(\Gamma_0 \Rightarrow \Delta_0)) = 1$. □

Various soundness and completeness theorems from the literature are obtained as particular instances of Theorem 2.3.5, by taking $\mathcal{F}$ to be the entire propositional language. For instance, Example 2.3.3 shows that classical propositional assignments coincide with **LK**-legal $\mathcal{CL}$-bivaluations. Theorem 2.3.5 thus provides proof of soundness and completeness of **LK** with respect to the usual semantics of propositional classical logic. Using a similar argument, soundness and completeness of $\mathbf{G_{PIL}}$ with respect to the non-deterministic semantics from [44] is also obtained as a particular instance of Theorem 2.3.5.

Alternative semantics for well-known logics are also obtained. For example, $\mathbf{G}_{Ł_3}$-legal bivaluations provide an alternative semantics to Łukasiewicz three-valued logic (Example 2.2.8). This semantics is two-valued, but not truth-functional. In Figure 2.1 we list the semantic constraints for $\mathbf{G}_{Ł_3}$-legal $\mathcal{CL}$-bivaluations. On the left of each such constraint we include the derivation rule that induces it. Another two-valued semantics for this logic was presented in [95], and was then used to construct a different calculus for it in [35].

$(\neg \Rightarrow)$	if $v(\varphi) = 1$, then $v(\neg\varphi) = 0$
$(\neg\neg \Rightarrow)$	if $v(\varphi) = 0$, then $v(\neg\neg\varphi) = 0$
$(\Rightarrow \neg\neg)$	if $v(\varphi) = 1$, then $v(\neg\neg\varphi) = 1$
$(\wedge \Rightarrow)$	if $v(\varphi) = 0$ or $v(\psi) = 0$, then $v(\varphi \wedge \psi) = 0$
$(\Rightarrow \wedge)$	if $v(\varphi) = 1$ and $v(\psi) = 1$, then $v(\varphi \wedge \psi) = 1$
$(\neg\wedge \Rightarrow)$	if $v(\neg\varphi) = 0$ and $v(\neg\psi) = 0$, then $v(\neg(\varphi \wedge \psi)) = 0$
$(\Rightarrow \neg\wedge)$	if $v(\neg\varphi) = 1$ or $v(\neg\psi) = 1$, then $v(\neg(\varphi \wedge \psi)) = 1$
$(\vee \Rightarrow)$	if $v(\varphi) = 0$ and $v(\psi) = 0$, then $v(\varphi \vee \psi) = 0$
$(\Rightarrow \vee)$	if $v(\varphi) = 1$ or $v(\psi) = 1$, then $v(\varphi \vee \psi) = 1$
$(\neg\vee \Rightarrow)$	if $v(\neg\varphi) = 0$ or $v(\neg\psi) = 0$, then $v(\neg(\varphi \vee \psi)) = 0$
$(\Rightarrow \neg\vee)$	if $v(\neg\varphi) = 1$ and $v(\neg\psi) = 1$, then $v(\neg(\varphi \vee \psi)) = 1$
$(\supset \Rightarrow)_{Ł_3}$	if $v(\neg\varphi) = 0$, $v(\psi) = 0$, and $(v(\varphi) = 1$ or $v(\neg\psi) = 1)$, then $v(\varphi \supset \psi) = 0$
$(\Rightarrow \supset)_{Ł_3}$	if $(v(\varphi) = 0$ or $v(\psi) = 1)$ and $(v(\neg\psi) = 0$ or $v(\neg\varphi) = 1)$, then $v(\varphi \supset \psi) = 1$
$(\neg \supset \Rightarrow)$	if $v(\varphi) = 0$ or $v(\neg\psi) = 0$, then $v(\neg(\varphi \supset \psi)) = 0$
$(\Rightarrow \neg \supset)$	if $v(\varphi) = 1$ and $v(\neg\psi) = 1$, then $v(\neg(\varphi \supset \psi)) = 1$

Figure 2.1: Semantic constraints induced by $\mathbf{G}_{Ł_3}$

2.4 Streamlining, Equivalence, and Gentzen's Axioms

After presenting his calculus for classical logic, Gentzen mentions that there are some simplifications possible for the sequent rules (see Section III, 2.2 of [54]). Following Gentzen's simplified variants, we present a calculus, denoted $Ax(\mathbf{LK})$, that consists of the following axioms (rules with no premises):[3]

$$\emptyset \,/\, p_1, p_2 \Rightarrow p_1 \wedge p_2 \qquad \emptyset \,/\, p_1 \wedge p_2 \Rightarrow p_1 \qquad \emptyset \,/\, p_1 \wedge p_2 \Rightarrow p_2$$
$$\emptyset \,/\, p_1 \vee p_2 \Rightarrow p_1, p_2 \qquad \emptyset \,/\, p_1 \Rightarrow p_1 \vee p_2 \qquad \emptyset \,/\, p_2 \Rightarrow p_1 \vee p_2$$
$$\emptyset \,/\, p_2 \Rightarrow p_1 \supset p_2 \qquad \emptyset \,/\, \Rightarrow p_1, p_1 \supset p_2 \qquad \emptyset \,/\, p_1, p_1 \supset p_2 \Rightarrow p_2$$
$$\emptyset \,/\, \Rightarrow p_1, \neg p_1 \qquad \emptyset \,/\, p_1, \neg p_1 \Rightarrow$$

In this section we show that such equivalent simplifications are not peculiar for **LK**, and are possible for every pure calculus. The results of the current section have several

[3]For some reason, Gentzen did not include the simplification of $(\Rightarrow \supset)$ to $\Rightarrow p_1, p_1 \supset p_2$ and $p_2 \Rightarrow p_1 \supset p_2$.

applications. The different manipulations that transform one calculus into another calculus which is equivalent will be proven useful for the construction of analytic calculi in Section 2.5.3. Also, the "axiomized" versions of sequent calculi make it simpler to reduce derivability in them to SAT, as we do in Chapter 3.

We present several useful streamlining principles that transform one calculus into another, without affecting the induced derivability relations.

Definition 2.4.1. Two calculi $\mathbf{G}_1$ and $\mathbf{G}_2$ are called *equivalent* if $\vdash^{\mathcal{F}}_{\mathbf{G}_1} = \vdash^{\mathcal{F}}_{\mathbf{G}_2}$ for every $\mathcal{F} \subseteq \mathcal{L}$. Equivalence is naturally defined also between single rules (and between a rule and a calculus) by identifying a rule r with the calculus $\{r\}$.

Lemma 2.4.2. The following holds:

1. $S \,/\, \Gamma \Rightarrow \psi, \Delta$ is equivalent to $S \,;\, \psi \Rightarrow \;/\, \Gamma \Rightarrow \Delta$.
2. $S \,/\, \Gamma, \psi \Rightarrow \Delta$ is equivalent to $S \,;\, \Rightarrow \psi \,/\, \Gamma \Rightarrow \Delta$.
3. $\{S \,;\, s_1 \,/\, s \;,\; S \,;\, s_2 \,/\, s\}$ is equivalent to $S \,;\, s_1 \cup s_2 \,/\, s$.

Proof. We show only the left-to-right direction of the third case. All other cases are handled similarly. Using Theorem 2.3.5, it suffices to show that every bivaluation that respects the rule $S \,;\, s_1 \cup s_2 \,/\, s$ also respects the rules $S \,;\, s_1 \,/\, s$ and $S \,;\, s_2 \,/\, s$. Let v be a bivaluation that respects the rule $S \,;\, s_1 \cup s_2 \,/\, s$. We prove that it respects the rule $S \,;\, s_1 \,/\, s$ ($S \,;\, s_2 \,/\, s$). Let $S = \{q_1, \dots, q_n\}$, $q'_1, \dots, q'_n$ respective subsequents of $q_1, \dots, q_n$, s' a subsequent of s_1 (s_2), and σ a substitution such that $\sigma(\mathit{frm}(\{q'_1, \dots, q'_n, s', s\})) \subseteq \mathit{dom}(v)$. Suppose $v(\sigma(q'_i)) = 1$ for every $1 \leq i \leq n$, and also that $v(\sigma(s')) = 1$. Clearly, $s' \subseteq s_1 \cup s_2$. Since v respects $S \,;\, s_1 \cup s_2 \,/\, s$, we have that $v(\sigma(s)) = 1$. □

Going back to Gentzen's axioms, we point out a useful application of Lemma 2.4.2. Call a rule *axiomatic* if it has an empty set of premises. In turn, call a calculus *axiomatic* if it consists solely of axiomatic rules. Lemma 2.4.2 allows us to convert every calculus to an axiomatic one, in a similar manner to the calculus $Ax(\mathbf{LK})$.

Theorem 2.4.3. Every calculus is equivalent to an axiomatic calculus.

Proof. Consider the following transformations of pure rules:

1. $S \,;\, \psi \Rightarrow \;/\, \Gamma \Rightarrow \Delta \longmapsto S \,/\, \Gamma \Rightarrow \psi, \Delta$
2. $S \,;\, \Rightarrow \psi \,/\, \Gamma \Rightarrow \Delta \longmapsto S \,/\, \Gamma, \psi \Rightarrow \Delta$
3. $S \,;\, \Gamma \Rightarrow \psi, \Delta \,/\, s \longmapsto \{S \,;\, \Gamma \Rightarrow \Delta \,/\, s \;,\; S \,;\; \Rightarrow \psi \,/\, s\}$
4. $S \,;\, \Gamma, \psi \Rightarrow \Delta \,/\, s \longmapsto \{S \,;\, \Gamma \Rightarrow \Delta \,/\, s \;,\; S \,;\, \psi \Rightarrow \;/\, s\}$

The first two are taken from items 1 and 2 of Lemma 2.4.2 (read from right to left). The last two are particular instance of item 3 of Lemma 2.4.2 (also read from right to left).

Given a calculus **G**, we apply these four transformations on the rules of **G** as long as it is possible. By Lemma 2.4.2, each step in this process results in a calculus which is equivalent to **G**. Observing that at least one transformation is applicable to any non-axiomatic rule, it remains to establish termination. For each rule $S\,/\,s$, let $\|S\,/\,s\| = \sum_{\Gamma\Rightarrow\Delta\in S}(|\Gamma| + |\Delta|)$. For every set R of rules, we associate the multiset M_R, given by $M_R = \lambda n \in \mathbb{N}.\ |\{r \in R \mid \|r\| = n\}|$. We prove that if R_2 is obtained from R_1 by one of the transformations, then $M_{R_2} \prec M_{R_1}$, where $\prec$ is the Dershowitz-Manna well-founded ordering over multisets of natural numbers [46]. Clearly, $R_2 = R_1 \setminus \{r\} \cup R$ for some set R that is obtained from r by one of the transformations. If the transformation is 1 or 2 then, w.l.o.g., r has the form $S \uplus \{\psi \Rightarrow\ \}\,/\,\Gamma \Rightarrow \Delta$ and R has the form $\{S\,/\,\Gamma \Rightarrow \psi, \Delta\}$ with $\Gamma \cup \Delta \neq \emptyset$. This means that M_{R_2} is obtained from M_{R_1} by replacing one copy of $\|S\,;\psi \Rightarrow\ /\,\Gamma \Rightarrow \Delta\|$ with a new copy of $\|S\,;\psi \Rightarrow\ /\,\Gamma \Rightarrow \Delta\| - 1$, and thus $M_{R_2} \prec M_{R_1}$. If the transformation is 3 or 4 then, w.l.o.g., r has the form $S \uplus \{\Gamma \Rightarrow \psi, \Delta\}\,/\,s$ where $\psi \notin \Delta$, and R has the form $\{S\,;\Gamma \Rightarrow \Delta\,/\,s, S\,;\ \Rightarrow \psi\,/\,s\}$. This means that M_{R_2} is obtained from M_{R_1} by replacing a copy of $\|S\,;\Gamma \Rightarrow \psi, \Delta\,/\,s\|$ with a copy of $\|S\,;\Gamma \Rightarrow \Delta\,/\,s\|$ and a copy of $\|S\,;\ \Rightarrow \psi\,/\,s\|$. Both are smaller than $\|S\,;\Gamma \Rightarrow \psi, \Delta\,/\,s\|$, and therefore $M_{R_2} \prec M_{R_1}$. □

This method for obtaining axiomatic calculi is applicable to every pure calculus, not only **LK**.

Example 2.4.4. The rule $\neg p_1 \Rightarrow\ ; p_2 \Rightarrow\ ;\ \Rightarrow p_1, \neg p_2\,/\,p_1 \supset p_2 \Rightarrow$ of $\mathbf{G}_{\text{Ł}_3}$ (Example 2.2.8) transforms into the equivalent axiomatic rules $\emptyset\,/\,p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$ and $\emptyset\,/\,\neg p_2, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$.

2.5 Analyticity

Roughly speaking, analyticity of a calculus provides a computable bound on the formulas that may appear in derivations of a sequent s from a set S of sequents. For propositional calculi, such a bound usually entails decidability and consistency (unprovability of the empty sequent). The special case of the subformula property is obtained when the set of subformulas of (formulas of) $S \cup \{s\}$ serves as this bound. Many useful calculi, however, do not admit this strict property, while still allowing some other effective bound. For example, in $\mathbf{G}_{\mathbf{C_1}}$ and $\mathbf{G}_{\text{Ł}_3}$ (Examples 2.2.9 and 2.2.8), there are sequents whose derivations require not only subformulas, but also negations of subformulas of the derived sequent.

In this section we provide a generalized definition of analyticity, that is parametrized by a distinguished set of unary connectives and a natural number. This generalized notion holds for a larger family of calculi, and still suffices to ensure decidability and consistency.

After defining this notion of analyticity and providing a semantic counterpart for it in Section 2.5.1, we introduce methods to identify and construct analytic calculi in Sections 2.5.2 and 2.5.3. Section 2.5.4 proves that these methods are indeed correct, that is, that the criteria that underlie them suffice for analyticity.

2.5.1 A Generalized Subformula Property

We start by generalizing the subformula relation between formulas, and prove that this generalized relation inherits the crucial properties of the usual one. This relation naturally induces a parametrized notion of analyticity, defined in Definition 2.5.5. After studying the properties of this new notion of analyticity, we provide it an equivalent semantic condition in Theorem 2.5.9, which, in addition to providing another viewpoint of this property, turns out to be very useful: first, it is the main tool for identifying and constructing analytic calculi in Sections 2.5.2 and 2.5.3; second, it has an important role in the decision procedure that we provide in Chapter 3.

In what follows, $\circledcirc$ denotes an arbitrary subset of unary connectives in $\diamondsuit^1_{\mathcal{L}}$ and k denotes an arbitrary *positive* integer. We denote the set of strings over $\circledcirc$ of length at most k by $\circledcirc^{\leq k}$ (e.g., $\{\neg, \circ\}^{\leq 2} = \{\epsilon, \neg, \circ, \neg\neg, \circ\circ, \neg\circ, \circ\neg\}$, where ϵ denotes the empty string). For convenience, we use the following notations: $\circ\mathcal{F} = \{\circ\varphi \mid \varphi \in \mathcal{F}\}$ for any unary connective $\circ$, $\circledcirc\varphi = \{\circ\varphi \mid \circ \in \circledcirc\}$ for any set $\circledcirc$ of unary connectives, $\circledcirc\mathcal{F} = \bigcup_{\circ\in\circledcirc} \circ\mathcal{F}$, $\circledcirc^{\leq k}\varphi = \left\{\bar{\circ}\varphi \mid \bar{\circ} \in \circledcirc^{\leq k}\right\}$, and $\circledcirc^{\leq k}\mathcal{F} = \bigcup_{\varphi\in\mathcal{F}} \circledcirc^{\leq k}\varphi$.

Definition 2.5.1. A formula φ is an *immediate* $\circledcirc$*-k-subformula* of a formula ψ if one of the followings hold:

- $\psi = \diamond(\psi_1, \dots, \psi_n)$ and $\varphi \in \circledcirc^{\leq k}\psi_i$ for some n-ary connective $\diamond \in \diamondsuit_{\mathcal{L}} \setminus \circledcirc$, formulas $\psi_1, \dots, \psi_n$, and $1 \leq i \leq n$.
- $\psi \in \circledcirc\varphi$.

The $\circledcirc$-k*-subformula* relation is the reflexive transitive closure of the immediate $\circledcirc$-k-subformula relation. We denote the set of $\circledcirc$-k-subformulas of a formula ψ by $sub^{\circledcirc}_k(\psi)$. This notation is naturally extended to sequents, sets of sequents, etc. When φ is a ($\circledcirc$-k-)subformula of ψ and $\varphi \neq \psi$, we call φ a *proper* ($\circledcirc$-k-)subformula of ψ.

When $\circledcirc = \emptyset$ (and so $\circledcirc^{\leq k} = \{\epsilon\}$), the $\circledcirc$-k-subformula relation amounts to the usual subformula relation. In such a case we call φ a *subformula* of ψ and denote $sub^{\circledcirc}_k$ by sub.

Example 2.5.2. $sub_1^{\{\neg\}}(\neg(p_1 \supset p_2)) = \{p_1, p_2, \neg p_1, \neg p_2, p_1 \supset p_2, \neg(p_1 \supset p_2)\}$ and $sub_2^{\{\neg\}}(\circ p_1) = \{p_1, \neg p_1, \neg\neg p_1, \circ p_1\}$.

We now explore the properties of this generalized relation. The first step is to define an adequate decreasing complexity measure cc on formulas. For every $\psi \in \mathcal{L}$, denote by $\bar{\circ}_\psi$ the longest (possibly empty) prefix of ψ that consists of $\circledcirc$-elements, and by b_ψ the formula for which $\psi = \bar{\circ}_\psi b_\psi$. Let $c : \mathcal{L} \to \mathbb{N}$ be a usual complexity measure for formulas (so that $c(\varphi) < c(\psi)$ whenever φ is a proper subformula of ψ). The function $cc : \mathcal{L} \to (\mathbb{N} \times \mathbb{N})$ is then given by $cc(\psi) = \langle c(b_\psi), |\bar{\circ}_\psi| \rangle$, where $|\bar{\circ}_\psi|$ denotes the length of $\bar{\circ}_\psi$.

Proposition 2.5.3. $cc(\varphi) < cc(\psi)$ whenever φ is a proper $\circledcirc$-k-subformula of ψ (where $<$ is the standard lexicographic order over $\mathbb{N} \times \mathbb{N}$).

Proof. We consider only the case that φ is an immediate $\circledcirc$-k-subformula of ψ. The claim then follows by standard induction. First, if $\psi = \diamond(\psi_1, \dots, \psi_n)$ and $\varphi \in \circledcirc^{\leq k}\psi_i$ for some $1 \leq i \leq n$ and $\diamond \notin \circledcirc$, then $c(b_\varphi) = c(b_{\psi_i}) \leq c(\psi_i) < c(\psi) = c(b_\psi)$, and so $cc(\varphi) < cc(\psi)$. Second, if $\psi \in \circledcirc\varphi$, then $\psi = \circ\varphi$ for some $\circ \in \circledcirc$, $\bar{\circ}_\psi = \circ\bar{\circ}_\varphi$, and $b_\psi = b_\varphi$. Hence, $c(b_\psi) = c(b_\varphi)$, but $|\bar{\circ}_\psi| = |\bar{\circ}_\varphi| + 1$, and so $cc(\varphi) < cc(\psi)$. $\square$

Using this complexity measure, it easily follows that the $\circledcirc$-k-subformula relation is anti-symmetric. Also, since every formula has finitely many immediate $\circledcirc$-k-subformulas, it also follows that $sub_k^{\circledcirc}(\psi)$ is finite for every $\psi \in \mathcal{L}$.

In addition, we have the following useful property of the generalized relation:

Lemma 2.5.4. $\sigma(sub_k^{\circledcirc}(\psi)) \subseteq sub_k^{\circledcirc}(\sigma(\psi))$ for every formula ψ and substitution σ.

We now define our generalized notion of analyticity. It is obtained from the usual subformula property by replacing the role of subformulas with $\circledcirc$-k-subformulas.

Definition 2.5.5. A calculus $\mathbf{G}$ is called $\circledcirc$-k-*analytic* if $S \vdash_{\mathbf{G}} s$ implies $S \vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(S \cup \{s\})} s$ for every set S of sequents and a sequent s.

Note that for every k, $\emptyset$-k-analytic calculi are calculi that enjoy the usual subformula property. We call such calculi $\emptyset$-*analytic*. Also note that whenever two calculi are equivalent (Definition 2.4.1), one is $\circledcirc$-k-analytic iff the other is.

We shall use the terms "analyticity" and "analytic" without any prefix whenever the prefix is clear from the context, or when describing calculi that are $\circledcirc$-k-analytic for some $\circledcirc$ and k.

Just like the usual subformula property, $\circledcirc$-k-analyticity of a pure calculus entails its decidability. Formally:

Definition 2.5.6. The *derivability problem* for an $\mathcal{L}$-calculus $\mathbf{G}$ is given by:

Input: A finite set S of $\mathcal{L}$-sequents and an $\mathcal{L}$-sequent s.

Question: Does $S \vdash_{\mathbf{G}} s$?

Obviously, one cannot expect to have decision procedures for the derivability problem for all pure calculi.[4] However, decidability is guaranteed whenever the calculus is $\circledcirc$-k-analytic.

Proposition 2.5.7. The derivability problem is decidable for every $\circledcirc$-k-analytic pure calculus.

Proof. For every $\circledcirc$-k-analytic calculus $\mathbf{G}$, finite set S of sequents, and sequent s, we have that $S \vdash_{\mathbf{G}} s$ iff $S \vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(S \cup \{s\})} s$. Since $sub_k^{\circledcirc}(S \cup \{s\})$ is finite, the latter can be checked by an exhaustive search of derivations of s from S in $\mathbf{G}$, that include only $sub_k^{\circledcirc}(S \cup \{s\})$-formulas. □

Moreover, $\circledcirc$-k-analyticity guarantees the consistency of the calculus, whenever the calculus is not trivial:

Proposition 2.5.8. The empty sequent is not derivable in any $\circledcirc$-k-analytic calculus, that does not include the rule $\emptyset \,/\, \Rightarrow$.

Proof. A proof of the empty sequent in a $\circledcirc$-k-analytic calculus would entail the existence of a proof that includes no formulas at all. This is only possible in the presence of the rule $\emptyset \,/\, \Rightarrow$, which cannot be included. □

Analyticity of sequent calculi is traditionally proved as a corollary of cut-admissibility. Indeed, if every rule in a pure calculus (except for (cut)) admits the local $\circledcirc$-k-subformula property (i.e., the premises consist only of $\circledcirc$-k-subformulas of the formulas in the conclusion), then cut-admissibility implies $\circledcirc$-k-analyticity. For example, the calculi $\mathbf{LK}$, $\mathbf{G_{PIL}}$, $\mathbf{G_{C_1}}$, $\mathbf{G_{Ł_3}}$ and $\mathbf{G}_4$ (Examples 2.2.7–2.2.10) admit cut-admissibility. Taking into account the structure of their logical rules, this directly entails that $\mathbf{LK}$ and $\mathbf{G_{PIL}}$ are $\emptyset$-analytic, and that $\mathbf{G_{C_1}}$, $\mathbf{G_{Ł_3}}$ and $\mathbf{G}_4$ are $\{\neg\}$-1-analytic.

There are cases, however, in which a sequent calculus does not enjoy cut-admissibility, although it is analytic. Examples include, e.g., sequent calculi for the modal logics $S5$ and B [85, 96, 101], bi-intuitionistic logic [84], and several calculi for paraconsistent logics [16]. Other ways are thus needed for proving $\circledcirc$-k-analyticity, independently from cut-admissibility.

[4] Example 2.7.9 below shows how to translate propositional Hilbert-type calculi to pure calculi. In particular, the undecidable Hilbert-type calculus from [80] translates to an undecidable pure calculus.

Next we provide a semantic characterization of analyticity which is independent of cut-admissibility. To apply this criterion, one has to consider bivaluations that are defined only on $\circledcirc$-k-subformulas of the derived sequent and the assumptions, and show that the existence of a countermodel in the form of such a partial bivaluation entails the existence of a full countermodel. Formally:

Theorem 2.5.9. An $\mathcal{L}$-calculus $\mathbf{G}$ is $\circledcirc$-k-analytic iff every $\mathbf{G}$-legal bivaluation v can be extended to a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation, provided that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas.

Proof. Suppose that $S \vdash_{\mathbf{G}} s$ but $S \not\vdash_{\mathbf{G}}^{\mathcal{F}} s$ for $\mathcal{F} = sub_k^{\circledcirc}(S \cup \{s\})$. By Theorem 2.3.5, there exists a $\mathbf{G}$-legal $\mathcal{F}$-bivaluation v such that $v(S) = 1$ and $v(s) = 0$, but $u(S) \leq u(s)$ for every $\mathbf{G}$-legal $\mathcal{L}$-bivaluation u. Therefore, v cannot be extended to a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation. In addition, $dom(v) = \mathcal{F}$ is finite and closed under $\circledcirc$-k-subformulas.

For the converse, suppose that v is a $\mathbf{G}$-legal bivaluation, $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas, and v cannot be extended to a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation. Let $s = \Gamma \Rightarrow \Delta$, where $\Gamma = \{\psi \in dom(v) \mid v(\psi) = 1\}$ and $\Delta = \{\psi \in dom(v) \mid v(\psi) = 0\}$. Then, $dom(v) = frm(s) = sub_k^{\circledcirc}(s)$ and $v(s) = 0$. We show that $u(s) = 1$ for every $\mathbf{G}$-legal $\mathcal{L}$-bivaluation u. Indeed, every such u does not extend v, and so $u(\psi) \neq v(\psi)$ for some $\psi \in dom(v)$. Then, $u(\psi) = 0$ if $\psi \in \Gamma$, and $u(\psi) = 1$ if $\psi \in \Delta$. In either case, $u(s) = 1$. By Theorem 2.3.5, $\not\vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(s)} s$ and $\vdash_{\mathbf{G}} s$. □

Often, a slightly weaker notion of analyticity is employed, by considering only cases where $S = \emptyset$. We say that a calculus $\mathbf{G}$ is *weakly* $\circledcirc$*-*k*-analytic* if $\vdash_{\mathbf{G}} s$ implies $\vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(s)} s$ for every sequent s. The proof of Theorem 2.5.9 shows that this seemingly weaker notion is actually equivalent to the stronger one: given that a calculus $\mathbf{G}$ is weakly $\circledcirc$-k-analytic, the second part of the proof shows that every $\mathbf{G}$-legal bivaluation whose domain is finite and closed under $\circledcirc$-k-subformulas can be extended to a full $\mathbf{G}$-legal bivaluation. The first part of the proof then establishes that $\mathbf{G}$ is $\circledcirc$-k-analytic, and not only weakly $\circledcirc$-k-analytic.

Example 2.5.10. A particular instance of Theorem 2.5.9 is the ability to define full classical assignments based only on the values that are assigned to atomic formulas. This fact is taken for granted, however in this general semantic framework it does not always hold. Recall the axiomatic version of **LK**, described in Section 2.4, and denoted by $Ax(\mathbf{LK})$. In [54], Gentzen only presents this calculus as a simplification, but does not actually use it, and even restricts the attention to it for the case where "we attached no importance to the Hauptsatz" (see Section III, 2.2. of [54]). And indeed, as Gentzen mentions, while equivalent to **LK**, $Ax(\mathbf{LK})$ does not admit cut-admissibility. Nevertheless, $Ax(\mathbf{LK})$ is

$\emptyset$-analytic, as can be proved using Theorem 2.5.9: similarly to Example 2.3.3, it can be shown that a bivaluation is $Ax(\mathbf{LK})$-legal if and only if it assigns values to compound formulas according to the classical truth tables. Since partial classical assignments can be extended to full ones, Theorem 2.5.9 ensures that $Ax(\mathbf{LK})$ admits $\emptyset$-analyticity.

Theorem 2.5.9 is also useful for proving that a certain calculus is not analytic:

Example 2.5.11. Consider a pure calculus, denoted here by $\mathbf{G}$, that consists of the following rules:

$$p_1 \Rightarrow \ / \circ p_1 \Rightarrow \qquad\qquad p_1 \Rightarrow \ / \Rightarrow \circ p_1$$

The $\mathbf{G}$-legal bivaluation v whose domain is $\{p_1\}$, such that $v(p_1) = 0$ cannot be extended to a full $\mathbf{G}$-legal bivaluation: according to the first rule, $v(\circ p_1) = 0$ must hold, and according to the second, $v(\circ p_1) = 1$ must hold. Indeed, $\mathbf{G}$ is not analytic, as the sequent $\Rightarrow p_1$ is derivable in it, but only using a cut on $\circ p_1$.

While the semantic characterization of $\circledcirc$-k-analyticity from Theorem 2.5.9 provides meaningful insights on this property, it is not effective for determining $\circledcirc$-k-analyticity, as in order to use it, one needs to go over all bivaluations whose domains are closed under $\circledcirc$-k-subformulas, and check whether they can be fully extended. Therefore, a decidable *syntactic* criterion for $\circledcirc$-k-analyticity is desired. In the next section we generalize the result of [21] in order to provide a sufficient syntactic criterion for $\circledcirc$-k-analyticity for a wider family of calculi. Calculi that admit this criterion are then used in Section 2.5.3 for providing a method to construct $\circledcirc$-k-analytic calculi.

2.5.2 Sufficient Criterion for Analyticity

In this section we generalize the coherence condition from [21], that was given for canonical calculi, and show that the generalized condition ensures analyticity. Unlike the ability to extend partial bivaluations, which is semantic in nature, the coherence property (as well at its generalization that we introduce here) is syntactic and decidable. Roughly speaking, canonical calculi are pure calculi in which each rule introduces exactly one connective in the conclusion, and all premises include only atomic formulas. Here we relax these requirements, and allow several connectives to be introduced at once, and not only from atomic formulas, as long as all premises include only $\circledcirc$-k-subformulas of the conclusion. This is defined as follows:

Definition 2.5.12. A rule r is called *$\circledcirc$-k-ordered* if every formula in its premises is a proper $\circledcirc$-k-subformula of some formula in its conclusion. Further, r is called *$\circledcirc$-k-directed* if it is $\circledcirc$-k-ordered, and its conclusion has the form $\Rightarrow \varphi$ or $\varphi \Rightarrow$ for some formula φ. A calculus is called *$\circledcirc$-k-ordered* (*$\circledcirc$-k-directed*) if it consists of *$\circledcirc$-k-ordered* ($\circledcirc$-k-directed) rules. For every k, we call $\emptyset$-k-ordered ($\emptyset$-k-directed) rules and calculi *$\emptyset$-ordered* (*$\emptyset$-directed*).

Example 2.5.13. The calculi **LK** and $\mathbf{G_{PIL}}$ are $\emptyset$-directed, while $\mathbf{G_{C_1}}$, $\mathbf{G_{Ł_3}}$ and $\mathbf{G}_4$ are $\{\neg\}$-1-directed.

In [21], the *coherence* property was defined for canonical calculi, and was used for proving analyticity in them. Roughly speaking, a canonical calculus is *coherent* if whenever two rules share the same formula in their conclusion, but on different sides, the empty sequent is derivable from their premises using only (CUT). We generalize this requirement for the case of $\circledcirc$-k-directed calculi:

Definition 2.5.14. A $\circledcirc$-k-directed calculus **G** is called *coherent* if for every two rules of **G** of the forms $S_1 \,/ \Rightarrow \varphi_1$ and $S_2 \,/\, \varphi_2 \Rightarrow$, and two substitutions σ_1, σ_2, if $\sigma_1(\varphi_1) = \sigma_2(\varphi_2)$, then the empty sequent is derivable from $\sigma_1(S_1) \cup \sigma_2(S_2)$ using only (CUT).

Note that for canonical calculi, this definition coincides with that of [21]. Also, it is decidable whether a given calculus is coherent or not: for each pair of rules $S_1 \,/ \Rightarrow \varphi_1$ and $S_2 \,/\, \varphi_2 \Rightarrow$, one can first rename the atomic variables so that no atomic variable occurs in both rules, and then it suffices to check the above condition for the most general unifier of φ_1 and φ_2. Finally, it is useful to notice that if a calculus **G** is coherent, $\circledcirc$-k-directed, or $\circledcirc$-k-ordered, then the same holds for any sub-calculus of **G** (i.e. any calculus that is obtained from **G** by deleting some of its rules).

Example 2.5.15. **LK**, $\mathbf{G_{PIL}}$, $\mathbf{G_{Ł_3}}$ and $\mathbf{G}_4$ are coherent, while $\mathbf{G_{C_1}}$ is not. Indeed, for the rules $p_1 \Rightarrow \,/ \Rightarrow \neg p_1$ and $p_1 \Rightarrow \,/\, \neg\neg p_1 \Rightarrow$ of $\mathbf{G_{C_1}}$, if $\sigma_1(p_1) = \neg p_1$ and $\sigma_2(p_1) = p_1$, we have $\sigma_1(\neg p_1) = \sigma_2(\neg\neg p_1)$, but the empty sequent cannot be derived from $\neg p_1 \Rightarrow \,; p_1 \Rightarrow$ using only (CUT).

Our notion of coherence, which generalizes that of [21], together with $\circledcirc$-k-directedness, suffices for $\circledcirc$-k-analyticity:

Theorem 2.5.16. Every coherent $\circledcirc$-k-directed calculus is $\circledcirc$-k-analytic.

This theorem is obtained as a corollary of Theorem 2.5.21 (see Section 2.5.3 below), which will be proved in Section 2.5.4. We now present some examples and applications.

Example 2.5.17. **LK** and $\mathbf{G_{PIL}}$ are coherent and $\emptyset$-directed, and hence they are $\emptyset$-analytic. $\mathbf{G_{Ł_3}}$ is coherent and $\{\neg\}$-1-directed, and hence it is $\{\neg\}$-1-analytic. Similarly, every canonical system (as defined in [21]) is $\emptyset$-directed, and hence every coherent canonical system is $\emptyset$-analytic.

Example 2.5.18 (Hierarchy of double negations)**.** The paper [63] studies an infinite family, denoted $\{L2^{n+2} \mid n \in \mathbb{N}\}$, of pure sequent calculi for non-classical logics that admit the double negation principle as well as its weaker forms (e.g., $\neg\neg\neg\psi \leftrightarrow \neg\psi$). For example, the calculus $L4$ is the same as the calculus $\mathbf{G}_4$ presented in Example 2.2.8. This calculus is coherent and $\{\neg\}$-1-directed, and hence, by Theorem 2.5.16, it is $\{\neg\}$-1-analytic. Moreover, it can be easily observed that for every n, $L2^{n+2}$ is coherent and $\{\neg\}$-$(n+1)$-directed, and thus by Theorem 2.5.16, it is $\{\neg\}$-$(n+1)$-analytic.

Example 2.5.19 (Dolev-Yao intruder deductions)**.** In [42], a formal deductive system for the Dolev-Yao intruder model was presented. Its language consists of two binary connectives: pairing, denoted $\langle \cdot, \cdot \rangle$, and encryption, denoted $[\cdot]_\cdot$ (where the argument in the subscript is the encryption key). Formulated as an Hilbert-type calculus, which we call $\mathcal{H}$, this system includes the rules of the first column in the following table:

	$\mathcal{H}$	$\mathbf{G}(\mathcal{H})$	$\mathbf{G}_{DY}$
Pairing	$p_1 \,;\, p_2 \,/\, \langle p_1, p_2 \rangle$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow \langle p_1, p_2 \rangle$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow \langle p_1, p_2 \rangle$
Unpairing	$\langle p_1, p_2 \rangle \,/\, p_1$	$\Rightarrow \langle p_1, p_2 \rangle \,/\, \Rightarrow p_1$	$p_1 \Rightarrow \,/\, \langle p_1, p_2 \rangle \Rightarrow$
	$\langle p_1, p_2 \rangle \,/\, p_2$	$\Rightarrow \langle p_1, p_2 \rangle \,/\, \Rightarrow p_2$	$p_2 \Rightarrow \,/\, \langle p_1, p_2 \rangle \Rightarrow$
Encryption	$p_1 \,;\, p_2 \,/\, [p_1]_{p_2}$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow [p_1]_{p_2}$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow [p_1]_{p_2}$
Decryption	$[p_1]_{p_2} \,;\, p_2 \,/\, p_1$	$\Rightarrow [p_1]_{p_2} \,;\, \Rightarrow p_2 \,/\, \Rightarrow p_1$	$p_1 \Rightarrow \,;\, \Rightarrow p_2 \,/\, [p_1]_{p_2} \Rightarrow$

The middle column of the table provides a pure sequent calculus, denoted $\mathbf{G}(\mathcal{H})$, that is obtained from $\mathcal{H}$ by applying the method sketched in Example 2.7.9 below. The right column includes a third sequent calculus, that we call $\mathbf{G}_{DY}$, obtained from $\mathbf{G}(\mathcal{H})$ by performing the streamlining rules of Lemma 2.4.2. All three calculi define the same logic. $\mathbf{G}_{DY}$ is coherent and $\emptyset$-directed, and thus by Theorem 2.5.16, it is $\emptyset$-analytic.

2.5.3 Constructing Analytic Calculi

Sequent calculi provide a wide range of possibilities for non-classical logics, and in particular, for sub-classical logics (logics that are contained in classical logic). By choosing a subset of derivation rules that are derivable in **LK**, one easily obtains a (system calculus for a) sub-classical logic. Various important and useful non-classical logics can be formalized in this way, with the most prominent example being intuitionistic logic. In general, the resulting logics come at first with no semantics, and might be unusable for computational purposes, since the new calculi might not be analytic. Indeed, even though **LK** is analytic, there is no guarantee that an arbitrary collection of classically derivable sequent rules constitutes an analytic sequent calculus.

While Theorem 2.5.16 allows us to prove that many calculi are $\circledcirc$-k-analytic (by observing that they are $\circledcirc$-k-directed and coherent), some calculi are left out. For example,

$\mathbf{G_{C_1}}$ is $\{\neg\}$-1-analytic, but it is not coherent (see Example 2.5.15). To capture $\mathbf{G_{C_1}}$ and other useful calculi, we introduce a more general method to prove $\circledcirc$-k-analyticity. More precisely, in this section we present a method for obtaining calculi that are *analytic by construction.*

As a motivating example, consider Sette's atomic paraconsistent logic $\mathbf{P_1}$ from [92], that allows contradictions on atomic formulas, but forbids them on compound ones. In $\mathbf{P_1}$ we have that every φ follows from $\{p, \neg p\}$, but not from $\{\psi, \neg\psi\}$ when ψ is compound. Since the explosion principle is manifested in **LK** through the rule $(\neg\Rightarrow)$, a natural way to design a sequent calculus for $\mathbf{P_1}$ is to allow applications of $(\neg\Rightarrow)$ only on compound formulas. We introduce a calculus which naturally achieves this goal, denoted $\mathbf{G_{P_1}}$. It is obtained from **LK** by replacing $(\neg\Rightarrow)$ with several weaker variants of it, namely, with its following *applications*:

$$\Rightarrow \neg p_1 \,/\, \neg\neg p_1 \Rightarrow \qquad\qquad \Rightarrow p_1 \wedge p_2 \,/\, \neg(p_1 \wedge p_2) \Rightarrow$$

$$\Rightarrow p_1 \vee p_2 \,/\, \neg(p_1 \vee p_2) \Rightarrow \qquad\qquad \Rightarrow p_1 \supset p_2 \,/\, \neg(p_1 \supset p_2) \Rightarrow$$

As we shall see in what follows, this type of construction is subject to the criterion that we propose in this section. Thus, the $\emptyset$-analyticity of our calculus is established in Example 2.5.23 below.

The general method for constructing $\circledcirc$-k-analytic calculi that we present is obtained by joining applications of rules of a certain basic coherent $\circledcirc$-k-directed calculus. The derivable rules that are collected to create new calculi will all have the form of *applications* of existing rules. For this, it is useful to observe a duality between rules and their applications: every pure rule is an application of itself (using the identity substitution and the empty context sequent), and every application of a pure rule constitutes a new, perhaps weaker, pure rule. In particular, we may apply Definition 2.5.12 to applications of rules, and have *$\circledcirc$-k-ordered applications* (i.e., applications in which every formula that occurs in the premises is a proper $\circledcirc$-k-subformula of some formula that occurs in the conclusion). Also observe that an application $\langle \sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n \,/\, \sigma(s) \cup c_1 \cup \ldots \cup c_n \rangle$ of a $\circledcirc$-k-directed rule $s_1, \ldots, s_n \,/\, s$ is itself $\circledcirc$-k-ordered iff every formula of the context sequent $c_1 \cup \ldots \cup c_n$ is a proper $\circledcirc$-k-subformula of the formula that occurs in s.

Example 2.5.20. The following are $\emptyset$-ordered, $\{\neg\}$-1-ordered and $\{\neg\}$-2-ordered applications of the rule $(\supset\Rightarrow)$ of **LK** (respectively):

$$\frac{p_1 \Rightarrow p_1 \wedge p_2 \qquad p_1, p_2 \Rightarrow}{p_1, (p_1 \wedge p_2) \supset p_2 \Rightarrow} \qquad\qquad \frac{\neg p_1 \Rightarrow p_1 \wedge p_2 \qquad \neg p_1, p_2 \Rightarrow}{\neg p_1, (p_1 \wedge p_2) \supset p_2 \Rightarrow}$$

$$\frac{\neg\neg p_3 \Rightarrow p_1 \wedge p_2, \neg(p_1 \wedge p_2) \qquad \neg\neg p_3, p_2 \supset p_3 \Rightarrow \neg(p_1 \wedge p_2)}{\neg\neg p_3, (p_1 \wedge p_2) \supset (p_2 \supset p_3) \Rightarrow \neg(p_1 \wedge p_2)}$$

Our main result for this section is the following theorem, that provides a method for obtaining calculi that are $\circledcirc$-k-analytic *by construction*.

Theorem 2.5.21. Let $\mathbf{G_B}$ be a $\circledcirc$-k-directed coherent calculus. Then, every calculus that consists of $\circledcirc$-k-ordered applications of rules of $\mathbf{G_B}$ is $\circledcirc$-k-analytic.

This theorem is proved in Section 2.5.4. We devote the reminder of the current section for presenting applications and consequences of Theorem 2.5.21.

Simple Consequences

First, observe that Theorem 2.5.16 is obtained as a corollary of Theorem 2.5.21:

Proof of Theorem 2.5.16. Every rule of $\mathbf{G_B}$ is a trivial $\circledcirc$-k-ordered application of itself, and, by Theorem 2.5.21, $\mathbf{G_B}$ itself is $\circledcirc$-k-analytic. □

Also, for calculi that are constructed this way, $\circledcirc$-k-analyticity is preserved when omitting a rule:

Corollary 2.5.22. Suppose that $\mathbf{G}$ consists of $\circledcirc$-k-ordered applications of rules of some $\circledcirc$-k-directed coherent calculus. Then every calculus $\mathbf{G}' \subseteq \mathbf{G}$ is $\circledcirc$-k-analytic.

Analyticity for our calculus $\mathbf{G_{P_1}}$ is now obtained using Theorem 2.5.21.

Example 2.5.23 (Atomic paraconsistent logic)**.** The calculus $\mathbf{G_{P_1}}$ described above for Sette's atomic paraconsistent logic can be constructed using the method of Theorem 2.5.21. Begin with $\mathbf{LK} \setminus \{(\neg \Rightarrow)\}$, and add the above $\emptyset$-ordered applications of $(\neg \Rightarrow)$ to allow left-introduction of negation only for compound formulas. By Theorem 2.5.21, this calculus is $\emptyset$-analytic. Alternatively, since $\mathbf{G_{P_1}}$ is directed and coherent, its analyticity also follows from Theorem 2.5.16. Note that $\mathbf{G_{P_1}}$ is equivalent to the calculus given in [4] for this Sette's logic.

More Examples

In some cases, when adding a new rule r to an existing calculus $\mathbf{G}$, some premises of r are already derivable in $\mathbf{G}$. For example, consider augmenting $\mathbf{G_{PIL}}$ with the rule $\bot \Rightarrow p_1 \,/\, \Rightarrow \bot \supset p_1$, which is an application of $(\Rightarrow \supset)$. Since $\bot \Rightarrow p_1$ is derivable in $\mathbf{G_{PIL}}$, it is a redundant premise: one can equivalently add the rule $\emptyset \,/\, \Rightarrow \bot \supset p_1$. The next proposition is used for omitting such redundant premises in the following examples.

Proposition 2.5.24. Let $\mathbf{G_B}$ be a $\circledcirc$-k-directed coherent calculus, $\mathbf{G}$ a calculus consisting solely of $\circledcirc$-k-ordered applications of rules of $\mathbf{G_B}$, and s a conclusion of a $\circledcirc$-k-ordered application of some rule of $\mathbf{G_B}$, all premises of which are derivable in $\mathbf{G}$. Then, $\mathbf{G} \cup \{\emptyset \,/\, s\}$ is $\circledcirc$-k-analytic.

Proof. Let S be a set of sequents such that $S\,/\,s$ is a $\circledcirc$-k-ordered application of some rule of $\mathbf{G_B}$, and for every $s' \in S$, $\vdash_{\mathbf{G}} s'$. Denote $\mathbf{G} \cup \{S\,/\,s\}$ by $\mathbf{G}_1$ and $\mathbf{G} \cup \{\emptyset\,/\,s\}$ by $\mathbf{G}_2$. By Theorem 2.5.21, $\mathbf{G}_1$ is $\circledcirc$-k-analytic. We prove that so is $\mathbf{G}_2$. Suppose that $S_0 \vdash_{\mathbf{G}_2} s_0$. We prove that $S_0 \vdash_{\mathbf{G}_2}^{sub_k^{\circledcirc}(S_0 \cup \{s_0\})} s_0$. First, we show that $S_0 \vdash_{\mathbf{G}_1} s_0$, by induction on the length of the proof of s_0 from S_0 in $\mathbf{G}_2$. If s_0 is not the conclusion of an application of $\emptyset\,/\,s$ then this is straightforward by the induction hypothesis. Otherwise, $s_0 = \sigma(s) \cup c$ for some substitution σ and context sequent c. Since $\vdash_{\mathbf{G}} s'$ for every $s' \in S$, by Lemma 2.2.11, we have $\vdash_{\mathbf{G}} \sigma(s') \cup c$ for every $s' \in S$, and in particular, $S_0 \vdash_{\mathbf{G}_1} \sigma(s') \cup c$. Now, apply $S\,/\,s$ and obtain $S_0 \vdash_{\mathbf{G}_1} s_0$. Since $\mathbf{G}_1$ is $\circledcirc$-k-analytic, we have $S_0 \vdash_{\mathbf{G}_1}^{sub_k^{\circledcirc}(S_0 \cup \{s_0\})} s_0$. By taking the proof of s_0 from S_0 in $\mathbf{G}_1$ and deleting every premise in every application of $S\,/\,s$, we obtain a proof of s_0 from S_0 in $\mathbf{G}_2$ in which only $sub_k^{\circledcirc}(S_0 \cup \{s_0\})$-formulas occur, which means that $S_0 \vdash_{\mathbf{G}_2}^{sub_k^{\circledcirc}(S_0 \cup \{s_0\})} s_0$. □

Remark 2.5.25. The requirement in Proposition 2.5.24 that both $\mathbf{G}$ and s originate from the same calculus $\mathbf{G_B}$ is crucial. Consider, for example, the case in which $\mathbf{G}$ consists of the rule $r_1 = \;\Rightarrow p_1 \,/\, \Rightarrow p_1 T p_2$, $\mathbf{G_B}$ consists of the rule $r_2 = p_2 \Rightarrow \;/\; p_1 T p_2 \Rightarrow$, and s is $p_1 T p_2 \Rightarrow p_2$. s is the conclusion of the $\emptyset$-ordered application $\langle p_2 \Rightarrow p_2, p_1 T p_2 \Rightarrow p_2 \rangle$ of r_2, whose premise is derivable in $\mathbf{G}$. Also, $\mathbf{G}$ and $\mathbf{G_B}$ are $\emptyset$-directed and coherent (and thus also $\emptyset$-analytic). However $\mathbf{G} \cup \{\emptyset\,/\,s\}$ is not $\emptyset$-analytic, as it raises the "Tonk" problem [87] (see also [17]).

Example 2.5.26. In [27], it was shown that $\mathbf{G_{C_1}}$ is $\{\neg\}$-1-analytic, as a corollary of cut-admissibility. Using the methods of this section, we provide a simpler proof of the $\{\neg\}$-1-analyticity of $\mathbf{G_{C_1}}$. For this purpose, we construct a calculus which is equivalent to $\mathbf{G_{C_1}}$, that we call $\mathbf{G_{C_1}}'$. Take $\mathbf{G_B}$ to be $\mathbf{LK}$, and $\mathbf{G}$ to be $\mathbf{LK} \setminus \{(\neg \Rightarrow)\}$. By Theorem 2.5.16, $\mathbf{G}$ is $\{\neg\}$-1-analytic. $\mathbf{G_{C_1}}'$ is obtained by augmenting $\mathbf{G}$ with the following axiomatic rules:

$\emptyset \,/\, \neg\neg p_1 \Rightarrow p_1$	
$\emptyset \,/\, p_1, \neg p_1, \neg(p_1 \wedge \neg p_1) \Rightarrow$	$\emptyset \,/\, \neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2$
$\emptyset \,/\, \neg(p_1 \vee p_2) \Rightarrow \neg p_1, p_2$	$\emptyset \,/\, \neg(p_1 \vee p_2) \Rightarrow \neg p_1, \neg p_2$
$\emptyset \,/\, \neg(p_1 \vee p_2) \Rightarrow p_1, \neg p_2$	$\emptyset \,/\, \neg(p_1 \supset p_2) \Rightarrow p_1, p_2$
$\emptyset \,/\, \neg(p_1 \supset p_2) \Rightarrow p_1, \neg p_2$	$\emptyset \,/\, \neg(p_1 \supset p_2) \Rightarrow \neg p_1, \neg p_2$

Every rule here has the form $\emptyset\,/\,s$, where s is the conclusion of a $\{\neg\}$-1-ordered application of the rule $(\neg \Rightarrow)$ of $\mathbf{G_B}$, whose premises are all derivable in $\mathbf{G}$. For example, $\neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2$ is the conclusion of the following $\{\neg\}$-1-ordered application of

$(\neg \Rightarrow)$, whose premise is derivable in **G**:

$$\frac{\Rightarrow p_1 \wedge p_2, \neg p_1, \neg p_2}{\neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2}$$

By Proposition 2.5.24, $\mathbf{G_{C_1}}'$ is $\{\neg\}$-1-analytic. Using Lemma 2.4.2, it is easy to see that $\mathbf{G_{C_1}}'$ is equivalent to $\mathbf{G_{C_1}}$, and furthermore, the $\{\neg\}$-1-analyticity of $\mathbf{G_{C_1}}'$ entails the $\{\neg\}$-1-analyticity of $\mathbf{G_{C_1}}$.

Example 2.5.27 (Extended Primal Infon Logic)**.** The calculus $\mathbf{G_{PIL}}$ is $\emptyset$-analytic, as was shown in Example 2.5.17. One of the most attractive properties of $\mathbf{G_{PIL}}$ is the fact that its derivability problem is solvable in linear time. It is possible to augment $\mathbf{G_{PIL}}$ with additional rules in order to make it somewhat closer to **LK**, without compromising its $\emptyset$-analyticity nor its linear time complexity. For this, we introduce an extended logic, called **EPIL**, which is defined by the following calculus, called $\mathbf{G_{EPIL}}$ (that is, $\mathcal{T} \vdash_{\mathbf{EPIL}} \varphi$ iff $\Gamma \vdash_{\mathbf{G_{EPIL}}} \varphi$ for some finite $\Gamma \subseteq \mathcal{T}$). $\mathbf{G_{EPIL}}$ is obtained from $\mathbf{G_{PIL}}$ by the addition of the following set of rules. These rules recover some natural properties of the classical connectives (none of them is derivable in $\mathbf{G_{PIL}}$):

$\emptyset \,/\, \Rightarrow \bot \supset p_1$	$\emptyset \,/\, p_1 \vee p_1 \Rightarrow p_1$	$\emptyset \,/\, \Rightarrow p_1 \supset p_1$
$\emptyset \,/\, \bot \vee p_1 \Rightarrow p_1$	$\emptyset \,/\, p_1, \neg p_1 \Rightarrow$	$\emptyset \,/\, \Rightarrow (p_1 \wedge p_2) \supset p_1$
$\emptyset \,/\, p_1 \vee \bot \Rightarrow p_1$	$\emptyset \,/\, p_1 \vee (p_1 \wedge p_2) \Rightarrow p_1$	$\emptyset \,/\, \Rightarrow (p_1 \wedge p_2) \supset p_2$
$\emptyset \,/\, (p_1 \wedge p_2) \vee p_1 \Rightarrow p_1$	$\emptyset \,/\, \Rightarrow p_2 \supset (p_1 \supset p_2)$	

Each of these rules has the form $\emptyset \,/\, s$, where s is the conclusion of a $\emptyset$-ordered application of a rule of **LK**, whose premises are all derivable in $\mathbf{G_{PIL}}$. By repeatedly applying Proposition 2.5.24, augmenting $\mathbf{G_{PIL}}$ with these axiomatic rules results in a $\emptyset$-analytic calculus. The fact that the extended calculus also has linear time complexity will be proved in the next Chapter (see Example 3.2.2).

2.5.4 Proof of Theorem 2.5.21

Let **G** be a calculus that consists of $\circledcirc$-k-ordered applications of rules of a $\circledcirc$-k-directed coherent calculus $\mathbf{G_B}$. We prove that **G** is $\circledcirc$-k-analytic. Using Theorem 2.5.9, it suffices to prove that every **G**-legal bivaluation v can be extended to a **G**-legal $\mathcal{L}$-bivaluation, provided that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas. Thus, in what follows, we fix an arbitrary **G**-legal bivaluation v such that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas.

We extend v iteratively: in each step we add a single formula to the domain of v. Thus, we construct a sequence of **G**-legal bivaluations that extend v, and use this sequence in order to define a **G**-legal $\mathcal{L}$-bivaluation that extends v.

Since the $\circledcirc$-k-subformula relation is a partial order, $sub_k^{\circledcirc}(\psi)$ is finite for every ψ, and $dom(v)$ is finite, there exists an enumeration $\psi_1, \psi_2, \dots$ of $\mathcal{L}$ such that:

1. If $\psi_i \in dom(v)$ and $\psi_j \notin dom(v)$ then $i < j$.
2. If ψ_i is a $\circledcirc$-k-subformula of ψ_j then $i \leq j$.

We define a sequence $v_0, v_1, \dots$ of bivaluations inductively by:

1. $v_0 = v$.
2. For every $i > 0$, v_i is defined over $dom(v) \cup \{\psi_1, \dots, \psi_i\}$ as follows:
   (a) $v_i(\varphi) = v_{i-1}(\varphi)$ for every $\varphi \in dom(v_{i-1})$.
   (b) If $\psi_i \notin dom(v_{i-1})$, then $v_i(\psi_i) = 1$ iff there exists a rule of the form $s_1, \dots, s_n \,/ \Rightarrow \varphi$ in $\mathbf{G_B}$, sequents $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, and a substitution σ such that $\sigma(frm(\{s'_1, \dots, s'_n\})) \subseteq dom(v_{i-1})$, $\sigma(\varphi) = \psi_i$ and $v_{i-1}(\sigma(s'_j)) = 1$ for every $1 \leq j \leq n$. Otherwise, $v_i(\psi_i) = 0$.

We show that each bivaluation in the sequence is **G**-legal. For this, the following lemma is needed:

Lemma 2.5.28. Let $\langle \{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \dots \cup c_n \rangle$ be a $\circledcirc$-k-ordered application of a $\circledcirc$-k-directed rule $r = s_1, \dots, s_n \,/\, s$, and let φ_s be the single formula in $frm(s)$. Then, all formulas in $sub_k^{\circledcirc}(\sigma(s'_i) \cup c_i)$ are proper $\circledcirc$-k-subformulas of $\sigma(\varphi_s)$ for every $1 \leq i \leq n$. In particular,

$$sub_k^{\circledcirc}(frm(\{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n, \sigma(s) \cup c_1 \cup \dots \cup c_n\})) \subseteq sub_k^{\circledcirc}(\sigma(s)).$$

Proof. Denote $\langle \{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \dots \cup c_n \rangle$ by $\hat{r}$. Suppose that ψ is a $\circledcirc$-k-subformula of some $\varphi \in \sigma(frm(s'_i)) \cup frm(c_i)$. We show that φ is a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. Since ψ is a $\circledcirc$-k-subformula of φ, it would then follow that ψ is also a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. If $\varphi = \sigma(\varphi')$ for some $\varphi' \in frm(s'_i)$, then since r is $\circledcirc$-k-directed, φ' is a proper $\circledcirc$-k-subformula of φ_s. By Lemma 2.5.4, φ is a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. Otherwise, $\varphi \in frm(c_i)$, and since $\hat{r}$ is $\circledcirc$-k-ordered, φ is a proper $\circledcirc$-k-subformula of some formula in $frm(\sigma(s) \cup c_1 \cup \dots \cup c_n)$. If φ is a proper $\circledcirc$-k-subformula of some formula in $frm(\sigma(s))$, then this formula must be $\sigma(\varphi_s)$. Otherwise, let θ be a formula in $frm(c_1, \dots, c_n)$ such that φ is a proper $\circledcirc$-k-subformula of θ, and θ has a maximal number of connectives. Since $\hat{r}$ is $\circledcirc$-k-ordered, θ must also be a proper $\circledcirc$-k-subformula of some formula $\theta' \in frm(\sigma(s) \cup c_1, \dots, c_n)$. By the maximality of θ, we have that $\theta' \in frm(\sigma(s))$, which means that $\theta' = \sigma(\varphi_s)$. Since φ is a proper $\circledcirc$-k-subformula of θ, we also have that φ is a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. $\square$

We show by induction on i that each v_i is $\mathbf{G}$-legal. For $i = 0$, this holds by our assumption regarding v. Let $i > 0$, and r be a rule of $\mathbf{G}$. Then, there exist a rule $s_1, \dots, s_n / s$ of $\mathbf{G_B}$, sequents $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, a substitution α, and sequents $c_1, \dots, c_n$ such that $r = \alpha(s'_1) \cup c_1, \dots, \alpha(s'_n) \cup c_n / \alpha(s) \cup c_1 \cup \dots \cup c_n$. Let $s''_1 \subseteq s'_1, \dots, s''_n \subseteq s'_n$, $c'_1 \subseteq c_1, \dots, c'_n \subseteq c_n$ and σ be a substitution such that $\sigma(frm(\{\alpha(s''_1) \cup c'_1, \dots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \dots \cup c_n\})) \subseteq dom(v_i)$. We show that $v_i(\{\sigma(\alpha(s''_j) \cup c'_j) \mid 1 \leq j \leq n\}) \leq v_i(\sigma(\alpha(s) \cup c_1, \dots, c_n))$. If $\psi_i \notin \sigma(frm(\{\alpha(s''_1) \cup c'_1, \dots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \dots \cup c_n)\})$ or $\psi_i \in dom(v_{i-1})$, then $\sigma(frm(\{\alpha(s''_1) \cup c'_1, \dots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \dots \cup c'_n\})) \subseteq dom(v_{i-1})$, and hence this holds by the induction hypothesis. Assume now that $\psi_i \in \sigma(frm(\{\alpha(s''_1) \cup c'_1, \dots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \dots \cup c_n)\})$ and $\psi_i \notin dom(v_{i-1})$. Let φ_s be the single formula in $frm(s)$. We first prove that $\psi_i = \sigma(\alpha(\varphi_s))$. Otherwise, $\sigma(\alpha(\varphi_s)) \in dom(v_{i-1})$. By Lemma 2.5.28, the set of formulas that occur in r is contained in $sub^{\circledcirc}_k(\alpha(\varphi_s))$, and by Lemma 2.5.4, we also have that for every formula φ that occurs in r, $\sigma(\varphi) \in \sigma(sub^{\circledcirc}_k(\alpha(\varphi_s))) \subseteq sub^{\circledcirc}_k(\sigma(\alpha(\varphi_s)))$. $dom(v_{i-1})$ is closed under $\circledcirc$-k-subformulas, and $\sigma(\alpha(\varphi_s)) \in dom(v_{i-1})$. Thus we have $\psi_i \in dom(v_{i-1})$, which is a contradiction.

Similarly, we show that $\sigma(frm(\alpha(s''_j) \cup c'_j)) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq n$. Indeed, let $\varphi \in \sigma(frm(\alpha(s''_j) \cup c'_j))$ and let $\varphi' \in frm(\alpha(s''_j) \cup c'_j)$ such that $\varphi = \sigma(\varphi')$. By Lemma 2.5.28, φ' is a proper $\circledcirc$-k-subformula of $\alpha(\varphi_s)$, and hence by Lemma 2.5.4, φ is a proper $\circledcirc$-k-subformula of $\psi_i = \sigma(\alpha(\varphi_s))$. In particular, $\varphi \neq \psi_i$. Since $\sigma(frm(\alpha(s''_j) \cup c'_j)) \subseteq dom(v_i)$, it follows that $\varphi \in dom(v_{i-1})$.

Now, suppose that $v_i(\sigma(\alpha(s''_j) \cup c'_j)) = 1$ for every $1 \leq j \leq n$. We prove that $v_i(\sigma(\alpha(s) \cup c_1 \cup \dots \cup c_n)) = 1$. If $v_i(\sigma(c'_1 \cup \dots \cup c'_n)) = 1$, then we are clearly done. Assume otherwise. Hence, we have $v_i(\sigma(\alpha(s''_j))) = 1$ for every $1 \leq j \leq n$. Since $\sigma(\alpha(frm(s''_j))) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq n$, we have $v_{i-1}(\sigma(\alpha(s''_j))) = 1$ for every such j. Distinguish two cases:

- $s = \ \Rightarrow \varphi_s$: Since $\sigma(\alpha(frm(s''_j))) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq n$, $\sigma(\alpha(\varphi_s)) = \psi_i$, and $v_{i-1}(\sigma(\alpha(s''_j))) = 1$ for every $1 \leq j \leq n$, by the definition of v_i we have $v_i(\psi_i) = 1$, and so $v_i(\sigma(\alpha(s))) = 1$.

- $s = \varphi_s \Rightarrow$: To prove that $v_i(\sigma(\alpha(s))) = 1$, we show that $v_i(\psi_i) = 0$. By the definition of v_i, it suffices to prove that for every rule of the form $q_1, \dots, q_m / \ \Rightarrow \varphi'$ in $\mathbf{G_B}$, sequents $q'_1 \subseteq q_1, \dots, q'_n \subseteq q_n$ and substitution σ' such that $\sigma'(frm(q'_j)) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq m$ and $\sigma'(\varphi') = \psi_i$, we have $v_{i-1}(\sigma'(q'_j)) = 0$ for some $1 \leq j \leq m$. Let $q_1, \dots, q_m / \ \Rightarrow \varphi'$ and σ' as above. Since $\mathbf{G_B}$ is coherent, the empty sequent is derivable from $\{\sigma(\alpha(s_1)), \dots, \sigma(\alpha(s_n)), \sigma'(q_1), \dots, \sigma'(q_m)\}$ using only (CUT). It can be shown by induction on this derivation that the same holds for $\{\sigma(\alpha(s'_1)), \dots, \sigma(\alpha(s'_n)), \sigma'(q'_1), \dots, \sigma'(q'_m)\}$, and in particular, we have $\sigma(\alpha(s'_1)), \dots, \sigma(\alpha(s'_n)), \sigma'(q'_1), \dots, \sigma'(q'_m) \vdash_{\mathbf{G}}^{dom(v_{i-1})} \Rightarrow$. By Theorem 2.3.5, since v_{i-1} is $\mathbf{G}$-legal and $v_{i-1}(\sigma(\alpha(s'_j))) = 1$ for every $1 \leq j \leq n$, we have $v_{i-1}(\sigma'(q'_j)) = 0$ for some $1 \leq j \leq m$.

Finally, let v' be the $\mathcal{L}$-bivaluation given by $v'(\psi_i) = v_i(\psi_i)$ for every $i > 0$. Clearly, v' extends v. To see that it is $\mathbf{G}$-legal, let $s_1, \dots, s_n / s \in \mathbf{G}$, $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, and σ be a substitution. Let $j = \max\{i \mid \psi_i \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))\}$. Then, $v'(\psi) = v_j(\psi)$ for every $\psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$. Since v_j is $\mathbf{G}$-legal, we have $v'(\{\sigma(s'_i) \mid 1 \leq i \leq n\}) = \min\{v_j(\sigma(s'_i)) \mid 1 \leq i \leq n\} \leq v_j(\sigma(s)) = v'(\sigma(s))$. □

2.6 Cut-admissibility

The most well-studied property of sequent calculi is the admissibility of the cut rule. When cut is admissible, the calculus is generally considered well-behaved, and reasoning about the calculus becomes much easier. Moreover, proof-search algorithms have no need to "guess" the cut formulas.

The purpose of this section is to study the connection between cut-admissibility and our generalized notion of analyticity. We show that for a wide sub-family of pure calculi, these properties are equivalent. After explicitly defining the property of cut-admissibility, we provide a semantic framework for pure calculi without the cut rule in Section 2.6.1. Then, Section 2.6.2 presents our main result, which is proven in Section 2.6.5. Sections 2.6.3 and 2.6.4 include examples and generalizations.

We now precisely define cut-admissibility.

Definition 2.6.1. A derivation of s from S in a calculus $\mathbf{G}$ is called *cut-limited* if in every application of (CUT), the cut formula is in $frm(S)$. We write $S \vdash_{\mathbf{G}}^{\mathrm{cf}} s$ if such a derivation exists. A calculus $\mathbf{G}$ enjoys *cut-admissibility* if $\vdash_{\mathbf{G}} = \vdash_{\mathbf{G}}^{\mathrm{cf}}$.

Note that what we call here cut-admissibility is actually known as *strong* cut-admissibility, in which cuts are allowed, but they are confined to apply only on formulas that appear in the set of assumptions [12]. Usual cut-admissibility, that we call here *weak cut-admissibility*, only requires that $\vdash_{\mathbf{G}} s$ iff $\vdash_{\mathbf{G}}^{\mathrm{cf}} s$ for every sequent s. For pure calculi, however, the two notions turn out to be equivalent (see [12]).

2.6.1 Semantics in the Absence of Cut

In this section we prove a soundness and completeness theorem for cut-limited derivations in pure calculi, with respect to a variant of the bivaluation semantics from Section 2.3.

We then obtain a sufficient semantic criterion for cut-admissibility, that will be used in order to prove the equivalence between cut-admissibility and analyticity in a wide sub-family of calculi.

Theorem 2.3.5 establishes a strong connection between derivations in pure calculi and bivaluations. When considering derivations without cut, however, bivaluations fall short, as the cut rule is sound for them. In order to obtain a semantics in which (CUT) is unsound, it must be possible that both $\Rightarrow \varphi$ and $\varphi \Rightarrow$ are satisfied. This is obtained by the addition of a third truth value, $1/2$, so that every sequent in which some formula is assigned $1/2$ is satisfied. This way, when assigning φ the value $1/2$, both sequents above are satisfied.

Thus, the semantics for cut-limited derivations is based on three truth-values: "false", "true", and "indeterminate", represented as 0, 1, and $1/2$, respectively.

Definition 2.6.2. An $\mathcal{L}$*-trivaluation* is a function v from $\mathcal{L}$ to $\{0, 1/2, 1\}$. A trivaluation v is extended to sequents by: $v(\Gamma \Rightarrow \Delta) = 1$ iff $v(\varphi) < 1$ for some $\varphi \in \Gamma$ or $v(\varphi) > 0$ for some $\varphi \in \Delta$. Otherwise, $v(\Gamma \Rightarrow \Delta) = 0$.[5] For a set S of sequents, by $v(S)$ we mean $\min \{v(s) \mid s \in S\}$.

Definition 2.6.3. A trivaluation v *respects* a rule $s_1, \dots, s_n \,/\, s$ if $v(\{\sigma(s'_1), \dots, \sigma(s'_n)\}) \leq v(\sigma(s))$ for every subsequents $s'_1, \dots, s'_n$ of $s_1, \dots, s_n$ (respectively) and substitution σ. v is called **G**-legal if it respects all rules of **G**.

Note that unlike bivaluations, we only consider trivaluations that are totally defined.

Example 2.6.4. Consider a trivaluation v such that $v(p_1) = v(p_2) = 1/2$, $v(p_1 \wedge p_2) = 1$, and $v(p_1 \vee p_2) = 0$. For every other formula, $v(\varphi) = 1/2$. v respects neither of the rules $\Rightarrow p_1, p_2 \,/\, \Rightarrow p_1 \vee p_2$ and $p_1, p_2 \Rightarrow \,/\, p_1 \wedge p_2 \Rightarrow$. However, v does respect their axiomatic counterparts $p_1 \wedge p_2 \Rightarrow p_1$, $p_1 \wedge p_2 \Rightarrow p_2$, $p_1 \Rightarrow p_1 \vee p_2$ and $p_2 \Rightarrow p_1 \vee p_2$.

When semantically describing the existence of a cut-limited derivation of a sequent s from a set S of sequents in a calculus **G**, we should take into account that (CUT) *is* sound when the cut formula is in *frm*(S). Thus, *frm*(S)-formulas must be assigned either 1 or 0. For this requirement, we introduce the following definition:

Definition 2.6.5. The *support* of a trivaluation v, denoted $\mathrm{supp}(v)$, is the set $\{\varphi \in \mathcal{L} \mid v(\varphi) \neq 1/2\}$. v is called:

- $\mathcal{F}$*-determined* (for $\mathcal{F} \subseteq \mathcal{L}$) if $\mathcal{F} \subseteq \mathrm{supp}(v)$; and
- *fully determined* if it is $\mathcal{L}$-determined.

---

[5] Note that trivaluations never assign $1/2$ to a sequent.

Remark 2.6.6. It might seem that there is no real difference between trivaluations and (partial) bivaluations: given a trivaluation v, a bivaluation v^* can be obtained from it simply by taking $v^* = \lambda\varphi \in \text{supp}(v).v(\varphi)$. Moreover, if v is **G**-legal for some calculus **G**, then so is v^*. The converse, however, does not always hold, and there lies the difference between these two formalisms. Consider, for example, the trivaluation v from Example 2.6.4. The domain of v^* would then be $\text{supp}(v) = \{p_1 \wedge p_2, p_1 \vee p_2\}$. v^* is (trivially) **LK**-legal, as for every substitution σ and rule r of **LK**, $\sigma(\mathit{frm}(r)) \nsubseteq \mathit{dom}(v)$. However, as seen in Example 2.6.4, v itself is not **LK**-legal.

Note that every trivaluation is $\emptyset$-determined, and also that $\mathcal{L}$-bivaluations are actually fully determined trivaluations, and thus they will be called this way in the reminder of this section.

For cut-limited derivations, we have the following variant of Theorem 2.3.5:

Theorem 2.6.7. $S \vdash^{\text{cf}}_{\mathbf{G}} s$ iff $v(S) \leq v(s)$ for every $\mathit{frm}(S)$-determined **G**-legal trivaluation v.

Proof.
Soundness: Soundness is proved analogously to the proof of Theorem 2.3.5. The only difference is that (CUT), in general, is unsound for trivaluations. However, assuming $S \vdash^{\text{cf}}_{\mathbf{G}} s$, we have that cuts are limited to formulas in $\mathit{frm}(S) \subseteq \text{supp}(v)$. For such formulas, v assigns either 1 or 0 and thus (CUT) is valid.
Completeness: Recall the definitions related to *ω-sequents* from the proof of Theorem 2.3.5. Let S be a set of sequents and s a sequent such that $S \nvdash^{\text{cf}}_{\mathbf{G}} s$. Call an ω-sequent $L \Rightarrow R$ *maximal unprovable* if the followings hold:

- $S \nvdash^{\text{cf}}_{\mathbf{G}} L \Rightarrow R$
- $S \vdash^{\text{cf}}_{\mathbf{G}} L, \varphi \Rightarrow R$ for every $\varphi \notin L$, and $S \vdash^{\text{cf}}_{\mathbf{G}} L \Rightarrow \varphi, R$ for every $\varphi \notin R$.

It is routine to extend s to a maximal unprovable ω-sequent $L \Rightarrow R$.

Define a trivaluation v as follows:

$$v = \lambda\varphi \in \mathcal{L}. \begin{cases} 1 & \varphi \in L \\ 0 & \varphi \in R \\ 1/2 & \mathit{otherwise} \end{cases}$$

First note that v is well defined, as $L \cap R = \emptyset$ (otherwise, $S \vdash^{\text{cf}}_{\mathbf{G}} L \Rightarrow R$). Clearly, $v(s) = 0$, and for every $\Gamma \Rightarrow \Delta \in S$ we have that $S \vdash^{\text{cf}}_{\mathbf{G}} \Gamma \Rightarrow \Delta$, which means that either $\Gamma \nsubseteq L$ or $\Delta \nsubseteq R$, and so $v(\Gamma \Rightarrow \Delta) = 1$.

v is shown to be **G**-legal as in the proof of Theorem 2.3.5, using the extensions of trivaluations to sequents. It is left to show that v is $\mathit{frm}(S)$-determined. Assume otherwise. Then we have some $\varphi \in \mathit{frm}(S) \setminus (L \cup R)$. Since $L \Rightarrow R$ is maximal unprovable, we must have $S \vdash^{\mathrm{cf}}_{\mathbf{G}} L \Rightarrow \varphi, R$ and $S \vdash^{\mathrm{cf}}_{\mathbf{G}} L, \varphi \Rightarrow R$. Applying (cut) (which is allowed, as $\varphi \in \mathit{frm}(S)$), we obtain $S \vdash^{\mathrm{cf}}_{\mathbf{G}} L \Rightarrow R$, which is a contradiction. □

Note that only *completeness* of cut-limited derivations with respect to $\mathit{frm}(S)$-determined trivaluations is used in the proof of Theorem 2.6.20.

Remark 2.6.8. The three valued semantics for the cut-free fragment of **LK** that is obtained from Theorem 2.6.7 is equivalent to the Nmatrix semantics in [68].

Theorem 2.6.7 gives rise to a sufficient semantic criterion for cut-admissibility, which is based on the following notion of *determination*:

Definition 2.6.9. We say that a trivaluation v' is a *determination* of a trivaluation v (alternatively, we say that v' *determines* v) if $v(\varphi) = v'(\varphi)$ for every $\varphi \in \mathrm{supp}(v)$. v' is called an $\mathcal{F}$*-determination of* v if, in addition, it is $\mathcal{F}$-determined. If v' is fully determined we call it a *full determination* of v.

It immediately follows from our definitions that:

Proposition 2.6.10. Suppose that v' determines v. Then for every sequent s, if $v'(s) = 1$ then $v(s) = 1$. The converse holds as well when v is $\mathit{frm}(s)$-determined.

The sufficient semantic criterion for cut-admissibility is given in the following corollary:

Corollary 2.6.11. If every **G**-legal trivaluation has a **G**-legal full determination, then **G** enjoys cut-admissibility.

Proof. Suppose $S \nvdash^{\mathrm{cf}}_{\mathbf{G}} s$. By Theorem 2.6.7, there exists some $\mathit{frm}(S)$-determined **G**-legal trivaluation v such that $v(S) = 1$ and $v(s) = 0$. Let v' be a **G**-legal full determination of v. By Proposition 2.6.10, $v'(S) = 1$ and $v'(s) = 0$. By Theorem 2.3.5 (and the fact that fully determined trivaluations are actually $\mathcal{L}$-bivaluations), we have $S \nvdash_{\mathbf{G}} s$. □

2.6.2 From Analyticity to Cut-admissibility

In this section we identify a sub-family of calculi in which analyticity and cut-admissibility are equivalent. First, note that analyticity may not imply cut-admissibility in general:

Example 2.6.12. Consider the calculus $Ax(\mathbf{LK})$ from Section 2.4. It was shown in Example 2.5.10 that this calculus is $\emptyset$-analytic. However, it does not admit cut-admissibility. For instance, the derivable sequent $p_1 \wedge p_2 \Rightarrow p_1 \vee p_2$ has no derivation without cut. This is proved using Theorem 2.6.7: recall the bivaluation v from Example 2.6.4. Clearly, $v(p_1 \wedge p_2 \Rightarrow p_1 \vee p_2) = 0$. Also, v is $Ax(\mathbf{LK})$-legal. By Theorem 2.6.7, $\nvdash^{\mathrm{cf}}_{\mathbf{G}} p_1 \wedge p_2 \Rightarrow p_1 \vee p_2$.

However, for the family of $\circledcirc$-k-directed calculi, these two fundamental properties coincide:

Theorem 2.6.13. Every $\circledcirc$-k-analytic $\circledcirc$-k-directed pure calculus enjoys cut-admissibility.

This theorem is a particular instance of Theorem 2.6.20 below, that will be proved in Section 2.6.5.

For all the calculi mentioned above (except $Ax(\mathbf{LK})$), this theorem allows one to obtain cut-admissibility as a consequence of $\circledcirc$-k-analyticity for some $\circledcirc$ and k. This also includes every calculus in the hierarchy of weak double negations (Example 2.5.18).

2.6.3 Some Applications

We outline two possible applications of Theorem 2.6.13 in cut-admissibility proofs:

Simpler Semantic Proofs of Cut-admissibility

Theorem 2.6.13 reduces the burden in proving cut-admissibility to establishing only *analytic cut-admissibility*. An application of (CUT) in a derivation of s from S is called a *$\circledcirc$-k-analytic cut* if the cut formula is in $sub_k^{\circledcirc}(S \cup \{s\})$. In turn, *$\circledcirc$-$k$-analytic cut-admissibility* concerns only the admissibility of non-$\circledcirc$-k-analytic cuts. Proving this property is often easier than showing full cut-admissibility. For example, it is straightforward to prove that **LK** is complete for the classical truth tables, when applications of (CUT) are restricted to be $\emptyset$-analytic. Indeed, assuming s is not derivable from S in **LK** using $\emptyset$-analytic cuts, one extends s to a maximal unprovable sequent s^* that consists solely of $sub(S \cup \{s\})$-formulas. Then, a countermodel v can be defined simply by setting $v(\varphi) = 1$ for every φ on the left side of s^*, and $v(\psi) = 0$ for every ψ on its right side. Using $\emptyset$-analytic cuts, it immediately follows that $frm(s^*) = sub(S \cup \{s\})$, which makes it easy to prove that v respects the classical truth tables, and can therefore be extended to a full classical countermodel. By Theorem 2.6.13, we may conclude that **LK** enjoys cut-admissibility. Providing a semantic proof of cut-admissibility without going through $\emptyset$-analytic cuts is possible, but more complicated.

Sufficient Criterion for Cut-admissibility

Theorem 2.6.13 extends the sufficient criterion for analyticity from Theorem 2.5.16 to be also sufficient for cut-admissibility.

Corollary 2.6.14. Every pure $\circledcirc$-k-directed coherent calculus enjoys cut-admissibility.

Proof. Let **G** be such a calculus. By Theorem 2.5.16, **G** is $\circledcirc$-k-analytic. By Theorem 2.6.13, it admits cut-admissibility. □

2.6.4 Strengthening The Result

Actually, Theorem 2.6.13 holds for a more general notion of analyticity, as the internal structure of the $\circledcirc$-k-subformula relation is not important to the proof of the theorem. The crucial properties of the $\circledcirc$-k-subformula relation that are actually needed to obtain Theorem 2.6.13 are the fact that it is an effective order relation (in the sense that it induces an enumeration of formulas) that conforms with substitutions. We thus further generalize the $\circledcirc$-k-subformula property, by assuming a given ordering of $\mathcal{L}$-formulas, denoted $\prec$, which has to satisfy certain properties, as defined next.

Notation 2.6.15. Given a binary relation R on $\mathcal{L}$, we denote by $R[\varphi]$ the set $\{\psi \in \mathcal{L} \mid \langle \psi, \varphi \rangle \in R\}$. This notation is naturally extended to sets ($R[\Gamma] = \bigcup_{\varphi \in \Gamma} R[\varphi]$), sequents ($R[\Gamma \Rightarrow \Delta] = R[\Gamma] \cup R[\Delta]$), and sets of sequents ($R[S] = \bigcup_{s \in S} R[s]$). By *order relation* we mean an irreflexive and transitive binary relation.

Definition 2.6.16. An order relation $\prec$ is called:

- *safe* if it is prefinite ($\prec[\varphi]$ is finite for every $\varphi \in \mathcal{L}$), and the function $\lambda\varphi \in \mathcal{L}.\prec[\varphi]$ is computable.

- *structural* if $\varphi \prec \psi$ implies $\sigma(\varphi) \prec \sigma(\psi)$ for every substitution σ.

Example 2.6.17. The usual proper subformula relation over $\mathcal{CL}$ is a structural safe order relation. The proper $\{\neg\}$-1-subformula relation is also an example of a structural and safe order relation. Moreover, for every $\circledcirc$ and k, the proper $\circledcirc$-k-subformula relation is a structural and safe order relation.

In what follows, $\prec$ denotes an arbitrary safe and structural order relation over $\mathcal{L}$, and $\preceq$ denotes its reflexive closure.

The above definition allows us to further generalize $\circledcirc$-k-analyticity, to a property that we call $\prec$-analyticity.

Definition 2.6.18. A calculus **G** is called $\prec$-*analytic* if $S \vdash_{\mathbf{G}} s$ iff $S \vdash_{\mathbf{G}}^{\preceq[S \cup \{s\}]} s$ for every set S of sequents and sequent s.

It is left to accommodate the definition of $\circledcirc$-k-directedness (Definition 2.5.12) to the generalized notion of analyticity that we consider in this section:

Definition 2.6.19. A rule S/s is called *$\prec$-ordered* if $frm(S) \subseteq \prec[s]$. It is called *$\prec$-directed* if, in addition, s has the form $\Rightarrow \varphi$ or $\varphi \Rightarrow$ for some formula φ. A calculus **G** is called *$\prec$-ordered* (*$\prec$-directed*) if all its rules are *$\prec$-ordered* (*$\prec$-directed*).

Generalizing Theorem 2.6.13, we get that cut-admissibility follows from $\prec$-analyticity in the family of $\prec$-directed pure calculi, for any structural and safe order relation $\prec$.

Theorem 2.6.20. Every $\prec$-analytic $\prec$-directed pure calculus enjoys cut-admissibility.

2.6.5 Proof of Theorem 2.6.20

We now prove Theorem 2.6.20, by utilizing Corollary 2.6.11. Thus, given a $\prec$-analytic $\prec$-directed pure calculus, we show that every **G**-legal trivaluation has a **G**-legal full determination. This is done in two steps: Lemma 2.6.21 below shows that it is possible to add a single formula to the support of a **G**-legal trivaluation. Then, Lemma 2.6.22 makes an iterative usage of this fact, thus obtaining a full determination.

Lemma 2.6.21. Let **G** be a $\prec$-analytic $\prec$-directed calculus, v a **G**-legal trivaluation and ψ a formula such that $\prec[\psi] \subseteq \text{supp}(v)$. Then v has a **G**-legal $\text{supp}(v) \cup \{\psi\}$-determination.

Proof. If $\psi \in \text{supp}(v)$, then this is trivial, as v determines itself. We therefore assume that $\psi \notin \text{supp}(v)$. Let $\Gamma_v = \{\varphi \in \prec[\psi] \mid v(\varphi) = 1\}$ and $\Delta_v = \{\varphi \in \prec[\psi] \mid v(\varphi) = 0\}$. We first show that $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \Delta_v$: Consider the $\preceq[\Gamma_v \Rightarrow \Delta_v]$-bivaluation u defined by $u(\varphi) = v(\varphi)$ for every $\varphi \in \preceq[\Gamma_v \Rightarrow \Delta_v]$. u is indeed a bivaluation, as $\preceq[\Gamma_v \Rightarrow \Delta_v] \subseteq \preceq[\psi] \subseteq \text{supp}(v)$. Clearly, u is **G**-legal, and $u(\Gamma_v \Rightarrow \Delta_v) = 0$. By Theorem 2.3.5, $\nvdash_{\mathbf{G}}^{\preceq[\Gamma_v \Rightarrow \Delta_v]} \Gamma_v \Rightarrow \Delta_v$, and since **G** is $\prec$-analytic, $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \Delta_v$. We define a trivaluation v', and show that it is a **G**-legal $\text{supp}(v) \cup \{\psi\}$-determination of v. v' is obtained from v by setting

$$v'(\psi) = \begin{cases} 1 & \nvdash_{\mathbf{G}} \Gamma_v, \psi \Rightarrow \Delta_v \\ 0 & \textit{otherwise} \end{cases}$$

Clearly, v' is a $\text{supp}(v) \cup \{\psi\}$-determination of v. We prove that it is **G**-legal. Let $s_1, \dots, s_n / s$ be a rule of **G**, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$ and σ a substitution. Suppose $v'(\sigma(s)) = 0$. We prove that $v'(\sigma(s'_i)) = 0$ for some $1 \leq i \leq n$. By our assumption, $frm(\sigma(s)) \subseteq \text{supp}(v') = \text{supp}(v) \cup \{\psi\}$. If $frm(\sigma(s)) \subseteq \text{supp}(v)$, then this follows from the fact that v is **G**-legal and v' determines v. Otherwise, $frm(\sigma(s)) \not\subseteq \text{supp}(v)$. Since **G** is $\prec$-directed, we have $frm(\sigma(s)) = \{\psi\}$. Also, for every $1 \leq i \leq n$, we have $frm(s'_i) \subseteq \prec[s]$, and in particular, $frm(\sigma(s'_i)) \subseteq \sigma(\prec[s]) \subseteq \prec[\sigma(s)] = \prec[\psi]$.

To show that there exists $1 \leq i \leq n$ such that $v'(\sigma(s'_i)) = 0$, we first prove that $\nvdash_{\mathbf{G}} \sigma(s) \cup (\Gamma_v \Rightarrow \Delta_v)$: by the above, either $\sigma(s) = (\psi \Rightarrow\ )$ or $\sigma(s) = (\ \Rightarrow \psi)$. In the first case, $v'(\psi) = 1$, which means that $\nvdash_{\mathbf{G}} \Gamma_v, \psi \Rightarrow \Delta_v$. In the second, $v'(\psi) = 0$, which means that $\vdash_{\mathbf{G}} \Gamma_v, \psi \Rightarrow \Delta_v$. Using (CUT), we have $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \psi, \Delta_v$. Either way, we have $\nvdash_{\mathbf{G}} \sigma(s) \cup (\Gamma_v \Rightarrow \Delta_v)$.

Since $\nvdash_{\mathbf{G}} \sigma(s) \cup (\Gamma_v \Rightarrow \Delta_v)$, we must have some $1 \leq i \leq n$ such that $\nvdash_{\mathbf{G}} \sigma(s'_i) \cup (\Gamma_v \Rightarrow \Delta_v)$. Suppose $s_i = \Gamma_i \Rightarrow \Delta_i$ and $s'_i = \Gamma'_i \Rightarrow \Delta'_i$. Then using the fact that $frm(\sigma(s'_i)) \subseteq \prec[\psi] \subseteq \Gamma_v \cup \Delta_v$, we have $\sigma(\Gamma'_i) \subseteq \Gamma_v$ and $\sigma(\Delta'_i) \subseteq \Delta_v$. Thus $v(\sigma(s'_i)) = 0$, and therefore $v'(\sigma(s'_i)) = 0$. □

Lemma 2.6.22. Let **G** be a $\prec$-analytic $\prec$-directed calculus and v a **G**-legal trivaluation. Then there exists a **G**-legal full determination of v.

Proof. For every trivaluation u and formula ψ, u_ψ denotes an arbitrary **G**-legal $\text{supp}(u) \cup \{\psi\}$-determination of u, if such exists (otherwise, u_ψ is undefined). Note that whenever u is **G**-legal and $\prec[\psi] \subseteq \text{supp}(u)$, Lemma 2.6.21 provides us with such a determination, in which case u_ψ is defined.

Let $\psi_1, \psi_2, ...$ be an enumeration of the formulas of $\mathcal{L}$, such that $i < j$ whenever $\psi_i \prec \psi_j$. Such enumeration exists by the safety of $\prec$. For every i, denote the set $\{\psi_1, ..., \psi_i\}$ by $\Phi_{\leq i}$ (in particular, $\Phi_{\leq 0} = \emptyset$).

Define a sequence $v^0, v^1, ...$ of trivaluations as follows: $v^0 = v$, and for every $i > 0$, $v^i = v^{i-1}_{\psi_i}$. We prove by induction on i that v^i is defined, and is a **G**-legal $\Phi_{\leq i}$-determination of v, and also of v^{i-1} (for $i \geq 1$).

For $i = 0$, this trivially holds. Now let $i > 0$. By the induction hypothesis, v^{i-1} is **G**-legal and is a $\Phi_{\leq i-1}$-determination of v. By the enumeration, $\prec[\psi_i] \subseteq \Phi_{\leq i-1} \subseteq \text{supp}(v^{i-1})$. Thus, v^i is defined, and is a **G**-legal $\text{supp}(v^{i-1}) \cup \{\psi_i\}$-determination of v^{i-1}, and thus also of v. Now, $\Phi_{\leq i} = \Phi_{\leq i-1} \cup \{\psi_i\} \subseteq \text{supp}(v^{i-1}) \cup \{\psi_i\} \subseteq \text{supp}(v^i)$, and therefore v^i is actually a $\Phi_{\leq i}$-determination of v^{i-1} and of v.

We now define v', a **G**-legal full determination of v. For every $\varphi \in \mathcal{L}$, let i_φ be the index of φ in the enumeration (that is, $\varphi = \psi_{i_\varphi}$). Define $v' = \lambda\varphi \in \mathcal{L}.v^{i_\varphi}(\varphi)$.

For every $\varphi \in \mathcal{L}$, $v'(\varphi) = v^{i_\varphi}(\varphi) \neq 1/2$, as v^{i_φ} is $\Phi_{\leq i_\varphi}$-determined. Also, for every $\varphi \in \text{supp}(v)$, $v'(\varphi) = v^{i_\varphi}(\varphi) = v(\varphi)$, as v^{i_φ} determines v. Finally, let $s_1, ..., s_n \,/\, s$ be a rule of **G**, $s'_1, ..., s'_n$ respective subsequents of $s_1, ..., s_n$, and σ a substitution. Let $k = \min\{i \mid frm(\sigma(\{s'_1, ..., s'_n, s\})) \subseteq \Phi_{\leq i}\}$. Then $v'(\varphi) = v^k(\varphi)$ for every $\varphi \in frm(\sigma(\{s'_1, ..., s'_n, s\}))$. Therefore, if $v'(\sigma(\{s'_1, ..., s'_n\})) = 1$, then $v^k(\sigma(\{s'_1, ..., s'_n\})) = 1$ as well. Since v^k is **G**-legal, $v^k(\sigma(s)) = 1$, and therefore $v'(\sigma(s)) = 1$. □

Putting all pieces together we have proved Theorem 2.6.20: Assuming that G is $\prec$-analytic and $\prec$-directed, by Lemma 2.6.22, every **G**-legal $\mathcal{L}$-trivaluation has a **G**-legal full determination, and cut-admissibility follows by Corollary 2.6.11. □

2.7 Single-conclusion Pure Calculi

Gentzen's calculus for propositional intuitionistic logic from [54], called LJ, is obtained from LK by an additional requirement according to which all sequents are single-conclusion sequents (sequents $\Gamma \Rightarrow \Delta$ with $|\Delta| \leq 1$). Obviously, LJ cannot be presented as a pure calculus due to this restriction. While there is no known analytic pure calculus for intuitionistic logic, we identify a sub-family of calculi in which such a restriction on context sequents turns out to be redundant.

We start by defining the family of *single conclusion calculi*, and their associated consequence relations:

Definition 2.7.1. A *single-conclusion pure rule* is a pure rule that consists only of single-conclusion sequents. A *single-conclusion pure calculus* is a pure calculus whose rules are all single-conclusion pure rules. A *(strict) single-conclusion derivation* of a (strict) single-conclusion sequent in a calculus **G** is defined as a usual derivation, in which only (strict) single-conclusion sequents occur.[6] A (strict) single conclusion sequent $\Gamma \Rightarrow \Delta$ is *(strictly) single-conclusion derivable* from a set S of (strict) single-conclusion sequents in a calculus **G** if it has a (strict) single-conclusion derivation from S in **G**. We denote the latter by $S \vdash^{\leq 1}_{\mathbf{G}} s$ ($S \vdash^{=1}_{\mathbf{G}} s$).

To identify a sub-family of calculi **G** in which $\vdash^{\leq 1}_{\mathbf{G}}$ ($\vdash^{=1}_{\mathbf{G}}$) and $\vdash_{\mathbf{G}}$ are equivalent, we introduce the following definitions:

Definition 2.7.2. A pure rule is called *definite* if at least one of its premises has an empty right side whenever the conclusion has an empty right side. A calculus is called *definite* if each of its rules is definite.

Example 2.7.3. Both $(\vee \Rightarrow)$ and $(\Rightarrow \vee)$ of **LK** are definite (all sequents of the former have empty right sides, and the right side of the conclusion of the latter is not empty). However, the rule $(\neg \Rightarrow\ )$ is not definite.

Definition 2.7.4. A rule is called a *Horn rule* if the sum of the number of formulas in the right side of the conclusion and the number of premises with a non-empty left side is at most one. A calculus is called a *Horn calculus* if each of its rules is a Horn rule.

Example 2.7.5. The rule $(\Rightarrow \vee)$ of **LK** (see Example 2.2.7) is Horn. In contrast, the rule $(\vee \Rightarrow)$ is not Horn, as both its premises have a non-empty left side.

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[6] A strict single-conclusion sequent is a sequent $\Gamma \Rightarrow \Delta$ with $|\Delta| = 1$.

For the family of single-conclusion calculi that are Horn and definite, we show that one can consider only derivations that contain single-conclusion sequents, without affecting the induced derivability relation.

Proposition 2.7.6. Let $\mathbf{G}$ be a pure Horn definite single-conclusion calculus, S a set of (strict) single-conclusion sequents and s a (strict) single-conclusion sequent. Then, $S \vdash_{\mathbf{G}} s$ iff $S \vdash_{\mathbf{G}}^{\leq 1} s$ ($S \vdash_{\mathbf{G}}^{=1} s$).

Proof. The left-to-right direction is obvious. To show right-to-left direction of the non-strict version, we prove by induction on the length of the derivation in $\mathbf{G}$ that if $S \vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$ then $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma \Rightarrow \mathcal{E}$ for some singleton or empty set $\mathcal{E} \subseteq \Delta$. We consider here only the case of an application of some rule of $\mathbf{G}$. The other cases are easy. Suppose that $\Gamma \Rightarrow \Delta$ is the conclusion of an application of a single-conclusion rule r of $\mathbf{G}$, whose conclusion is $\Gamma_0 \Rightarrow \Delta_0$. Then, there exist subsequents $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$ of the premises of r, a substitution σ and sets $\Gamma'_1, \dots, \Gamma'_n, \Delta'_1, \dots, \Delta'_n$ of formulas, such that $\Gamma = \sigma(\Gamma_0) \cup \Gamma'_1 \cup \dots \cup \Gamma'_n$, $\Delta = \sigma(\Delta_0) \cup \Delta'_1 \cup \dots \cup \Delta'_n$, and $S \vdash_{\mathbf{G}} \sigma(\Gamma_i), \Gamma'_i \Rightarrow \sigma(\Delta_i), \Delta'_i$ for every $1 \leq i \leq n$ with shorter derivations. Since $\mathbf{G}$ is Horn, one of the following holds:

- $\Gamma_i = \emptyset$ for every $1 \leq i \leq n$. In this case, we have $S \vdash_{\mathbf{G}} \Gamma'_i \Rightarrow \sigma(\Delta_i), \Delta'_i$ for every $1 \leq i \leq n$. The induction hypothesis entails that for every $1 \leq i \leq n$, $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_i \Rightarrow \mathcal{E}_i$ for some singleton or empty set $\mathcal{E}_i \subseteq \sigma(\Delta_i) \cup \Delta'_i$. If $\mathcal{E}_i \subseteq \Delta'_i$ for some i, then we obtain $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma \Rightarrow \mathcal{E}_i$ using (WEAK), and $\mathcal{E}_i \subseteq \Delta'_i \subseteq \Delta'_1 \cup \dots \cup \Delta'_n$. Otherwise, for every $1 \leq i \leq n$, we have $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_i \Rightarrow \mathcal{E}_i$ for some singleton $\mathcal{E}_i \subseteq \sigma(\Delta_i)$. Since $\mathbf{G}$ is a single conclusion calculus, for every $1 \leq i \leq n$ there exists a formula ψ_i such that $\mathcal{E}_i = \sigma(\Delta_i) = \{\sigma(\psi_i)\}$. Hence for every $1 \leq i \leq n$, we have $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_i \Rightarrow \sigma(\psi_i)$. In this case, we can apply r with (trivial) subsequents $\Rightarrow \sigma(\psi_i)$ and context sequents $\Gamma'_i \Rightarrow$, and obtain $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma \Rightarrow \sigma(\Delta_0)$ ($|\Delta_0| \leq 1$ since r is a single-conclusion rule).

- $\Gamma_i \neq \emptyset$ for a single $1 \leq i \leq n$ and $\Delta_0 = \emptyset$. Since $\mathbf{G}$ is definite, $\Delta_j = \emptyset$ for some $1 \leq j \leq n$. Now, if $j \neq i$ then since $\Gamma_i \neq \emptyset$ and $\mathbf{G}$ is Horn, $\Gamma_j = \emptyset$ as well. In this case, $S \vdash_{\mathbf{G}} \Gamma'_j \Rightarrow \Delta'_j$ and by the induction hypothesis, $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_j \Rightarrow \mathcal{E}_j$ for some $\mathcal{E}_j \subseteq \Delta'_j \subseteq \Delta$ such that $|\mathcal{E}_j| \leq 1$. Using (WEAK) we get that $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma \Rightarrow \mathcal{E}_j$. Hence we may assume that $\Delta_i = \emptyset$. Now, the induction hypothesis entails that for every $j \neq i$, $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_j \Rightarrow \mathcal{E}_j$ for some singleton or empty set $\mathcal{E}_j \subseteq \sigma(\Delta_j) \cup \Delta'_j$, and $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_i, \sigma(\Gamma_i) \Rightarrow \mathcal{E}_i$ for some singleton or empty set $\mathcal{E}_i \subseteq \Delta'_i$. If $\mathcal{E}_j \subseteq \Delta'_j$ for some $j \neq i$, then we obtain $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma \Rightarrow \mathcal{E}_j$ using (WEAK) and $\mathcal{E}_j \subseteq \Delta$. Otherwise, for every $j \neq i$, we have $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_j \Rightarrow \mathcal{E}_j$ for some singleton $\mathcal{E}_j \subseteq \sigma(\Delta_j)$. Since $\mathbf{G}$ is a single conclusion calculus, for every $j \neq i$ there exists a formula ψ_j such that $\mathcal{E}_j = \sigma(\Delta_j) = \{\sigma(\psi_j)\}$. Hence for every $j \neq i$, we have $S \vdash_{\mathbf{G}}^{\leq 1} \Gamma'_j \Rightarrow \sigma(\psi_j)$. In this case, we can apply r with context sequents $\Gamma'_i \Rightarrow \mathcal{E}_i$ and $\Gamma'_j \Rightarrow$ for $j \neq i$ and obtain $S \vdash^{\leq 1}_{\mathbf{G}} \Gamma \Rightarrow \mathcal{E}_i$, where $\mathcal{E}_i \subseteq \Delta'_i \subseteq \Delta$.

The proof of the right-to-left direction of the strict version is similar: one proves by induction on the length of the derivation in $\mathbf{G}$ that if $S \vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$ then $S \vdash^{=1}_{\mathbf{G}} \Gamma \Rightarrow \mathcal{E}$ for some singleton $\mathcal{E} \subseteq \Delta \cup \{\psi\}$, where ψ is the formula on the right side of s. □

Example 2.7.7. The single-conclusion calculus $\mathbf{G_{PIL}}$ is both Horn and definite, and hence $S \vdash_{\mathbf{G_{PIL}}} s$ iff $S \vdash^{\leq 1}_{\mathbf{G_{PIL}}} s$ whenever $S \cup \{s\}$ consists of single-conclusion sequents.

Example 2.7.8. The paper [20] studies the family of canonical single-conclusion sequent calculi, by adjusting the theory of canonical calculi [21] to a single-conclusion setting. For canonical calculi that are Horn and definite, there is no difference between the ordinary version and the single-conclusion version.

Example 2.7.9. Any Hilbert-type calculus H (without side conditions on rule applications) can be translated to a pure sequent calculus $\mathbf{G}_H$, by taking a rule of the form $\Rightarrow \psi_1 \,;\, \dots \,;\, \Rightarrow \psi_n \,/\, \Rightarrow \psi$ for each Hilbert-type derivation rule $\psi_1, \dots, \psi_n \,/\, \psi$ (where $n = 0$ for axioms). For example, the axiom $\varphi \supset (\psi \supset \varphi)$ is translated to the axiomatic rule $/ \Rightarrow \varphi \supset (\psi \supset \varphi)$, and *modus ponens* is translated to the rule $\Rightarrow \varphi \,;\, \Rightarrow \varphi \supset \psi \,/\, \Rightarrow \psi$. It is easy to show that ψ is derivable from Γ in H iff $\vdash^{=1}_{\mathbf{G}_H} \Gamma \Rightarrow \psi$. Since $\mathbf{G}_H$ is always Horn and definite, the latter holds iff $\vdash_{\mathbf{G}_H} \Gamma \Rightarrow \psi$.

Chapter 3

SAT-based Decision Procedure

As shown in Section 2.5, the derivability problem of a given pure calculus is decidable whenever the calculus is $\circledcirc$-k-analytic for some $\circledcirc$ and k. However, the mere decidability of this problem does not provide any efficient decision procedure. Moreover, a great deal of ingenuity is often required for developing proof-search algorithms for sequent calculi (see, e.g., [45]). In this chapter we show that for $\circledcirc$-k-analytic pure calculi, it is possible to replace proof-search by SAT solving. This is done using a polynomial-time reduction from the derivability problem to the complement of SAT. While SAT is NP-complete, it is considered "easy" when it comes to real-world applications. Indeed, there are many off-the-shelf SAT solvers, that, despite an exponential worst-case time complexity, are considered extremely efficient (see, e.g., [56]). The reduction that we propose here is *uniform*, and does not employ any calculus-specific optimizations. Instead, it shifts all heuristics and optimizations to the realm of SAT-solvers.

In Section 3.1 we present a reduction the derivability problem in a given $\circledcirc$-k-analytic pure calculus to the complement of SAT, and prove its correctness, as well as its polynomial time complexity. In Section 3.2 we identify a subfamily of $\circledcirc$-k-analytic pure calculi for which the reduction induces a linear time decision procedure. Finally, in Section 3.3 we describe an implementation of the reduction, and discuss its performance in Section 3.3.3.

Publications Related to this Chapter

This chapter is mainly based on [72, 103, 104]. However, the results of [72] are generalized here to the notion of analyticity used in Chapter 2.

3.1 A Polynomial Reduction from Derivability to UNSAT

We utilize the semantic view of pure sequent calculi (see Section 2.3), and reduce the derivability problem in a given analytic sequent calculus to small countermodel search, which can be be easily given in terms of a SAT instance.

SAT instances are taken to be CNFs represented as sets of clauses, where clauses are sets of literals (that is, atomic variables and their negations, denoted by overlines). The set $\{x_\psi \mid \psi \in \mathcal{L}\}$ is used as the set of atomic variables in the SAT instances. The translation of sequents to SAT instances is naturally given by:

$$\texttt{SAT}^{+}(\Gamma \Rightarrow \Delta) = \{\{\overline{x_\psi} \mid \psi \in \Gamma\} \cup \{x_\psi \mid \psi \in \Delta\}\}$$
$$\texttt{SAT}^{-}(\Gamma \Rightarrow \Delta) = \{\{x_\psi\} \mid \psi \in \Gamma\} \cup \{\{\overline{x_\psi}\} \mid \psi \in \Delta\}$$

This translation captures the semantic interpretation of sequents. Indeed, given an $\mathcal{L}$-bivaluation v and a classical assignment u that assigns true to x_ψ iff $v(\psi) = 1$, we have that for every $\mathcal{L}$-sequent s: $v(s) = 1$ iff u satisfies $\texttt{SAT}^{+}(s)$, and $v(s) = 0$ iff u satisfies $\texttt{SAT}^{-}(s)$. Now, for a bivaluation to be **G**-legal for some calculus **G**, it should satisfy the semantic restrictions arising from the rules of **G**. These restrictions can be directly encoded as SAT instances (as done, e.g., in [67] for the particular case of the classical truth tables).

In the reduction that we present, we assume that the given calculus is axiomatic. If it is not, it can be transformed into an equivalent axiomatic calculus (see Theorem 2.4.3).

Definition 3.1.1. The SAT instance associated with a given axiomatic $\mathcal{L}$-calculus **G**, a subset $\circledcirc$ of $\diamondsuit^1_{\mathcal{L}}$, a natural number $k \geq 0$, a set of $\mathcal{L}$-sequents S and an $\mathcal{L}$-sequent s, denoted $\texttt{SAT}^{\circledcirc}_{k}(\mathbf{G}, S, s)$, consists of the following clauses:

1. $\texttt{SAT}^{+}(s')$ for every $s' \in S$
2. $\texttt{SAT}^{-}(s)$
3. $\texttt{SAT}^{+}(\sigma(s'))$ for every sequent s' and substitution σ such that $\emptyset / s' \in \mathbf{G}$ and $\sigma(\mathit{frm}(s')) \subseteq \mathit{sub}^{\circledcirc}_{k}(S \cup \{s\})$

Example 3.1.2. Consider the $\{\neg\}$-1-analytic calculus $\mathbf{G}_{\text{Ł}_3}$ for Łukasiewicz three-valued logic. Its axiomatic version, $Ax(\mathbf{G}_{\text{Ł}_3})$, contains the rules $\emptyset / p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$ and $\emptyset / \neg p_2, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$ (see Example 2.4.4). Accordingly, $\texttt{SAT}^{\{\neg\}}_{1}(Ax(\mathbf{G}_{\text{Ł}_3}), S, s)$ includes the clauses $\{\overline{x_{\psi_1}}, \overline{x_{\psi_1 \supset \psi_2}}, x_{\neg\psi_1}, x_{\psi_2}\}$ and $\{\overline{x_{\neg\psi_2}}, \overline{x_{\psi_1 \supset \psi_2}}, x_{\neg\psi_1}, x_{\psi_2}\}$ for every formula of the form $\psi_1 \supset \psi_2$ in $\mathit{sub}^{\{\neg\}}_{1}(S \cup \{s\})$.

Next, we prove the correctness of this reduction.

Theorem 3.1.3. For any $\circledcirc$-k-analytic axiomatic $\mathcal{L}$-calculus $\mathbf{G}$, we have $S \vdash_{\mathbf{G}} s$ iff $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$ is unsatisfiable.

Proof. Suppose that $S \not\vdash_{\mathbf{G}} s$. By Theorem 2.3.5, there exists a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation v such that $v(S) > v(s)$. The classical assignment u that assigns true to a variable x_ψ iff $v(\psi) = 1$ satisfies $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$.

For the converse, let u be a classical assignment satisfying the SAT instance $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$. Consider the $sub_k^{\circledcirc}(S \cup \{s\})$-bivaluation v defined by $v(\psi) = 1$ iff u assigns true to x_ψ. v is $\mathbf{G}$-legal, and $v(S) > v(s)$. By Theorem 2.3.5, we have $S \not\vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(S\cup\{s\})} s$. Since $\mathbf{G}$ is $\circledcirc$-k-analytic, we may conclude that $S \not\vdash_{\mathbf{G}} s$. □

We show that this reduction is computable in *polynomial* time. The exact exponent is of course constant, and is determined by the structure of the given calculus.

Definition 3.1.4. The *$\circledcirc$-k-complexity* of an axiomatic rule $\emptyset \,/\, s$, denoted $c_k^{\circledcirc}(\emptyset \,/\, s)$, is the minimal cardinality of a set $\Gamma \subseteq frm(s)$ such that $frm(s) \subseteq sub_k^{\circledcirc}(\Gamma)$. The *$\circledcirc$-$k$-complexity* of an axiomatic calculus $\mathbf{G}$, denoted $c_k^{\circledcirc}(\mathbf{G})$, is given by $\max\{c_k^{\circledcirc}(r) \mid r \in \mathbf{G}\}$. If $\circledcirc = \emptyset$, we denote $c_k^{\circledcirc}$ by c.

Example 3.1.5. $c(\emptyset \,/\, p_1, p_2 \Rightarrow p_1 \wedge p_2) = 1$, $c(\emptyset \,/\, p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2) = 2$ and $c_1^{\{\neg\}}(\emptyset \,/\, p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2) = 1$. By repeating these calculations on the other rules of $\mathbf{LK}$ and $\mathbf{G}_{Ł_3}$, we obtain that $c(Ax(\mathbf{LK})) = 1$, $c(Ax(\mathbf{G}_{Ł_3})) = 2$, and $c_1^{\{\neg\}}(Ax(\mathbf{G}_{Ł_3})) = 1$.

Theorem 3.1.6. Let $\mathbf{G}$ be an axiomatic $\mathcal{L}$-calculus. Given S and s, the SAT instance $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$ is computable in $O(n^m)$ time, where n is the length of the string representing S and s, and m is the $\circledcirc$-k-complexity of $\mathbf{G}$.

Proof. The following algorithm computes $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$:

1. Build a parse tree for the input using standard techniques. As usual, every node represents an occurrence of some subformula in $S \cup \{s\}$.

2. Using, e.g., the linear time algorithm from [37], compress the parse tree into an ordered dag by maximally unifying identical subtrees. After the compression, the nodes of the dag represent subformulas of $S \cup \{s\}$, rather than occurrences. Hence we may identify nodes with their corresponding formulas.

3. Traverse the dag. For every $\bar{\circ} \in \circledcirc^{\leq k}$ and node v that has a parent that is labeled with an element from $\diamondsuit_{\mathcal{L}} \setminus \circledcirc$, add a new path ending with v, such that the concatenation of the path is $\bar{\circ}$, if such a path does not exist. To do this it is possible to maintain in each node v a constant-size list of all elements of $\circledcirc^{\leq k}$ that end with v. Note that after these additions, the nodes of the dag one-to-one correspond to $sub_k^{\circledcirc}(S \cup \{s\})$.

4. $\mathtt{SAT}^{-}(s)$ is obtained by traversing the dag and generating $\{x_\psi\}$ for every ψ on the left side of s and $\{\overline{x_\psi}\}$ for every ψ on the right side of s.

5. For every $s' \in S$, $\mathtt{SAT}^{+}(s')$ is obtained similarly.

6. $\bigcup\left\{\mathtt{SAT}^{+}(\sigma(s')) \mid \emptyset \,/\, s' \in \mathbf{G}, \sigma(\mathit{frm}(s')) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})\right\}$ is generated by looping over all rules in $\mathbf{G}$. For each rule $\emptyset \,/\, s'$, let $\varphi_1, \dots, \varphi_{m'}$ ($m' \le m$) be formulas such that $\mathit{frm}(s')$ consists only of $\circledcirc$-k-subformulas of $\varphi_1, \dots, \varphi_{m'}$. Go over all m'-tuples of nodes in the dag. For each m' nodes $v_1, \dots, v_{m'}$ check whether $v_1, \dots, v_{m'}$ match the pattern given by $\varphi_1, \dots, \varphi_{m'}$, and if so, construct a mapping h from the formulas in $\mathit{sub}_k^{\circledcirc}(s')$ to their matching nodes. Then, construct a clause consisting of a literal $\overline{x_{h(\varphi)}}$ for every φ on the left side of s', and a literal $x_{h(\varphi)}$ for every φ on the right side of s'. Note that only a constant depth of the sub-dags rooted at $v_1, \dots, v_{m'}$ is considered—that is the complexity of $\varphi_1, \dots, \varphi_{m'}$, in addition to nodes on paths that represent elements of $\circledcirc^{\le k}$. These are independent of the input $S \cup \{s\}$. To see that we generate exactly all required clauses, note that a substitution σ satisfies $\sigma(\mathit{frm}(s')) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})$ iff $\sigma(\{\varphi_1, \dots, \varphi_{m'}\}) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})$. Thus, there exists a substitution σ satisfying $\sigma(\mathit{frm}(s')) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})$ iff there are m' nodes matching the patterns given by $\varphi_1, \dots, \varphi_{m'}$.

Steps 1,2,3,4 and 5 require linear time. Each pattern matching in step 6 is done in constant time, and so handling a rule r with $c_k^{\circledcirc}(r) = m$ takes $O(n^m)$ time. Thus step 6 requires $O(n^m)$ time. □

Remark 3.1.7. We employ the same standard computation model of analysis of algorithms used in [44]. An efficient implementation of this algorithm cannot afford the variables x_ψ to literally include a full string representation of ψ. Thus we assume that each node has a key that can be printed and manipulated in constant time (e.g., its memory address).

Corollary 3.1.8. For any $\circledcirc$-k-analytic pure calculus $\mathbf{G}$, the derivability problem for $\mathbf{G}$ is in co-NP.

3.2 Linear Time Decision Procedure

Theorem 3.1.6 shows that when $c_k^{\circledcirc}(\mathbf{G}) = 1$, the SAT instance $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$ can be generated in *linear* time. In such cases, it is natural to identify calculi whose SAT instances can be also decided in linear time. This is the case, for example, for SAT instances consisting of *Horn clauses* [47] – clauses with at most one positive literal. It is routine to verify that for every Horn calculus (see Definition 2.7.4), its axiomatic version consists solely of single-conclusion sequents. In this case, the semantic constraints that are induced by the calculus produce Horn clauses in the reduction above. When such calculi also have a $\circledcirc$-k-complexity of 1, the entire decision procedure (reducing + solving) can be done in linear time. Formally:

Proposition 3.2.1. Let $\mathbf{G}$ be a $\circledcirc$-k-analytic Horn $\mathcal{L}$-calculus such that $c_k^{\circledcirc}(\mathbf{G}) = 1$. The derivability problem for $\mathbf{G}$ can be decided in linear time using a HORNSAT solver, for the case where the set of premises consists of single-conclusion sequents.

Proof. By Theorem 2.4.3, there exists an axiomatic calculus $\mathbf{G}'$ that is equivalent to $\mathbf{G}$. It is easy to verify that $\mathbf{G}'$ consists of single-conclusion sequents, and that when S also consists of single-conclusion sequents, $\mathtt{SAT}_k^{\circledcirc}(\mathbf{G}', S, s)$ consists of Horn clauses for every s. □

Example 3.2.2. [34] presents a reduction from the derivability problem for $\mathbf{G_{PIL}}$ to HORNSAT. This reduction is a particular instance of the reduction presented above. The derivability problem for $\mathbf{G_{EPIL}}$ (see Example 2.5.27) is also decidable in linear time, as $\mathbf{G_{EPIL}}$ is a $\emptyset$-analytic Horn calculus and $c(\mathbf{G_{EPIL}}) = 1$. We may also require that the disjunction of $\mathbf{G_{PIL}}$ is symmetric by adding the pure axiomatic rule $r = \emptyset \,/\, p_1 \vee p_2 \Rightarrow p_2 \vee p_1$. Obviously, the resulting calculus is still Horn. In addition, using the semantic criterion of Theorem 2.5.9, it is possible to show that it is $\emptyset$-analytic. However, $c(r) = 2$, and so the resulting calculus will no longer have a complexity measure of 1, but of 2. The algorithm described in Theorem 3.1.6 will then require quadratic time, and thus the entire decision procedure will also require quadratic time.

Example 3.2.3. The calculus $\mathbf{G}_{DY}$ (Example 2.5.19) is Horn, and the complexity of its axiomatic version is 1. and $c(\mathbf{G}_{DY}) = 1$. Thus, the reduction to SAT induces a linear time decision procedure for it.

Example 3.2.4. The linear time fragment of dual-Horn clauses can be utilized as well. For example, consider the ($\emptyset$-analytic) calculus $\mathbf{G_{PIL}}_d$ that consists of the rules $(\vee \Rightarrow)$, $(\Rightarrow \vee)$, $(\wedge \Rightarrow)$ of **LK** and the following rules for "dual primal implication":

$$(\prec \Rightarrow) \qquad p_1 \Rightarrow \,/\, p_1 \prec p_2 \Rightarrow \qquad\qquad (\Rightarrow \prec) \qquad \Rightarrow p_1 \,;\, p_2 \Rightarrow \;/\; \Rightarrow p_1 \prec p_2$$

This calculus is coherent and $\emptyset$-directed. By Theorem 2.5.16, it is $\emptyset$-analytic. Also, the complexity measure of its axiomatic version is 1, and the SAT-instance generated by Definition 3.1.1 consists of dual-Horn clauses. Thus the derivability problem for $\mathbf{G_{PIL}}_d$ can be decided in linear time.

3.3 Implementation of The Decision Procedure

In this section we describe our implementation of the decision procedure, in a tool called *Gen2sat*, available in [102]. *Gen2sat* is implemented in Java, and uses the SAT-solver sat4j [75]. For a given calculus **G** and an input sequent s, *Gen2sat* decides whether s is derivable in **G**. If s is not derivable, the tool provides a countermodel. If it is derivable, the tool provides a sub-calculus in which s is already derivable (using the explanation for the lack of a countermodel given by sat4j).

We start with a high-level description of the Gen2sat and its usage in Section 3.3.1. Section 3.3.2 includes some implementation details. We conclude with Section 3.3.3, where we compare the performance of Gen2sat to that of MetTeL, which is another generic tool for non-classical logics.

3.3.1 Features and Usage

There is a variety of tools developed in the spirit of logic engineering, such as MultLog [30], TINC [40], NESCOND [83], LoTREC [53], and finally MetTeL [98], which generates a theorem prover for a given logic, as well as a source code for the prover, that can be further optimized. The aim of Gen2sat is similar, allowing the user to specify the logic and automatically obtain a decision procedure. In contrast to MetTeL which uses tableaux, in Gen2sat the logic is given by a sequent calculus. Moreover, the core of Gen2sat is the above reduction to SAT, thus it leaves the "hard work" and heuristic considerations of optimizations to state of the art SAT solvers, allowing the user to focus solely on the *logical* considerations. The current implementation of Gen2sat supports $\circledcirc$-k-analytic pure calculi for $k = 1$, and derivation problems with no premises (e.g., does $\vdash_{\mathbf{G}} s$?).

Gen2sat can be run both via a web interface and from the command line. In the web-based version the user fills in a form; in the command line a property file is passed as an argument. From the command line, Gen2sat is called by: `java -jar gen2sat.jar <path>`. The form has the following fields:

Connectives A comma separated list of connectives, each specified by its symbol and arity, separated by a colon.

Rules Each pure rule is specified in a separate line that starts with "rule:". The rule itself has two parts separated by "/": the premises, which is a semicolon separated list of sequents, and the conclusion, which is a sequent.

Analyticity For the usual subformula property this field is left empty. For $\circledcirc$-1-analyticity, it contains a comma separated list of the elements of $\circledcirc$.

Input file

```
connectives: P:2, E:2
rule: =>a; =>b / =>aPb
rule: a=> / aPb=>
rule: b=> / aPb=>
rule: =>a; =>b / =>aEb
rule: =>b; a=> / aEb=>
analyticity:
inputSequent: (((m1 P m2 ) E k) E k),k=>m1
```

Output

```
provable
There's a proof that uses only these rules:
[=>b; a=> / a E b=>, a=> / a P b=>]
```

Input file

```
connectives: AND:2,OR:2,IMPLIES:2,TOP:0
rule: =>p1; =>p2 / =>p1 AND p2
rule: p1,p2=> / p1 AND p2=>
rule: =>p1,p2 / =>p1 OR p2
rule: =>p2 / =>p1 IMPLIES p2
rule: =>p1; p2=> / p1 IMPLIES p2=>
rule: / => TOP
analyticity:
inputSequent: =>p IMPLIES p
```

Output

```
unprovable
Countermodel:
p=false, p IMPLIES p=false
```

Figure 3.1: Examples for inputs to Gen2sat

Input sequent The sequent whose derivability should be decided.

The web-based version includes predefined forms for some propositional logics (e.g. classical logic, primal infon logic and more). In addition, it allows the user to import sequent calculi from *Paralyzer*.[1]

If the input sequent s is unprovable in the input calculus $\mathbf{G}$, Gen2sat outputs a countermodel, in the form of a $\mathbf{G}$-legal $sub_1^{\circledcirc}$-bivaluation v such that $v(s) = 0$. If it is provable, Gen2sat recovers a sub-calculus in which the sequent is already provable (the full proof is unobtainable due to the semantic approach of Gen2sat). Thus, for a provable sequent Gen2sat outputs a subset of rules that suffice to prove the sequent.

Figure 3.1 presents examples for the usage of Gen2sat. In the left example, the input contains a sequent calculus for the Dolev-Yao intruder model (Example 2.5.19). The connectives E and P correspond to encryption and pairing. The sequent is provable, meaning that given two messages m_1 and m_2 that are paired and encrypted twice with k, the intruder can discover m_1 if it knows k. In the right example, the input file contains a sequent calculus for primal infon logic (Example 2.2.10), where the implication connective is not reflexive, and hence the input sequent is unprovable. Both calculi are $\emptyset$-analytic, and hence the analyticity field is left empty.

3.3.2 Implementation Details

Gen2sat is implemented in Java and uses sat4j [75] as its underlying SAT solver. Since its algorithm is a "one-shot" reduction to SAT, no changes are needed in the SAT solver

[1]Paralyzer is a tool that transforms Hilbert-type calculi of a certain general form into equivalent analytic sequent calculi. It was described in [41] and can be found at `http://www.logic.at/people/lara/paralyzer.html`.

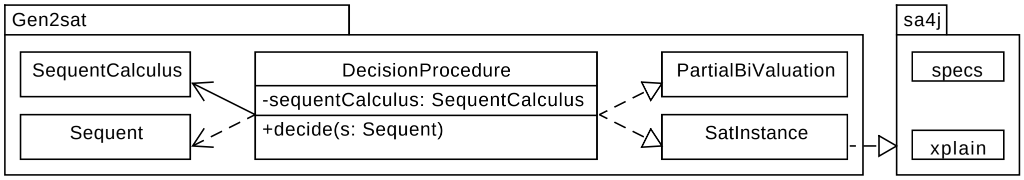

Figure 3.2: A partial class diagram of Gen2sat

itself. In particular, sat4j can be easily replaced by other available solvers. Figure 3.2 includes a partial class diagram of Gen2sat. The two main modules of sat4j that we use are `specs`, which provides the solver itself, and `xplain`, which searches for an unsat core. The main class of Gen2sat is `DecisionProcedure`, that is instantiated with a specific `SequentCalculus`. Its main method `decide` checks whether the input sequent is provable. Given a `Sequent` s and a calculus $\mathbf{G}$, `decide` first transforms $\mathbf{G}$ to an equivalent axiomatic calculus, and then generates a `SatInstance` stating that s has a countermodel, by applying the rules of the calculus on the relevant formulas, as described in Theorem 3.1.6. `SatInstance` is the only class that uses sat4j directly, and thus it is the only class that will change if another SAT solver is used.

For satisfiable instances, the `specs` module returns a satisfying assignment, which is directly translated to a countermodel in the form of a `PartialBivaluation`. For unsatisfiable instances, the `xplain` module generates a subset of clauses that is itself unsatisfiable. Tracking back to the rules that induced these clauses, we are able to recover a smaller sequent calculus in which s is already provable. Note however, that the smaller calculus need not be analytic, and then the correctness, that relies on Theorem 3.1.3 might fail. Nevertheless, correctness is preserved in this case, as the "if" part of Theorem 3.1.3 holds even for non-analytic calculi. Thus, although Gen2sat does not provide a proof of the sequent, we do obtain useful information about the rules that were used in it.

3.3.3 Performance

Gen2sat is generic, as the sequent calculus in which derivability should be checked is a part of the input, and not a part of the code of the tool. The purpose of this section is to check whether the approach that underlies its implementation is efficient. For this purpose, we compared the performance of Gen2sat with that of another tool that has a similar purpose, called MetTeL [98]. Like Gen2sat, MetTeL is a generic prover for non-classical logics. However, the approach undertaken in MetTeL is completely different. First, it operates on tableau calculi, rather than on sequent calculi. Second, in the core of MetTeL there are efficient implementations of useful techniques for tableau proof-search, that are applicable for a wide variety of tableau calculi. This is in contrast to

Gen2sat's underlying semantic approach, that goes through a reduction to SAT.

We describe an evaluation performed on Gen2sat and MetTeL, aimed at checking to what extent these two generic tools can be used in practice. The results, as we shall see, are encouraging, as both tools perform well on relatively large problem sets. Also, it seems that some improvements can be utilized to get even better running times, without sacrificing genericity. Another general goal of this evaluation is to identify families of logics, or logic problems that are executed better in each of the tools.

As a case study, we consider Łukasiewicz three-valued logic $Ł_3$ from Example 2.2.8, that employs three truth values: t, f, and u (for "true", "false" and "undetermined", respectively). Valid formulas in $Ł_3$ are the formulas that are always assigned the truth value t. Its implication-free fragment is identical to Kleene's three-valued logic [66]. As a consequence, it does not have implication-free valid formulas. $Ł_3$ is decidable, like every propositional logic that is defined using a finite-valued logical matrix.

We start by describing the different implementations of this logic in both tools. This is followed by a description of the problems that were tested. Then, we provide the actual results of this case study, and discuss the various differences that were observed between the tools.

Calculi

The paper [59] presents a tableau calculus for $Ł_3$ (henceforth denoted $\mathcal{T}$ for "Tableau"), which is available in the online version of MetTeL. The sequent calculus $\mathbf{G}_{Ł_3}$ for this logic from Example 2.2.8 (henceforth denoted $\mathcal{S}$ for "Sequent"), can easily be implemented in Gen2sat, as it is $\{\neg\}$-1-analytic and pure. The most straightforward comparison would be between MetTeL's implementation of the first and Gen2sat's implementation of the second. However, since our goal is to compare the underlying *approaches* rather than specific *calculi*, we believe that it is important to use the same calculus in both frameworks. For this purpose, we have translated the sequent calculus $\mathcal{S}$ to a tableau calculus (henceforth denoted $\mathcal{ST}$ for "Sequent–Tableau"), using the general technique outlined in [12]. To summarize, we have considered three implementations of $Ł_3$:

- $\mathcal{T}$ the tableau calculus from [59], implemented in MetTeL. Its specification in MetTeL is shown in Figure 3.3.
- $\mathcal{S}$ the sequent calculus from [13], implemented in Gen2sat. Its specification in Gen2sat is shown in Figure 3.4.
- $\mathcal{ST}$ a translation of $\mathcal{S}$ to a tableau calculus, implemented in MetTeL. Its specification in MetTeL is shown in Figure 3.5.

Note that the calculus $\mathcal{T}$ is *three-valued* (corresponding to the three values of $Ł_3$). In order to check the validity of a given formula φ, one needs to apply $\mathcal{T}$ both on $F:\varphi$ and

```
specification Lukasiewicz;
syntax Lukasiewicz{
     sort valuation;
     sort formula;
     valuation true = 'T' formula;
     valuation unknown = 'U' formula;
     valuation false = 'F' formula;
     formula true = 'true';
     formula false = 'false';
     formula negation = '~' formula;
     formula conjunction = formula '&' formula;
     formula disjunction = formula '|' formula;
     formula implication = formula '->' formula;
}
tableau Lukasiewicz{
     T P  F P /  priority 0 $;
     T P  U P /  priority 0 $;
     U P  F P /  priority 0 $;
     U P  F P /  priority 0 $;
     T ~P / F P priority 1 $;
     U ~P / U P priority 1 $;
     F ~P / T P priority 1 $;
     T (P & Q) / T P  T Q priority 1 $;
     F (P & Q) /  F P $| F Q priority 2 $;
     U (P & Q) /  T P  U Q $| U P  T Q $| U P  U Q  priority 3 $;
     T (P | Q) / T P $| T Q priority 2 $;
     F (P | Q) /  F P  F Q priority 1 $;
     U (P | Q) /  F P  U Q $| U P  F Q $| U P  U Q  priority 3 $;
     F (P -> Q) /  T P  F Q priority 1 $;
     U (P -> Q) / U P  F Q $|  T P  U Q  priority 2 $;
     T (P -> Q) / T Q $|  F P $| U P  U Q  priority 3 $;
     T false / priority 0 $;
     U false / priority 0 $;
     U true / priority 0 $;
     F true / priority 0 $;
}
```

Figure 3.3: Definition of $\mathcal{T}$ in MetTeL

```
connectives: &:2,|:2,->:2,!:1
rule: =>p1; =>p2 / => p1 & p2
rule: p1,p2=> /  p1 & p2 =>
rule: =>p1,p2 / => p1 | p2
rule: p1=>; p2=> / p1 | p2 =>
rule: a=> / !! a=>
rule: =>a / => !! a
rule: !A, !B=> / !(A | B)=>
rule: =>!A; =>!B / => !(A | B)
rule: !A=>; !B=> / !(A & B)=>
rule: =>!A, !B / => !(A & B)
rule: /! A, A=>
rule: ! A => ; B =>; => A,! B / A -> B=>
rule: A=>B; ! B=>! A / => A -> B
rule: A, ! B=> / ! (A -> B)=>
rule: =>A; =>!B / => ! (A -> B)
analyticity: !
details: false
```

Figure 3.4: Definition of $\mathcal{S}$ in Gen2sat

on $U : \varphi$. Only if both turn out to be unsatisfiable, then the formula is valid. Obviously, once one of them is found satisfiable, there is no need to check the second. In contrast, when using $\mathcal{S}$, checking the validity of a formula φ amounts to applying the calculus once on the sequent $\Rightarrow \varphi$.

Taking performance rather than verbosity in Gen2sat for this evaluation, we have compiled a non-verbose version of the tool, that does not include any information besides a final answer of whether the sequent is derivable or not. The reason for this is that when profiling Gen2sat, we noticed that the bottleneck is in the `specs` module of sat4j, which is only needed to explain the result of the tool, but not to decide derivability. These considerations lead to a total of five variants of the implementations that we consider:

- $\mathcal{S}_m$ the implementation of $\mathcal{S}$ in the non-verbose version of Gen2sat.
- $\mathcal{S}$ the implementation of $\mathcal{S}$ in the usual (slower) version of Gen2sat.
- $\mathcal{ST}$ the implementation of $\mathcal{ST}$ in MetTeL.
- $\mathcal{T}$-F the implementation of $\mathcal{T}$ in MetTeL, applied on inputs of the form $F : \varphi$.
- $\mathcal{T}$-U the implementation of $\mathcal{T}$ in MetTeL, applied on inputs of the form $U : \varphi$.

Problem Classes

We have generated instances of different sizes from two classes of problems. The first is a class of random problems, generated by MetTeL itself. The second class is more

```
specification ST;
syntax ST{
     sort valuation;
     sort formula;
     valuation true = 'T' formula;
     valuation false = 'F' formula;
     formula negation = '!' formula;
     formula conjunction = formula '&' formula;
     formula disjunction = formula '|' formula;
     formula implication = formula '->' formula;
}

tableau ST{
     T P  F P / priority 0 $;
     T (P & Q) / T P  T Q priority 1 $;
     F (P & Q) /  F P $| F Q priority 2 $;
     T (P | Q) / T P $| T Q priority 2 $;
     F (P | Q) /  F P  F Q priority 1 $;
     F (!(!(P))) / F P priority 1 $;
     T (!(!(P))) / T P priority 1 $;
     F !(P) / T P priority 1 $;
     T (!(P | Q)) / T !P  T !Q priority 1 $;
     F (!(P | Q)) / F !P $| F !Q priority 2 $;
     T (!(P & Q)) / T !P $| T !Q priority 2 $;
     F (!(P & Q)) / F !P  F !Q priority 1 $;
     T (P->Q) / F P  F !Q $| F P  T !P $| T Q  F !Q $| T Q  T !P priority 3 $;
     F (P->Q) / T P  F Q  F !P  $| T !Q  F Q  F !P priority 2 $;
     T !(P->Q) / T P  T !Q priority 1 $;
     F !(P->Q) / F P $| F !Q priority 2 $;
}
```

Figure 3.5: Definition of $\mathcal{ST}$ in MetTeL

structured, and provides more control on the derivability of sequents.

- **Random Problems**: Using the random problem generator included with MetTeL, we have generated random formulas of depths 10, 15 and 20, with 50 formulas for each depth. Most random formulas turned out to be not valid in Ł$_3$, and so this benchmark mainly tests the tools for non-valid formulas.

- **Rothenberg's Problems**: In order to gain more control on the derivability of formulas, we have used the four problem classes from [88]:
  (1) $(A^n \vee B^n) \supset (A \vee B)^n$ (2) $(A \vee B)^n \supset (A^n \vee B^n)$
  (3) $(n \cdot (A \wedge B)) \supset ((n \cdot A) \wedge (n \cdot B))$ (4) $((n \cdot A) \wedge (n \cdot B)) \supset (n \cdot (A \wedge B))$
  where: $A^0 = \top$, $A^{n+1} = A \odot A^n$, $0 \cdot A = \bot$, $(n+1) \cdot A = A \oplus (n \cdot A)$, $A \odot B = \neg(\neg A \oplus \neg B)$ and $A \oplus B = \neg A \supset B$. We have only considered the language $\{\wedge, \vee, \supset, \neg\}$, and so we have defined $\top$ as $p \supset p$ and $\bot$ as $\neg\top$. We produced formulas for $0 \leq n \leq 300$ of intervals of 5. These problems were designed to test provers for infinite-valued Łukasiewicz logic [76], and are all valid in it, as well as in Ł$_3$. We generated non-valid formulas simply by adding a negation. In [88], problems of the first and third class are said to be easy, while problems of the second and forth class are considered hard. There are several explanations to this classification in [88] (e.g., hard problems require cuts and branching proofs), that are backed by experimental results of several implementations of calculi for infinite-valued Łukasiewicz logic.

Results

The experiments were made on a dedicated Linux machine with four dual-core 2.53 Ghz AMD Opteron 285 processors and 8GB RAM. The Java heap limit was 4GB. Figures 3.6 to 3.8 exhibit the main results. A timeout of 10000 ms was imposed on all problems, and anything higher appears in these figures as '11000'.

- **Random Problems**: We generated problems of depth 10, 15, and 20. For each depth, 50 problems were generated, and their running times were measured on the five different implementations. For each formula depth and implementation, the median running time over the 50 problems was calculated, along with the lowest and highest 25%. For example, when measuring the performance of $\mathcal{S}_m$ on the 50 problems of depth 10, the lowest 25% of running times were all below 14ms, and the highest 25% of them were above 33ms. The median was 22ms. The results are presented in Figure 3.6, that includes a boxplot for each formula depth and implementation, in which the rectangle marks the range between the lowest and

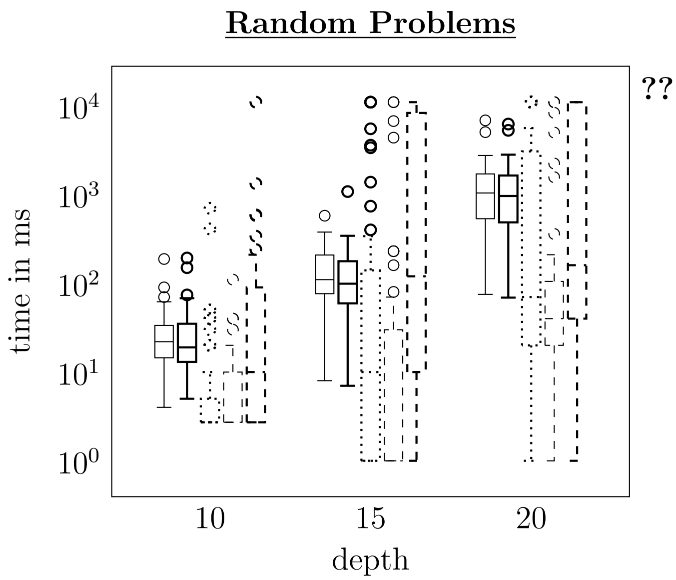

Figure 3.6: Running times on random problems

highest 25% of running times, and the middle line marks the median. Every dot above the segments is considered an *outlier*. For example, for depth 20, the medians of the aforementioned implementations are 1042ms, 964ms, 70ms, 40ms, and 160ms.

- **Rothenberg's Problems**: Figures 3.7 and 3.8 present running times on Rothenberg's problems. The x-axis shows the size of the problem (ranging from 0 to 300), and the y-axis shows running times in ms (ranging from 0ms, through 5000ms to 10000ms). Each of the 5 implementations is represented by its own line chart on the graphs. For example, on provable problems of class 1, $\mathcal{S}$ has reached the timeout around $N = 150$, and the running time for $N = 100$ was almost 5000ms.

Discussion

- **Random Problems**: On random problems, $\mathcal{ST}$ outperformed all other implementations of $Ł_3$. In particular, it performed better than $\mathcal{S}$, thus showing that when considering random problems with this particular calculus (recall that $\mathcal{ST}$ is a translation of $\mathcal{S}$ to tableau), MetTeL outperformed Gen2sat. It was also observed, however, that MetTeL implementations were less stable. Thus, although having a better result on most of the inputs, some inputs reached the timeout. As for depths that are not presented in Figure 3.6, we note that for depths higher than 20, Gen2sat always reached the timeout, while MetTeL almost never did. We stress that these results do not necessarily indicate any downside of Gen2sat's *approach*,

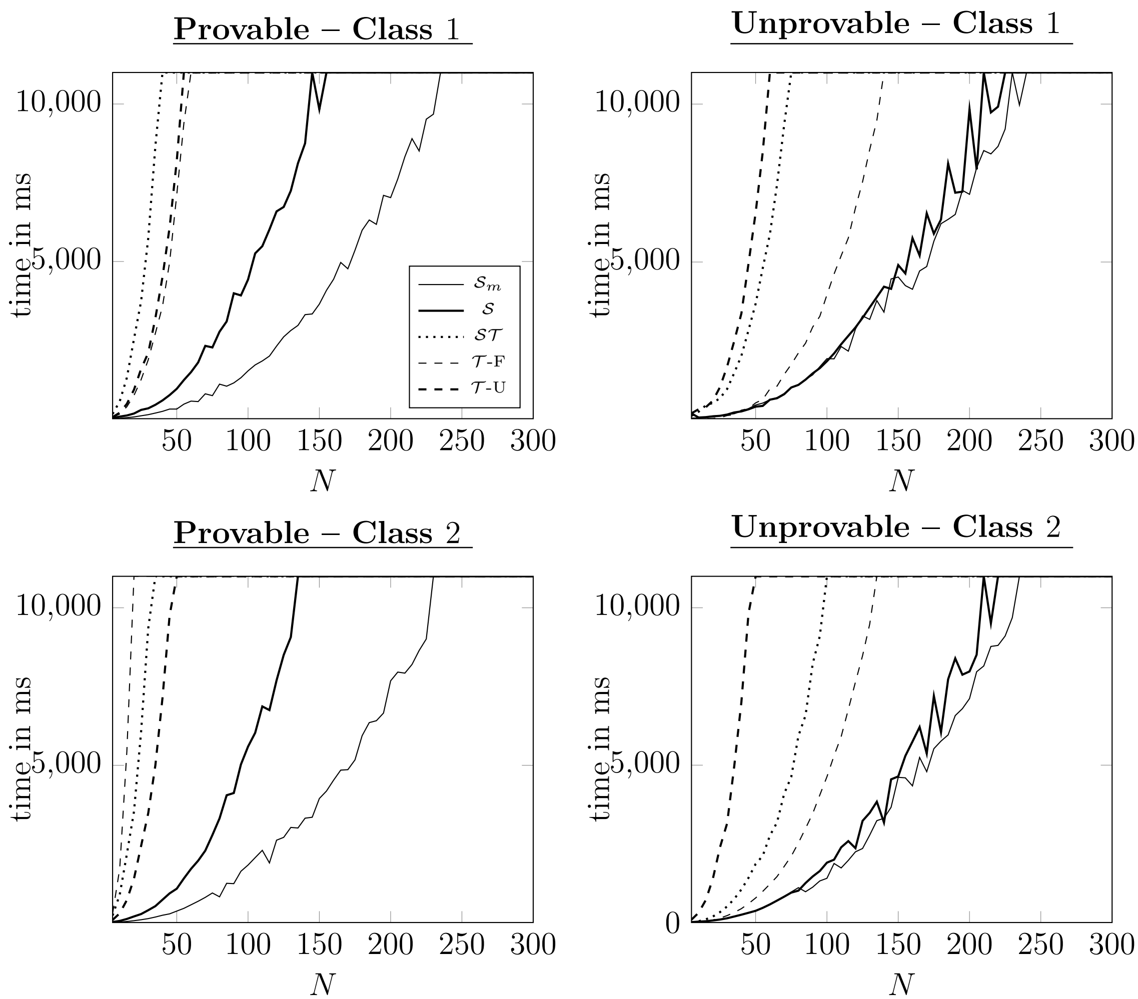

Figure 3.7: Running times on provable and unprovable instances of classes 1–2 from Rothenberg's problems. N is the size of the Rothenberg problem.

but may indicate a room for improvement in its *implementation*. Indeed, most of the running time was not inside the SAT-solver, but in the preprocessing done by Gen2sat. The reduction implemented in Gen2sat does not produce too many SAT clauses (recall that according to Theorem 3.1.6, the SAT instance for this calculus is linear in the size of the problem), but preliminary review of the code showed that the process of producing them can be improved. As an example for one of these possible improvements, we describe the effect of the `hashCode()` function in Java. A crucial point in the reduction that Gen2sat implements is that each variable corresponds to a *formula* of the input, but not to an *occurrence* of a formula. For this reason, Gen2sat employs *sets* rather than *lists*. Manipulating sets in Java involves many calls to the `hashCode()` function of each object. Since logical formulas are

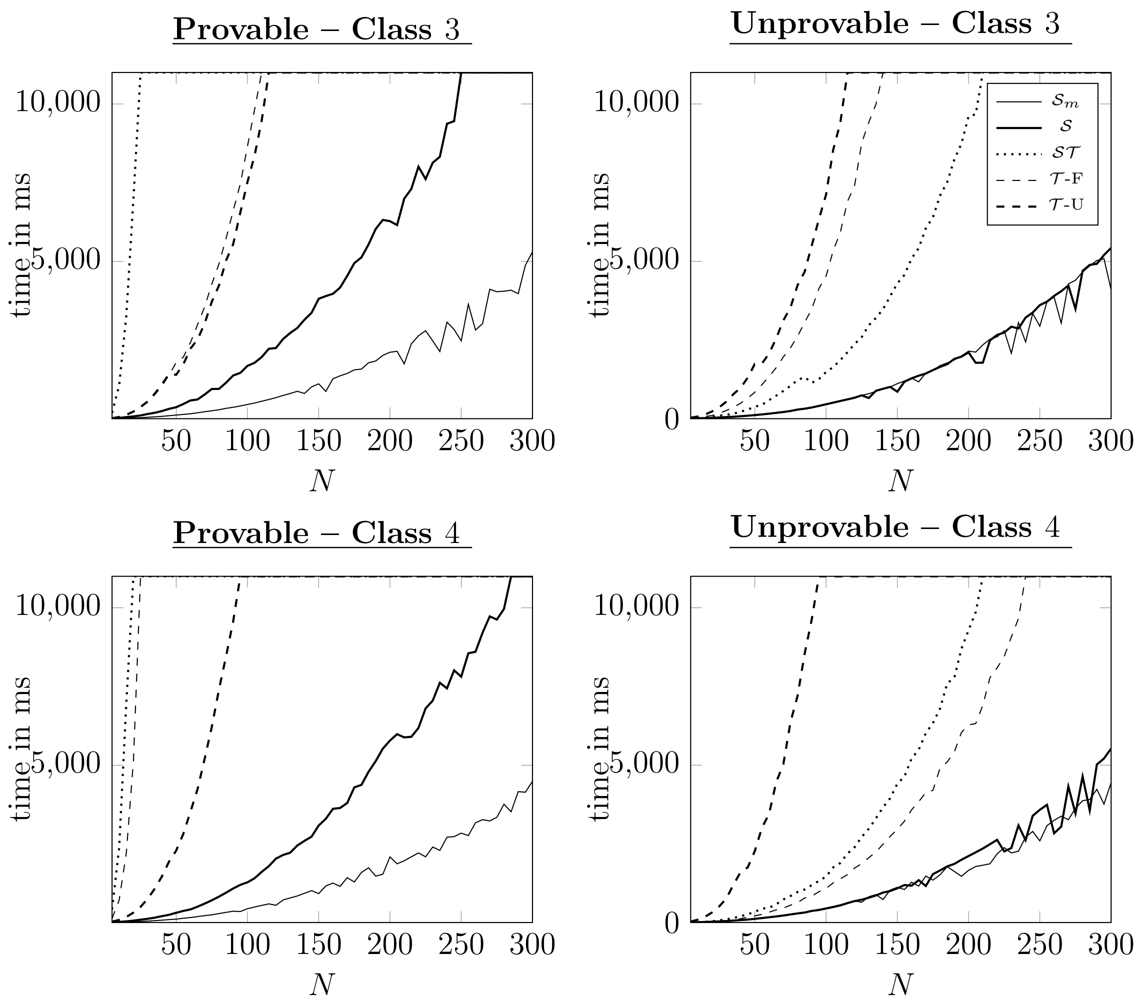

Figure 3.8: Running times on provable and unprovable instances of classes 3–4 from Rothenberg's problems. N is the size of the Rothenberg problem.

essentially trees, each such call to `hashCode()` initiates a recursion. Our preliminary profiling shows that by caching these hash codes, a significant performance improvement could be gained.

- **Rothenberg's Problems**: Here the implementation of $\mathcal{S}$ in Gen2sat performed better than all other implementations. In every problem class, both Gen2sat implementations of $\mathcal{S}$ (verbose and non-verbose) performed better than the MetTeL implementations of $\mathcal{ST}$ and $\mathcal{T}$. Notably, there was a big difference between the performances of the verbose and non-verbose versions of Gen2sat ($\mathcal{S}$ and $\mathcal{S}_m$, respectively), but only on *provable instances.* The reason is that on such instances, the largest amount of computation time is spent on calls to the `xplain` module of sat4j, that is disabled in the non-verbose version. On unprovable problems, for which this module is not called, the running-time difference between the two versions of Gen2sat was negligible.

Comparing the different implementations of MetTeL between themselves, we did not get consistent results. Focusing on $\mathcal{T}$ however, we did see that problems of the form $U : \varphi$ are processed slower than problems of the form $F : \varphi$, whenever φ was not valid. In all these formulas, it was possible to assign F to the Rothenberg formula, but not U. This is not surprising, as the rules for U in $\mathcal{T}$ involve three-way branching, that significantly increases the search space for MetTeL. When φ was valid, however, F-problems and U-problems either performed similarly, or U-problems were processed faster. Thus, when using the prover generated by MetTeL for $\mathcal{T}$, it is better to first use it with an F-label and only if it was not satisfiable, run it again with U.

On the other hand, almost all the rules in $\mathcal{ST}$ have one premise, which explains the better performance of this calculus over $\mathcal{T}$. Moreover, few fine grained priority values improved the performance for this calculus. For example, raising the priority value of `T (P->Q)` from 3 to 4, and that of `F (P->Q)` from 2 to 3 resulted in some improvement in running times.

Note that both MetTeL and Gen2sat performed better on unprovable problems than on provable ones. When using the non-verbose version of Gen2sat, however, the difference was negligible.

A plausible explanation to the gap between Gen2sat's performance on random problems and its performance on Rothenberg's problems is that the textual representation of the random problem was much longer (mainly due to the number of variables used in them, which is almost always much larger than the number of variables in Rothenberg's problems). Since Gen2sat's bottleneck was the pre-solving stage, we suspect that the actual input size has such a distinguished effect.

- **On Easy and Hard Problems**

  Figure 3.9 shows that Rothenberg's original classification of hard vs. easy problems does not hold for the provers MetTeL and Gen2sat generate for Łukasiewicz three-valued logic. In $\mathcal{S}$, $\mathcal{S}_m$ and $\mathcal{T}$-U, we have that classes 3 and 4 were easier than classes 1 and 2. In $\mathcal{ST}$, the differences were minor. Only in $\mathcal{T}$-F, the classification of [88] survived, and classes 1 and 3 were easier than classes 2 and 4.

  The fact that the original classification did not survive the transition from infinite-valued Łukasiewicz logic to the three-valued one, is not surprising. First, these are two different logics, and second, the calculi for them are much simpler than the calculi for the infinite-valued version. For example, the sequent calculus that we consider here is cut-free, while only hyper-sequent calculi that are cut-free are available in the infinite case. In the three-valued case, however, we uncovered a different classification, according to which classes 1 and 2 are harder than classes 3 and 4. This is consistent with the fact that the problems of classes 3 and 4 are less complex than those of 1 and 2. At least in Gen2sat, where the complexity of the input has a big effect on the parsing stage, this is to be expected.

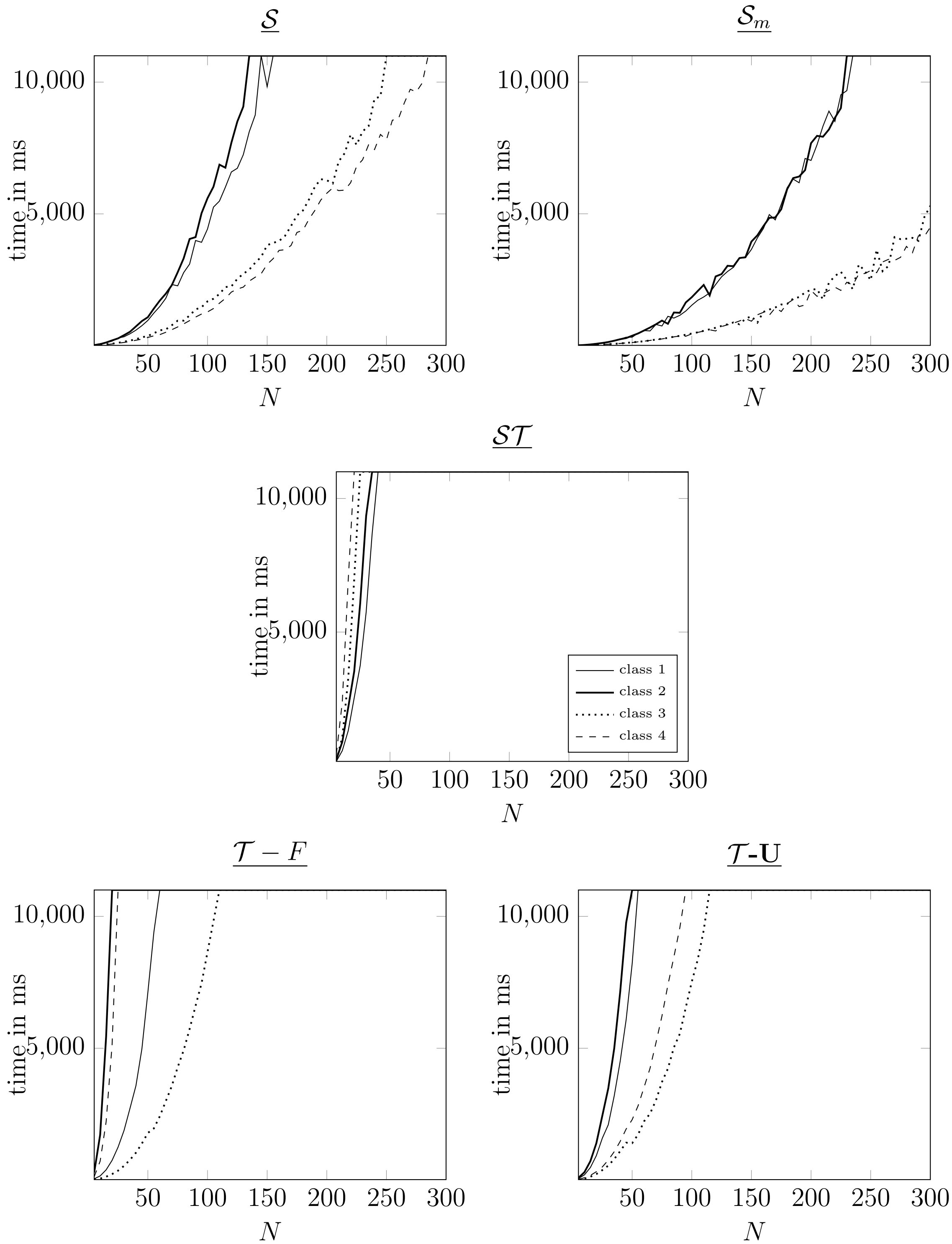

Figure 3.9: Easy and Hard Rothenberg problems: N is the size of the Rothenberg problem.

Chapter 4

Extending Pure Calculi with Modal Operators

As we have seen, the framework of pure sequent calculi is a very powerful one. When paired with the general definition of analyticity (Definition 2.5.5), effective calculi for many propositional non-classical logics are obtained. Even more, the reduction of Chapter 3 provides a concrete, uniform decision procedure for each of them. However, there are useful non-classical logics that seem to go beyond the reach of $\circledcirc$-k-analytic pure calculi. For example, the usual sequent rules for the modal operators $\Box$ and $\Diamond$ in various modal logics (e.g. K, KTB, S5 etc.) operate on the context sequent, and thus are not pure.

In this section, we consider the extension of pure sequent calculi with impure rules for introducing box-like modal operators. Our investigation is not limited to a single modal operator, and thus the systems that we study are *multimodal*. Moreover, the base logic need not be classical, and can be any logic that is described by a pure calculus. We prove a soundness and completeness theorem for the resulting calculi with respect to a Kripke-style semantics, that generalizes the bivaluations of Section 2.3. This semantics is then used in order prove the following result: if a pure calculus is $\circledcirc$-k-analytic, then it remains so after the addition of the considered rules for modal operators, excluding some degenerate cases (to be described below). The semantics is also used to extend the reduction of Chapter 3 to pure calculi that are augmented with a special kind of modal operators.

Figure 4.1 in Section 4.1 includes well-known modal rules, whose addition to arbitrary pure calculi are studied in this chapter. Section 4.2 elevates the bivaluations semantics of pure calculi from Section 2.3 to Kripke-style semantics, thus obtaining a strong soundness and completeness theorem. In Section 4.3 this theorem is used in order to prove that analyticity of a pure calculus survives the addition of the modal rules. Section 4.4

generalizes the reduction from Chapter 3 to a special kind of modal operators. Finally, in Section 4.5 we study some equivalences between some of the modal rules of Figure 4.1.

Publications Related to this Chapter

This chapter is mainly based on [72]. However, the results of [72] are generalized here to the notion of analyticity from Section 2.5, and to more modal operators.

4.1 Impure Rules for Modal Operators

In what follows, $\mathcal{L}$ continues to denote an arbitrary propositional language, and $\boxdot$ denotes a finite set of unary connectives, called *modal operators*, such that $\boxdot \cap \Diamond_{\mathcal{L}} = \emptyset$. We denote by $\mathcal{L}_{\boxdot}$ the propositional language obtained by augmenting $\mathcal{L}$ with the modal operators in $\boxdot$. The notations $\Box\mathcal{F}$ and $\boxdot\mathcal{F}$ are adaptations of the similar notations from Section 2.5, and are extended to sequents and sets of sequents in the obvious way (e.g. $\Box(\Gamma \Rightarrow \Delta) = \Box\Gamma \Rightarrow \Box\Delta$, $\boxdot S = \{\Box s \mid \Box \in \boxdot, s \in S\}$).

Unlike the connectives of $\mathcal{L}$, that may appear in any pure rule, the modal operators are manipulated according to a predefined set of rules, as given in Table 4.1. These rules are known to correspond to several classical modal logics [64, 101]. The upper part of Figure 4.1 includes the rules (K), (4), (45), (B), (B4), and (PF) (The rule (PF) is less common, and corresponds to the modal logic of functional Kripke models). The middle part of the table includes rules that correspond to the seriality axiom D. Each rule (X) of the above six rules is given a "serial version" (D_X). The bottom part of the table includes the rule (T), that corresponds to reflexive Kripke models. Note that, with the exception of (T), these are not pure rules, as their applications do not allow arbitrary context sequents. The table also includes the semantic property that corresponds to each rule. This will be used in Section 4.2, where we prove soundness and completeness.

To keep the discussion modular, we assume a given function M specifying the derivation rules that manipulate each $\Box \in \boxdot$. For every $\Box \in \boxdot$, $\mathsf{M}(\Box)$ is either a singleton consisting of one of the rules from the upper part of Table 4.1, or a pair consisting of such a rule (X) together with either (T) or a matching (D_X) rule. (Note that there is no need to consider the combination of both (T) and a (D_X)-rule, since all (D_X)-rules are derivable in the presence of (T).) We exclude the combination of (PF) and (T), as their combined frame conditions do not correspond to the derivations they allow.[1]

[1]Note that if classical negation is definable, the meaning of $\Box$ becomes trivial in this combination: on the one hand, the inclusion of (T) immediately entails the derivability of $\Box\varphi \Rightarrow \varphi$. On the other hand, the $\varphi \Rightarrow \Box\varphi$ can be proven using the rules ($\neg \Rightarrow$) and ($\Rightarrow \neg$) of **LK**, together with (T), (PF) and (CUT).

$$(\textsc{k})\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad (\textsc{pf})\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Box\Delta}\ (\textsc{functional})$$

$$(4)\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi}\ (\textsc{transitive}) \qquad (45)\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi, \Box\Delta}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi, \Box\Delta}\ \left(\substack{\textsc{transitive \&} \\ \textsc{euclidean}}\right)$$

$$(\textsc{b})\ \frac{\Gamma \Rightarrow \varphi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Delta}\ (\textsc{symmetric}) \qquad (\textsc{b}4)\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi, \Box\Delta_1, \Box\Delta_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi, \Box\Delta_1, \Delta_2}\ \left(\substack{\textsc{symmetric \&} \\ \textsc{transitive}}\right)$$

Additional seriality rules:

$$(\textsc{d}_\textsc{k})\ \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow} \qquad (\textsc{d}_\textsc{pf})\ \frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Box\Delta} \qquad (\textsc{d}_4)\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow}$$

$$(\textsc{d}_{45})\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\Delta} \qquad (\textsc{d}_\textsc{b})\ \frac{\Gamma \Rightarrow \Box\Delta}{\Box\Gamma \Rightarrow \Delta} \qquad (\textsc{d}_{\textsc{b}4})\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Box\Delta_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}$$

Additional reflexivity rule:

$$(\textsc{t})\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}$$

Figure 4.1: Application schemes of sequent rules for a modal operator $\Box$

Given a pure calculus **G** for $\mathcal{L}$, we obtain the calculus $\mathbf{G}_{\mathsf{M}}$ for $\mathcal{L}_{\boxdot}$ by augmenting **G** with the rules determined by $\mathsf{M}(\Box)$ for each $\Box \in \boxdot$. For a set $\mathcal{F} \subseteq \mathcal{L}_{\boxdot}$ of formulas, we write $S \vdash^{\mathcal{F}}_{\mathbf{G}_{\mathsf{M}}} s$ (or $S \vdash_{\mathbf{G}_{\mathsf{M}}} s$ when $\mathcal{F} = \mathcal{L}_{\boxdot}$) if there is a derivation of a sequent s from a set S of sequents in $\mathbf{G}_{\mathsf{M}}$ consisting only of $\mathcal{F}$-sequents.

Example 4.1.1. The following is an application of $(\Rightarrow \wedge)$ in $\mathcal{L}_{\boxdot}$:

$$\frac{p_3 \Rightarrow \Box p_1, \Box\Box p_3 \qquad p_3 \Rightarrow \Box p_2, \Box\Box p_3}{p_3 \Rightarrow (\Box p_1) \wedge (\Box p_2), \Box\Box p_3}$$

The context sequent in both premises is $p_3 \Rightarrow \Box\Box p_3$.

Example 4.1.2. Sequent calculi for classical modal logics are obtained by taking $\boxdot = \{\Box\}$, and augmenting **LK** with the appropriate rules for the modal operators. For example, calculi for the modal logics K and KD are obtained by respectively taking $\mathsf{M}(\Box) = \{(\text{K})\}$ or $\mathsf{M}(\Box) = \{(\text{K}), (\text{D}_\text{K})\}$. The logics $S4$ and $S5$ are captured by respectively taking $\mathsf{M}(\Box) = \{(4), (\text{T})\}$ and $\mathsf{M}(\Box) = \{(\text{B4}), (\text{T})\}$.

Example 4.1.3. The quotations employed in primal infon logic [44] are unary connectives of the form q `said`, where q ranges over a finite set of principals. Sequent rules for quotations are obtained by taking $\boxdot$ to include these connectives, and setting $\mathsf{M}(q\ \textit{said}) = \{(\text{PF}), (\text{D}_{\text{PF}})\}$ for every principal q. Augmenting the calculus $\mathbf{G_{PIL}}$ (Example 2.2.10) with M, we get that $\vdash_{\mathbf{G_{PIL}}_{\mathsf{M}}} \Gamma \Rightarrow \psi$ iff ψ is derivable from Γ in the Hilbert-type system for primal infon logic given in [44]. This can be shown by induction on the lengths of the derivations. By augmenting $\mathbf{G_{EPIL}}$ (Example 2.5.27) with M, we obtain the calculus $\mathbf{G_{EPIL}}_{\mathsf{M}}$, which augments primal infon logic (with quotations) with some natural classically valid tautologies, without compromising the linear time algorithm (see Example 4.4.7 below).

4.2 Semantics

In this section we generalize the semantics from Section 2.3 and elevate it to a Kripke-style semantics. Given a pure $\mathcal{L}$-calculus **G** and a specification M of rules for the modal operators $\boxdot$, there is a complete syntactic separation between the connectives of the original calculus and the modal operators: first, these are disjoint sets; and second, the rules of **G** contain no modal operators, and the rules of the modal operators from Figure 4.1 do not contain any elements from $\diamondsuit_{\mathcal{L}}$. We keep this separation also in the semantics. Semantics of the connectives from $\diamondsuit_{\mathcal{L}}$ is local: it will be governed by the bivaluation semantics in each possible world separately. In contrast, semantics of the modal operators is global: it will be governed by their usual meaning in Kripke models.

Similarly to what has been done for bivaluations, we consider *partial* Kripke models, in order to achieve a semantic counterpart of analyticity for the augmented calculi.

Definition 4.2.1. A *biframe* for M is a tuple $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ where:

1. W is a set of elements called *worlds*. Henceforth, we may identify $\mathcal{W}$ with this set (e.g., when writing $w \in \mathcal{W}$ instead of $w \in W$).

2. $\mathcal{R}$ is a function assigning a binary relation on W (called *accessibility relation*) to every $\Box \in \boxdot$. We write $\mathcal{R}_\Box$ instead of $\mathcal{R}(\Box)$, and $\mathcal{R}_\Box[w]$ for $\{w' \in W \mid w\mathcal{R}_\Box w'\}$. For every every $\Box \in \boxdot$, the relation $\mathcal{R}_\Box$ should have particular properties according to $\mathsf{M}(\Box)$ as indicated in Table 4.1.[2]

3. $\mathcal{V}$ is a function assigning a bivaluation to every $w \in W$, such that for every $w \in W$, $\Box \in \boxdot$ and formula ψ: if $\Box\psi \in dom(\mathcal{V}(w))$ and $\psi \in dom(\mathcal{V}(w'))$ for every $w' \in \mathcal{R}_\Box[w]$ then $\mathcal{V}(w)(\Box\psi) = \min\{\mathcal{V}(w')(\psi) \mid w' \in \mathcal{R}_\Box[w]\}$.[3] We write $\mathcal{V}_w$ instead of $\mathcal{V}(w)$.

Furthermore, if $dom(\mathcal{V}_w) = \mathcal{F}$ for every $w \in W$, we call $\mathcal{W}$ an $\mathcal{F}$*-biframe* for M.

Example 4.2.2. When $\mathsf{M}(\Box) = \{(\text{B4}), (\text{T})\}$, every biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ for M must have that $\mathcal{R}_\Box$ is reflexive, symmetric and transitive. If $(\text{D}_\text{X}) \in \mathsf{M}(\Box)$ for some (X) then $\mathcal{R}_\Box$ is serial. If $(\text{T}) \in \mathsf{M}(\Box)$ then $\mathcal{R}_\Box$ is reflexive.

Notation 4.2.3. Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ be a biframe for M. For a set $W' \subseteq W$ we denote $\min\{\mathcal{V}_{w'}(\psi) \mid w' \in W'\}$ by $\mathcal{V}_{W'}(\psi)$. This notation is extended to sequents and sets of sequents in the natural way (e.g., $\mathcal{V}_{W'}(S) = \min\{\mathcal{V}_{w'}(s) \mid s \in S, w' \in W'\}$). We denote by $dom(\mathcal{W})$ the intersection of all sets $dom(\mathcal{V}_w)$ for every $w \in W$. In particular, we have $\mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi)$ for every $w \in \mathcal{W}$ and $\psi, \Box\psi \in \mathcal{L}_\boxdot$ such that $\Box\psi \in dom(\mathcal{V}_w)$ and $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$.

Remark 4.2.4. There are two main differences between our definition of biframes and the usual definition of Kripke-style models. First, since we are in a multimodal setting, we have a separate accessibility relation for every $\Box \in \boxdot$. Moreover, the semantic properties of $\mathcal{R}_\Box$ may be different for each $\Box$. Second, for each world w, $\mathcal{V}_w$ may have a different domain. Accordingly, the semantics of $\Box$ is only enforced when the relevant formulas are in the appropriate domains. While the first difference was made to obtain generality, the second is crucial for the success of the decision procedure that we present in Section 4.4.

Next, we adopt the semantic viewpoint of pure rules in order to retain the connection between sequent calculi and their semantics, that was given in Definition 2.3.2.

[2]An accessibility relation $\mathcal{R}_\Box$ is called *transitive* if wRu and uRv imply wRv; *symmetric* if wRu implies uRw; *functional* if wRu and wRv imply $u = v$; *euclidian* if wRu and wRv imply uRv; *reflexive* if wRw for every $w \in \mathcal{W}$; and *serial* if for all $w \in W$, we have wRu for some u.

[3]Recall that $\min \emptyset = 1$.

Definition 4.2.5. A biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M is called $\mathbf{G}$-*legal* for an $\mathcal{L}$-calculus $\mathbf{G}$ if $\mathcal{V}_w$ is $\mathbf{G}$-legal for every $w \in W$ (see Definition 2.3.2).

We turn to proving soundness and completeness. Note that the rule (4) and its two variants (45) and (B4) are not sound for every partial biframe, even with the necessary frame conditions. For example, the sequent $\Box\varphi \Rightarrow \Box\Box\varphi$ is derivable from (4) using only these two formulas. However, this sequent is not valid in the following $\{\Box\varphi, \Box\Box\varphi\}$-biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$, in which $\mathcal{R}_\Box$ is transitive: $W = \{w_1, w_2\}$, $\mathcal{R}_\Box = \{\langle w_1, w_2\rangle, \langle w_2, w_2\rangle\}$, $\mathcal{V}_{w_1}(\Box\varphi) = 1$, $\mathcal{V}_{w_1}(\Box\Box\varphi) = 0$, $\mathcal{V}_{w_2}(\Box\varphi) = 0$, $\mathcal{V}_{w_2}(\Box\Box\varphi) = 0$. Thus, in the presence of any one of the rules (4), (45) and (B4), we require that $\mathcal{F}$ is "closed" with respect to $\boxdot$, that is, $\varphi \in \mathcal{F}$ whenever $\Box\varphi \in \mathcal{F}$. Note that this requirement is needed only for soundness.

Theorem 4.2.6 (Soundness)**.** Let $\mathbf{G}$ be an $\mathcal{L}$-calculus, $\mathcal{F}$ a set of $\mathcal{L}_\boxdot$-formulas, S a set of $\mathcal{F}$-sequents and s an $\mathcal{F}$-sequent. Suppose that for every $\Box \in \boxdot$, if $\{(4), (45), (\text{B}4)\} \cap \mathsf{M}(\Box) \neq \emptyset$ then $\psi \in \mathcal{F}$ whenever $\Box\psi \in \mathcal{F}$. If $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} s$ then $\mathcal{V}_W(S) \leq \mathcal{V}_W(s)$ for every $\mathbf{G}$-legal $\mathcal{F}$-biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M.

Proof. Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ be a $\mathbf{G}$-legal $\mathcal{F}$-biframe for M. Suppose that $\mathcal{V}_W(S) = 1$. We prove that $\mathcal{V}_W(s) = 1$ by induction on the length of the derivation of s from S in $\mathbf{G}_\mathsf{M}$ (that consists only of $\mathcal{F}$-sequents). If $s \in S$, or s is the conclusion of an application of a non-modal rule, then this is shown like in the proof of Theorem 2.3.5. If s is the conclusion of an application of some rule in $\mathsf{M}(\Box)$, then the proof carries on according to the identity of this rule. We explicitly prove the cases of (K), (4) and (T). The other cases are handled similarly.

1. If s is the conclusion of an application of (K) for some $\Box \in \boxdot$, then s has the form $\Box\Gamma \Rightarrow \Box\varphi$ for some $\Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} \Gamma \Rightarrow \varphi$ with a shorter derivation. Suppose for contradiction that $\mathcal{V}_w(\Box\Gamma \Rightarrow \Box\varphi) = 0$ for some $w \in W$. Then, $\mathcal{V}_w(\Box\varphi) = 0$, and $\mathcal{V}_w(\Box\psi) = 1$ for every $\psi \in \Gamma$. In particular, there must exist a world $w' \in \mathcal{R}_\Box[w]$ such that $\mathcal{V}_{w'}(\varphi) = 0$, and $\mathcal{V}_{w'}(\psi) = 1$ for every $\psi \in \Gamma$, which contradicts the induction hypothesis, according to which $\mathcal{V}_{w'}(\Gamma \Rightarrow \varphi) = 1$.

2. If s is the conclusion of an application of (4) for some $\Box \in \boxdot$, then s has the form $\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi$ for some $\Gamma_2 \subseteq \mathcal{F}$, $\varphi \in \mathcal{F}$ and Γ_1 such that $\Box\Gamma_1 \subseteq \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} \Box\Gamma_1, \Gamma_2 \Rightarrow \varphi$ with a shorter derivation. In addition, since $(4) \in \mathsf{M}(\Box)$, we have $\Gamma_1 \subseteq \mathcal{F}$ as well. Suppose for contradiction that $\mathcal{V}_w(\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi) = 0$ for some $w \in W$. Then, $\mathcal{V}_w(\Box\varphi) = 0$, and $\mathcal{V}_w(\Box\psi) = 1$ for every $\psi \in \Gamma_1 \cup \Gamma_2$. In particular, there must exist a world $w' \in \mathcal{R}_\Box[w]$ such that $\mathcal{V}_{w'}(\varphi) = 0$, and $\mathcal{V}_{w'}(\psi) = 1$ for every $\psi \in \Gamma_2$. Now, let $\psi \in \Gamma_1$ and $w'' \in \mathcal{R}_\Box[w']$. Since $(4) \in \mathsf{M}(\Box)$, we

have that $\mathcal{R}_\Box$ is transitive, which means that $w'' \in \mathcal{R}_\Box[w]$. Therefore, $\mathcal{V}_{w''}(\psi) = 1$ for every such w'', and hence $\mathcal{V}_{w'}(\Box\psi) = 1$ for every such ψ. We therefore have $\mathcal{V}_{w'}(\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi) = 0$, contradicting the induction hypothesis.

3. If s is the conclusion of an application of (T) for some $\Box \in \boxdot$, then s has the form $\Gamma, \Box\varphi \Rightarrow \Delta$ for some $\Gamma, \Delta \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} \Gamma, \varphi \Rightarrow \Delta$ with a shorter derivation. Let $w \in W$. By the induction hypothesis, $\mathcal{V}_w(\Gamma, \varphi \Rightarrow \Delta) = 1$, which means that either $\mathcal{V}_w(\psi) = 0$ for some $\psi \in \Gamma$, $\mathcal{V}_w(\psi) = 1$ for some $\psi \in \Delta$, or $\mathcal{V}_w(\varphi) = 0$. In the first two cases, we have $\mathcal{V}_w(\Gamma, \Box\varphi \Rightarrow \Delta) = 1$ as well. In the third case, since (T) $\in \mathsf{M}(\Box)$ we have that $\mathcal{R}_\Box$ is reflexive. This, together with the fact that $\mathcal{V}_w(\varphi) = 0$, means that $\mathcal{V}_w(\Box\varphi) = 0$, and hence $\mathcal{V}_w(\Gamma, \Box\varphi \Rightarrow \Delta) = 1$. $\Box$

We turn to completeness. Similarly to Theorem 2.3.5, we follow the canonical construction of a countermodel, whose worlds are maximal unprovable sequents, but adjust it to the case where only formulas from a certain set $\mathcal{F}$ are allowed in derivations. When $\mathcal{F}$ is infinite, this requires the use of ω-sequents (defined as in the proof of Theorem 2.3.5).

Theorem 4.2.7 (Completeness)**.** Let $\mathbf{G}$ be an $\mathcal{L}$-calculus, $\mathcal{F}$ a set of $\mathcal{L}_\boxdot$-formulas, S a set of $\mathcal{F}$-sequents and s an $\mathcal{F}$-sequent. If $S \nvdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} s$ then $\mathcal{V}_W(S) > \mathcal{V}_W(s)$ for some $\mathbf{G}$-legal $\mathcal{F}$-biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M.

Proof. We say that an ω-sequent $L \Rightarrow R$ is *M-S-$\mathcal{F}$-maximal unprovable* if the followings hold:

- $L \cup R \subseteq \mathcal{F}$
- $S \nvdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L \Rightarrow R$
- $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L, \psi \Rightarrow R$ for every $\psi \in \mathcal{F} \setminus L$ and $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L \Rightarrow \psi, R$ for every $\psi \in \mathcal{F} \setminus R$.

We denote the set of M-S-$\mathcal{F}$ maximal unprovable ω-sequents by $W(\mathsf{M}, S, \mathcal{F})$. Using (ID) and (CUT), it is easy to see that $L \cup R = \mathcal{F}$ and $L \cap R = \emptyset$ for every $L \Rightarrow R \in W(\mathsf{M}, S, \mathcal{F})$. In addition, it is a routine matter to show that every ω-sequent $L \Rightarrow R$ such that $L \cup R \subseteq \mathcal{F}$ and $S \nvdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L \Rightarrow R$ can be extended to a M-S-$\mathcal{F}$-maximal unprovable ω-sequent.

For every $L \Rightarrow R \in W(\mathsf{M}, S, \mathcal{F})$ and $\Box \in \mathsf{M}$, define the following sets:

$$L_1^{\Box} = \{\varphi \in \mathcal{F} \mid \Box\varphi \in L\}$$

$$L_2^{\Box} = \begin{cases} \emptyset & \{(4), (45), (\text{B4})\} \cap \mathsf{M}(\Box) = \emptyset \\ \Box\{\varphi \in \mathcal{F} \mid \Box\varphi \in L\} & \textit{otherwise} \end{cases}$$

$$R_1^{\Box} = \begin{cases} \emptyset & \{(45), (\text{B4})\} \cap \mathsf{M}(\Box) = \emptyset \\ \Box\{\varphi \in \mathcal{F} \mid \Box\varphi \in R\} & \textit{otherwise} \end{cases}$$

$$R_2^{\Box} = \begin{cases} \emptyset & \{(\text{B}), (\text{B4})\} \cap \mathsf{M}(\Box) = \emptyset \\ \mathcal{F} \cap \Box R & \textit{otherwise} \end{cases}$$

$$R_3^{\Box} = \begin{cases} \emptyset & (\text{PF}) \notin \mathsf{M}(\Box) \\ \{\varphi \in \mathcal{F} \mid \Box\varphi \in R\} & \textit{otherwise} \end{cases}$$

$$A_{L\Rightarrow R}^{\Box} = \left\{L' \Rightarrow R' \in W(\mathsf{M}, S, \mathcal{F}) \mid L_1^{\Box}, L_2^{\Box} \Rightarrow R_1^{\Box}, R_2^{\Box}, R_3^{\Box} \subseteq L' \Rightarrow R'\right\}$$

Using these sets, we define the following countermodel $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$, where:

1. $W = W(\mathsf{M}, S, \mathcal{F})$.

2. for every $\Box \in \boxdot$, we define $\mathcal{R}_\Box$ by specifying the set $\mathcal{R}_\Box[L \Rightarrow R]$ for every $L \Rightarrow R \in W$:

   (a) if $(\text{PF}) \notin \mathsf{M}(\Box)$ then $\mathcal{R}_\Box[L \Rightarrow R]$ is $A_{L\Rightarrow R}^{\Box}$.

   (b) If $(\text{PF}) \in \mathsf{M}(\Box)$ then $\mathcal{R}_\Box[L \Rightarrow R]$ consists of a single arbitrary element from $A_{L\Rightarrow R}^{\Box}$, unless $A_{L\Rightarrow R}^{\Box}$ is empty, in which case so is $\mathcal{R}_\Box[L \Rightarrow R]$.

3. For every $\psi \in \mathcal{F}$ and $L \Rightarrow R \in W$, $\mathcal{V}_{L\Rightarrow R}(\psi) = 1$ if $\psi \in L$ and $\mathcal{V}_{L\Rightarrow R}(\psi) = 0$ otherwise.

We first show that $\mathcal{V}_W(S) > \mathcal{V}_W(s)$. For every $\Gamma \Rightarrow \Delta \in S$ and $L \Rightarrow R \in W$, since $S \vdash_{\mathbf{G}_{\mathsf{M}}}^{\mathcal{F}} \Gamma \Rightarrow \Delta$ and $S \nvdash_{\mathbf{G}_{\mathsf{M}}}^{\mathcal{F}} L \Rightarrow R$, there exist some $\psi \in \Gamma \setminus L$ (and then $\mathcal{V}_{L\Rightarrow R}(\psi) = 0$) or $\psi \in \Delta \setminus R$ (and then $\mathcal{V}_{L\Rightarrow R}(\psi) = 1$). Either way, $\mathcal{V}_{L\Rightarrow R}(\Gamma \Rightarrow \Delta) = 1$. In addition, since $s \subseteq L_s \Rightarrow R_s$ for some $L_s \Rightarrow R_s \in W(\mathsf{M}, S, \mathcal{F})$, we have $\mathcal{V}_{L_s\Rightarrow R_s}(s) = 0$.

It remains to prove that $\mathcal{W}$ is a **G**-legal $\mathcal{F}$-biframe for M.

- **G**-legal: For every $L \Rightarrow R \in W$, the bivaluation $\mathcal{V}_{L\Rightarrow R}$ is shown to be **G**-legal similarly to the proof of Theorem 2.3.5.

- biframe: let $\Box \in \boxdot$, $\psi, \Box\psi \in \mathcal{F}$, and $L \Rightarrow R \in W$. If $\mathcal{V}_{L\Rightarrow R}(\Box\psi) = 1$ and $L' \Rightarrow R' \in \mathcal{R}_\Box[L \Rightarrow R]$, then we have $\Box\psi \in L$, which means that $\psi \in L'$, and hence $\mathcal{V}_{L'\Rightarrow R'}(\psi) = 1$. For the converse, suppose that $\mathcal{V}_{L\Rightarrow R}(\Box\psi) = 0$. Then

$\Box\psi \in R$. We prove that $S \not\vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L_1^\Box, L_2^\Box \Rightarrow \psi, R_1^\Box, R_2^\Box, R_3^\Box$, extend this sequent to an element $L' \Rightarrow R'$ of $\mathcal{R}_\Box[L \Rightarrow R]$, and then obtain that $\mathcal{V}_{L' \Rightarrow R'}(\psi) = 0$ (as $\psi \in R'$). Assume for contradiction that $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L_1^\Box, L_2^\Box \Rightarrow \psi, R_1^\Box, R_2^\Box, R_3^\Box$. Then there exist finite $\Gamma_1 \subseteq L_1^\Box$, $\Gamma_2 \subseteq L_2^\Box$, $\Delta_1 \subseteq R_1^\Box$, $\Delta_2 \subseteq R_2^\Box$ and $\Delta_3 \subseteq R_3^\Box$, such that $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} \Gamma_1, \Gamma_2 \Rightarrow \psi, \Delta_1, \Delta_2, \Delta_3$. Let $\Delta_2' = \{\varphi \in \mathcal{F} \mid \Box\varphi \in \Delta_2\}$. By applying the only rule in $\mathsf{M}(\Box) \cap \{(\textsc{k}), (4), (45), (\textsc{b}), (\textsc{b}4), (\textsc{pf})\}$, we obtain $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\psi, \Delta_1, \Delta_2', \Box\Delta_3$. Clearly, $\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\psi, \Delta_1, \Delta_2', \Box\Delta_3 \subseteq L \Rightarrow R$, and so $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L \Rightarrow R$, which is a contradiction. Now, $L_1^\Box, L_2^\Box \Rightarrow \psi, R_1^\Box, R_2^\Box, R_3^\Box$ can be extended to some element $L' \Rightarrow R'$ of W, and every such extension is an element of $A^\Box_{L \Rightarrow R}$. Thus we have some $L' \Rightarrow R' \in \mathcal{R}_\Box[L \Rightarrow R]$ such that $\psi \in R'$, and so $\mathcal{V}_{L' \Rightarrow R'}(\psi) = 0$.

- for M: let $\Box \in \boxdot$. We show that $\mathcal{R}_\Box$ has the properties that are induced by M. We separately consider each of the cases:

  - Suppose $(\textsc{d}_\text{X}) \in \mathsf{M}(\Box)$ for some (X). We show that $\mathcal{R}_\Box$ is serial. Similarly to the proof above that $S \not\vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L_1^\Box, L_2^\Box \Rightarrow \psi, R_1^\Box, R_2^\Box, R_3^\Box$, it can be shown that $S \not\vdash^{\mathcal{F}}_{\mathbf{G}_\mathsf{M}} L_1^\Box, L_2^\Box \Rightarrow R_1^\Box, R_2^\Box, R_3^\Box$, by applying $(\textsc{d}_\text{X})$ rather than (X), for the only $(\textsc{d}_\text{X}) \in \mathsf{M}(\Box)$, and that $L_1^\Box, L_2^\Box \Rightarrow R_1^\Box, R_2^\Box, R_3^\Box$ can be extended to some element $L' \Rightarrow R'$ in W such that $(L \Rightarrow R)\mathcal{R}_\Box(L' \Rightarrow R')$.
  - Suppose $(\textsc{t}) \in \mathsf{M}(\Box)$. We show that $\mathcal{R}_\Box$ is reflexive. Let $L \Rightarrow R \in W$. We show that $(L \Rightarrow R)\mathcal{R}_\Box(L \Rightarrow R)$, that is, $L_1^\Box, L_2^\Box \Rightarrow R_1^\Box, R_2^\Box, R_3^\Box \subseteq L \Rightarrow R$. Let $\psi \in L_1^\Box$, and assume for contradiction that $\psi \notin L$, that is, $\psi \in R$. Since $\psi \in L_1^\Box$, we have that $\Box\psi \in L$, and therefore, $\Box\psi \Rightarrow \psi \subseteq L \Rightarrow R$, which is impossible, as $(\textsc{t}) \in \mathsf{M}(\Box)$. The proofs that $L_2^\Box \subseteq L$ and $R_1^\Box \subseteq R$ are trivial. Now let $\psi \in R_2^\Box$, and assume for contradiction that $\psi \notin R$, that is, $\psi \in L$. Since $\psi \in R_2^\Box$, we have that $\psi = \Box\psi'$ for some $\psi' \in R$, and that $\psi \in \mathcal{F}$. This means that $\psi \Rightarrow \psi' \subseteq L \Rightarrow R$, which is again impossible by the presence of $(\textsc{t})$ in $\mathsf{M}(\Box)$. Finally, since $(\textsc{t}) \in \mathsf{M}(\Box)$, we must have that $(\textsc{pf}) \notin \mathsf{M}(\Box)$, which means that $R_3^\Box = \emptyset \subseteq R$.

  In the following cases, $L_a \Rightarrow R_a$, $L_b \Rightarrow R_b$ and $L_c \Rightarrow R_c$ denote arbitrary elements of W.

  - Suppose $(4) \in \mathsf{M}(\Box)$. We show that $\mathcal{R}_\Box$ is transitive. Suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_a)_1^\Box, (L_a)_2^\Box \Rightarrow (R_a)_1^\Box, (R_a)_2^\Box, (R_a)_3^\Box \subseteq L_c \Rightarrow R_c$. Since $(4) \in \mathsf{M}(\Box)$, we must have $(R_a)_1^\Box = (R_a)_2^\Box = (R_a)_3^\Box = \emptyset$. Now, let $\psi \in (L_a)_1^\Box$. Then both

$\psi \in \mathcal{F}$ and $\Box\psi \in L_a$, which means that $\Box\psi \in (L_a)_2^\Box \subseteq L_b$. Together with the fact that $\psi \in \mathcal{F}$, we have $\psi \in (L_b)_1^\Box \subseteq L_c$. Next, let $\psi \in (L_a)_2^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$, and $\psi \in L_b$. Therefore, $\psi \in (L_b)_2^\Box \subseteq L_c$.

– Suppose (45) $\in \mathsf{M}(\Box)$. We show that $\mathcal{R}_\Box$ is transitive and euclidean.

  * Transitivity: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_a)_1^\Box, (L_a)_2^\Box \Rightarrow (R_a)_1^\Box, (R_a)_2^\Box, (R_a)_3^\Box \subseteq L_c \Rightarrow R_c$. Since (45) $\in \mathsf{M}(\Box)$, we must have $(R_a)_2^\Box = (R_a)_3^\Box = \emptyset$. Similarly to the case of (4), $(L_a)_1^\Box, (L_a)_2^\Box \subseteq (L_c)$. Now let $\psi \in (R_a)_1^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$ and $\psi \in R_b$. Therefore, $\psi \in (R_b)_1^\Box \subseteq R_c$.
  * Euclideaness: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_b)_1^\Box, (L_b)_2^\Box \Rightarrow (R_b)_1^\Box, (R_b)_2^\Box, (R_b)_3^\Box \subseteq L_c \Rightarrow R_c$. Since (45) $\in \mathsf{M}(\Box)$, we have $(R_b)_2^\Box = (R_b)_3^\Box = \emptyset$. Let $\psi \in (L_b)_1^\Box$. Then $\Box\psi \in L_b$ and $\psi \in \mathcal{F}$. Hence $\Box\psi \notin R_b$, and therefore $\Box\psi \notin (R_a)_1^\Box$. Since we have $\psi \in \mathcal{F}$, this must mean that $\Box\psi \notin R_a$, and hence $\Box\psi \in L_a$. Again, since $\psi \in \mathcal{F}$, $\psi \in (L_a)_1^\Box \subseteq L_c$. Next, let $\psi \in (L_b)_2^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$ and $\psi \in L_b$. In particular, $\psi \notin R_b$. Since $(R_a)_1^\Box \subseteq R_b$, we also have $\psi \notin (R_a)_1^\Box$. Together with the fact that $\psi' \in \mathcal{F}$, we have $\psi \notin R_a$. This, in turn, means that $\psi \in L_a$, which, together with $\psi' \in \mathcal{F}$, means that $\psi \in (L_a)_2^\Box \subseteq L_c$. The fact that $(R_b)_1^\Box \subseteq R_c$ is proven symmetrically.

– Suppose (B) $\in \mathsf{M}(\Box)$. We show that $\mathcal{R}_\Box$ is symmetric. Suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$. We prove that $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_a \Rightarrow R_a)$, that is, $(L_b)_1^\Box, (L_b)_2^\Box \Rightarrow (R_b)_1^\Box, (R_b)_2^\Box, (R_b)_3^\Box \subseteq L_a \Rightarrow R_a$. Since (B) $\in \mathsf{M}(\Box)$, we have $(L_b)_2^\Box = (R_b)_1^\Box = (R_b)_3^\Box = \emptyset$. Let $\psi \in (L_b)_1^\Box$. Then $\Box\psi \in L_b \subseteq \mathcal{F}$, and hence $\Box\psi \notin R_b$, and in particular, $\Box\psi \notin (R_a)_2^\Box$. Since $\Box\psi \in \mathcal{F}$, we have also $\psi \notin R_a$, which means that $\psi \in L_a$. Next, let $\psi \in (R_b)_2^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in R_b \subseteq \mathcal{F}$. Hence $\psi' \notin L_b$, and in particular, $\psi' \notin (L_a)_1^\Box$. Since $\psi' \in \mathcal{F}$, we also have $\psi \notin L_a$, which means that $\psi \in R_a$.

– Suppose (B4) $\in \mathsf{M}(\Box)$. We show that $\mathcal{R}_\Box$ is transitive and symmetric.

  * Transitivity: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_a)_1^\Box, (L_a)_2^\Box \Rightarrow (R_a)_1^\Box, (R_a)_2^\Box, (R_a)_3^\Box \subseteq L_c \Rightarrow R_c$ First, note that $(R_a)_3^\Box = \emptyset$. Second, $(L_a)_1^\Box, (L_a)_2^\Box \subseteq L_c$ and $(R_a)_1^\Box \subseteq R_c$ are shown similarly to the case of (45). Let $\psi \in (R_a)_2^\Box \subseteq R_b$. Then $\psi \in \mathcal{F}$, and $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$. Hence $\psi \in (R_b)_1^\Box \subseteq R_c$.

  * Symmetry: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$. We prove that $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_a \Rightarrow R_a)$, that is, $(L_b)^\Box_1, (L_b)^\Box_2 \Rightarrow (R_b)^\Box_1, (R_b)^\Box_2, (R_b)^\Box_3 \subseteq L_a \Rightarrow R_a$. First, note that $(R_a)^\Box_3 = \emptyset$. Second, $(L_b)^\Box_1 \subseteq L_a$ and $(R_b)^\Box_2 \subseteq L_a$ are shown similarly to the case of (b). Let $\psi \in (L_b)^\Box_2$. Then $\psi \in L_b$, and $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$. In particular, $\psi \notin R_b$, and hence also $\psi \notin (R_a)^\Box_1$. Together with the fact that $\psi' \in \mathcal{F}$, we have that $\psi \notin R_a$, which means that $\psi \in L_a$. The fact that $(R_b)^\Box_1 \subseteq R_a$ is shown symmetrically.

– Suppose (pf) $\in \mathsf{M}(\Box)$. By the definition of $\mathcal{R}_\Box$ in this case, $\mathcal{R}_\Box$ is functional.

□

4.3 Analyticity

The definition of a $\circledcirc$-k-subformula (Definition 2.5.1) applies also for the language $\mathcal{L}_{\boxdot}$, which is a propositional language. Thus, the elements of $\boxdot$ are taken to be additional propositional connectives. It is important to note, however, that $\boxdot \cap \diamondsuit_\mathcal{L} = \emptyset$, and in particular, $\boxdot \cap \circledcirc = \emptyset$.

Example 4.3.1. Considering the language $\mathcal{CL}_{\boxdot}$, with $\boxdot = \{\Box\}$. $sub_1^{\{\neg\}}(\Box(p \wedge q)) = \{p, q, \neg p, \neg q, p \wedge q, \neg(p \wedge q), \Box(p \wedge q)\}$.

Thus, also in $\mathcal{L}_{\boxdot}$, the $\circledcirc$-k-subformula property induces the $\circledcirc$-k-analyticity property for pure calculi that are augmented with the above rules for modal operators: just like in the pure case, a calculus $\mathbf{G}_\mathsf{M}$ is $\circledcirc$-k-analytic if $S \vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(S \cup \{s\})} s$ whenever $S \vdash_{\mathbf{G}} s$.

In this section we show that, excluding some degenerate cases (to be described below), $\circledcirc$-k-analyticity is preserved when augmenting a pure calculus with the above rules for modal operators. Semantics will play a major role here, as what will actually be shown is how to use the possibility to extend partial bivaluations in order to extend partial biframes.

For this, we focus on a slightly restricted sub-family of calculi, namely *standard* calculi, thus ruling out some degenerate cases. Roughly speaking, a calculus is called standard if whenever an atomic formula occurs in one of its rules, it also occurs as a subformula in the same rule. This is formally defined as follows:

Definition 4.3.2. An atomic variable p is called *shared* in a rule r if at least one of the followings hold:

1. p is a proper subformula of some formula in the conclusion of r.

2. p is a proper subformula of every $\psi \in \mathit{frm}(s)$ for some non-empty premise s of r.

A rule is called *standard* if all atomic variables that occur in it are shared in it. A calculus is called *standard* if each of its rules is standard.

Example 4.3.3. All calculi considered in examples above are standard. In contrast, p_3 is not shared in the rule $\Rightarrow p_1, p_3 \,/\, \Rightarrow p_1 \vee p_2$, and so every calculus that includes this rule is not standard. Aside from such tailored examples, we are not aware of any non-standard calculus from the literature.

The main result of this section is:

Theorem 4.3.4. Let $\mathbf{G}$ be a standard $\mathcal{L}$-calculus. If $\mathbf{G}$ is $\circledcirc$-k-analytic then so is $\mathbf{G}_{\mathsf{M}}$.

Note that if $\mathbf{G}_{\mathsf{M}}$ is $\circledcirc$-k-analytic, then $\mathbf{G}$ must also be $\circledcirc$-k-analytic: given that S and s do not include any symbol from $\boxdot$, we have that if $S \vdash_{\mathbf{G}} s$ then also $S \vdash_{\mathbf{G}_{\mathsf{M}}} s$. The $\circledcirc$-k-analyticity of $\mathbf{G}_{\mathsf{M}}$ then ensures that there is a derivation of s from S in $\mathbf{G}_{\mathsf{M}}$ that consists only of $\mathit{sub}_k^{\circledcirc}(S \cup \{s\})$-formulas. This derivation cannot contain applications of $\mathsf{M}(\Box)$ for any $\Box \in \boxdot$, and hence it is also a derivation in $\mathbf{G}$.

We now present some examples for applications of Theorem 4.3.4, before turning to its proof.

Example 4.3.5. All sequent calculi for classical modal logics that are obtained from **LK** by the adding the rules of Table 4.1 are known to be $\emptyset$-analytic. Theorem 4.3.4 makes this fact a direct consequence of the $\emptyset$-analyticity of **LK**.

Example 4.3.6. $\mathbf{G_{PIL}}$ and $\mathbf{G_{EPIL}}$ are standard. Since they are $\emptyset$-analytic, so are $\mathbf{G}_{\mathbf{PIL}\mathsf{M}}$ and $\mathbf{G}_{\mathbf{EPIL}\mathsf{M}}$ (see Example 4.1.3). In contrast, the Hilbert-type system for primal infon logic in [44] admits a similar property that concerns local formulas (see Definition 4.4.1 below) rather than subformulas.

Example 4.3.7. One can add modal operators to the paraconsistent logic $\mathbf{C_1}$ (see Example 2.2.9), by augmenting $\mathbf{G_{C_1}}$ with one of the rules for modal operators. The $\{\neg\}$-1-analyticity of the calculus $\mathbf{G_{C_1}}$ then entails the $\{\neg\}$-1-analyticity of the extended calculus.

Example 4.3.8. The paper [89] augments Łukasiewicz three-valued logic (Example 2.2.8) with several modal operators. Proof-theoretically, [89] focuses on Hilbert-type calculi, that are rarely analytic. In contrast, such a goal could be achieved by augmenting $\mathbf{G}_{\mathbf{Ł_3}}$ with some of the modal rules from Figure 4.1. Using Theorem 4.3.4, the $\{\neg\}$-1-analyticity of $\mathbf{G}_{\mathbf{Ł_3}}$ entails the $\{\neg\}$-1-analyticity of the resulting calculi.

Next, we prove Theorem 4.3.4. We use Theorems 4.2.6 and 4.2.7, and show how to extend partial biframes into full ones. The general notion of biframes (that allows for different domains in each world), and the enforced semantics of the connectives from $\boxdot$, all make the extension of partial biframes more challenging than partial bivaluations. The following definitions are therefore needed. First, we introduce a more delicate notion of closure under $\circledcirc$-k-subformulas.

Definition 4.3.9. A set of $\mathcal{L}_{\boxdot}$-formulas is called $\circledcirc$-k-*closed* if whenever it contains a formula of the form $\circ\varphi$ for some $\circ \in \circledcirc$, it also contains φ, and whenever it contains a formula of the form $\diamond(\varphi_1, \dots, \varphi_n)$ for some $\diamond \in \Diamond_{\mathcal{L}} \setminus \circledcirc$, it also contains $\circledcirc^{\leq k}\psi_i$ for every $1 \leq i \leq n$.

Every set that is closed under $\circledcirc$-k-subformulas is also $\circledcirc$-closed. However, the converse may not hold. For example, the set $\{(\Box p_1) \wedge (\Box p_2), \Box p_1, \Box p_2\}$ is $\emptyset$-k-closed for any k, but it is not closed under $\emptyset$-k-subformulas, as p_1 and p_2 are missing.

Next, we generalize this property from sets of formulas to biframes.

Definition 4.3.10. A biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M is called $\circledcirc$-k-*closed* if the followings hold for every $w \in W$:

- $dom(\mathcal{V}_w)$ is $\circledcirc$-k-closed and finite.
- For every $\Box \in \boxdot$, if $\Box\psi \in dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_{\Box}[w]$.

Similarly to the case of pure calculi, the ability to extend partial models is essential also when introducing modal operators. We thus explicitly define what does it mean to extend a biframe.

Definition 4.3.11. A biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M *extends* a biframe $\langle W', \mathcal{R}', \mathcal{V}'\rangle$ for M if $W = W'$, $\mathcal{R} = \mathcal{R}'$, and $\mathcal{V}_w$ extends $\mathcal{V}'_w$ for every $w \in W$.

Finally, the main part of the proof of Theorem 4.3.4 is the following lemma, that is proven in the next section. From this lemma, the theorem immediately follows, with the help of Theorem 4.2.7.

Lemma 4.3.12. Let $\mathbf{G}$ be a standard $\circledcirc$-k-analytic $\mathcal{L}$-calculus, and $\mathcal{W}$ a $\mathbf{G}$-legal $\circledcirc$-k-closed biframe for M. Then, $\mathcal{W}$ can be extended to a $\mathbf{G}$-legal $\mathcal{L}_{\boxdot}$-biframe for M.

Before proving the lemma, we show how it is used to prove Theorem 4.3.4.

Proof of Theorem 4.3.4. Suppose that $\mathbf{G}$ is $\circledcirc$-k-analytic. Assume $S \vdash_{\mathbf{G}_{\mathsf{M}}} s$ for a set S of $\mathcal{L}_{\boxdot}$-sequents and an $\mathcal{L}_{\boxdot}$-sequent s. Let $S' \subseteq S$ be a finite set such that $S' \vdash_{\mathbf{G}_{\mathsf{M}}} s$. We prove that $S' \vdash_{\mathbf{G}_{\mathsf{M}}}^{sub_k^{\circledcirc}(S'\cup\{s\})} s$, and conclude that $S \vdash_{\mathbf{G}_{\mathsf{M}}}^{sub_k^{\circledcirc}(S\cup\{s\})} s$. Assume otherwise. By Theorem 4.2.7, there exists a $\mathbf{G}$-legal $sub_k^{\circledcirc}(S' \cup \{s\})$-biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ for M such that $\mathcal{V}_W(S') > \mathcal{V}_w(s)$. $\mathcal{W}$ is also $\circledcirc$-k-closed, and by Lemma 4.3.12, it can be extended to a $\mathbf{G}$-legal $\mathcal{L}_{\boxdot}$-biframe $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$ for M. After this extension, we still have $\mathcal{V}'_W(S') > \mathcal{V}'_w(s)$. Theorem 4.2.6 implies that $S' \not\vdash_{\mathbf{G}_{\mathsf{M}}} s$, which is a contradiction. □

Remark 4.3.13. The proof of Theorem 4.3.4 only considers biframes with a domain that is closed under $\circledcirc$-k-subformulas. Lemma 4.3.12, however, holds also for biframes that are $\circledcirc$-k-closed, even if they are not closed under $\circledcirc$-k-subformulas. This strengthening is needed in Section 4.4, where we extend the reduction of Chapter 3.

4.3.1 Proof of Lemma 4.3.12

Lemma 4.3.12 is basically an extension lemma. It ensures the ability to extend partial biframes into full ones. For the extension method that we propose here, the following property of $\circledcirc$-k-closed sets is useful:

Lemma 4.3.14. If $\mathcal{F} \subseteq \mathcal{L}_{\boxdot}$ is $\circledcirc$-k-closed, $\varphi, \psi \in \mathcal{L}$, and φ is a $\circledcirc$-k-subformula of ψ, then $\sigma(\psi) \in \mathcal{F}$ implies $\sigma(\varphi) \in \mathcal{F}$.

We add all formulas of the language to the domain of the biframe, not one by one – but many at a time. For each formula, we need to know the value of other formulas (usually, its $\circledcirc$-k-subformulas) in order to determine its value. In each step, we add to the domain of the biframe all the formulas for which we are able to determine this value. The following lemma shows that every step in this process is possible:

Lemma 4.3.15. Let $\mathbf{G}$ be a standard $\circledcirc$-k-analytic $\mathcal{L}$-calculus and $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ a $\mathbf{G}$-legal $\circledcirc$-k-closed biframe for M. Then:

1. Given $p \in At$, $\mathcal{W}$ can be extended to a $\mathbf{G}$-legal $\circledcirc$-k-closed biframe $\mathcal{W}'$ for M, such that $p \in dom(\mathcal{W}')$.

2. $\mathcal{W}$ can be extended to a $\mathbf{G}$-legal $\circledcirc$-k-closed biframe $\mathcal{W}'$ for M, such that $\boxdot dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$.

3. $\mathcal{W}$ can be extended to a $\mathbf{G}$-legal $\circledcirc$-k-closed biframe $\mathcal{W}'$ for M, such that $\circledcirc dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$, and for every $\diamond \in \diamondsuit^n_{\mathcal{L}} \setminus \circledcirc$, $\diamond(\varphi_1, \dots, \varphi_n) \in dom(\mathcal{W}')$ whenever $\circledcirc^{\leq k}\{\varphi_1, \dots, \varphi_n\} \subseteq dom(\mathcal{W})$.

Proof.

1. Let $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}' \rangle$, where $\mathcal{V}'$ is the function assigning to every $w \in W$, the $dom(\mathcal{V}_w) \cup \{p\}$-bivaluation $\mathcal{V}'_w$ obtained by extending $\mathcal{V}_w$ with the value 0 (say) for p if $p \notin dom(\mathcal{V}_w)$. Clearly, $\mathcal{W}'$ is a $\circledcirc$-k-closed biframe for M that extends $\mathcal{W}$, and $p \in dom(\mathcal{W}')$. It remains to show that $\mathcal{W}'$ is $\mathbf{G}$-legal. Let $w \in W$, $s_1, \dots, s_n / s \in \mathbf{G}$, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, and σ a substitution such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}'_w)$. We prove that $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$. If $p \notin \sigma(frm(\{s'_1, \dots, s'_n, s\}))$ or $p \in dom(\mathcal{V}_w)$, then this follows from the fact that $\mathcal{V}_w$ is $\mathbf{G}$-legal. We show that these are actually the only two options for p. Indeed, if $p \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$, then $p = \sigma(p')$ for some atomic variable $p' \in frm(\{s'_1, \dots, s'_n, s\})$. Since $\mathbf{G}$ is standard, p' is a proper subformula of some $\varphi \in frm(\{s'_1, \dots, s'_n, s\})$. Since $\sigma(\varphi) \in dom(\mathcal{V}'_w)$ and $\sigma(\varphi) \neq p$, we have $\sigma(\varphi) \in dom(\mathcal{V}_w)$. By Lemma 4.3.14, $p \in dom(\mathcal{V}_w)$.

2. For every $w \in W$, let $\mathcal{F}_w = dom(\mathcal{V}_w) \cup \boxdot dom(\mathcal{W})$. Let $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}' \rangle$, where $\mathcal{V}'$ is the function assigning to every $w \in W$, the $\mathcal{F}_w$-bivaluation $\mathcal{V}'_w$ defined by:
$$\mathcal{V}'_w(\psi) = \begin{cases} \mathcal{V}_w(\psi) & \psi \in dom(\mathcal{V}_w) \\ \mathcal{V}_{\mathcal{R}_\Box[w]}(\varphi) & \psi = \Box\varphi \in \mathcal{F}_w \setminus dom(\mathcal{V}_w) \end{cases}$$
We show first that $\mathcal{W}'$ is a biframe for M. Let $w \in W$. Let $\Box\psi \in dom(\mathcal{V}'_w)$ such that $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. If $\Box\psi \in dom(\mathcal{V}_w)$, then since $\mathcal{W}$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. Hence since $\mathcal{W}$ is a biframe for M, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$. If $\Box\psi \notin dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{W})$, and by the definition of $\mathcal{V}'$ in this case, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$.

   Obviously, $\mathcal{W}'$ extends $\mathcal{W}$ and $\boxdot dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$. It remains to show that $\mathcal{W}'$ is $\circledcirc$-k-closed and $\mathbf{G}$-legal.

   (a) $\circledcirc$-k-closed: For every $w \in W$, $dom(\mathcal{V}_w)$ is $\circledcirc$-k-closed and finite. Since we only added a finite number of formulas, all from $\boxdot\mathcal{L}_{\boxdot}$, $dom(\mathcal{V}'_w)$ is also $\circledcirc$-k-closed and finite for every $w \in W$. Now, suppose that $\Box\psi \in dom(\mathcal{V}'_w)$. If $\Box\psi \in dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$ since $\mathcal{W}$ is $\circledcirc$-k-closed. If $\Box\psi \notin dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$, and in particular $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$.

   (b) $\mathbf{G}$-legal: Let $w \in W$, $s_1, \dots, s_n / s \in \mathbf{G}$, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, and σ a substitution such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}'_w)$. We prove that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}_w)$, and then $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$ follows from the fact that $\mathcal{W}$ is $\mathbf{G}$-legal. Indeed, let $\psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$. If $\psi \notin \boxdot\mathcal{L}_{\boxdot}$, then $\psi \in dom(\mathcal{V}_w)$. If

$\psi \in \boxdot\mathcal{L}_{\boxdot}$, then $\psi = \sigma(p)$ for some atomic variable $p \in \mathit{frm}(\{s'_1, \dots, s'_n, s\})$. Since $\mathbf{G}$ is standard, p is a proper subformula of some compound $\mathcal{L}$-formula $\varphi \in \mathit{frm}(\{s'_1, \dots, s'_n, s\})$. Since φ is a compound $\mathcal{L}$-formula, we have $\sigma(\varphi) \notin \boxdot\mathcal{L}_{\boxdot}$, and hence $\sigma(\varphi) \in \mathit{dom}(\mathcal{V}_w)$. By Lemma 4.3.14, since $\mathit{dom}(\mathcal{V}_w)$ is $\circledcirc$-k-closed, $\psi \in \mathit{dom}(\mathcal{V}_w)$.

3. We define $\mathcal{W}'$ in several steps.

Embedding $\mathcal{L}$ in $\mathcal{L}_{\boxdot}$: Let σ_0 be some bijection from At to $At \cup \boxdot\mathcal{L}_{\boxdot}$. As a substitution, σ_0 is naturally extended to apply on all $\mathcal{L}$-formulas. It is straightforward to verify that its extension is a bijection from $\mathcal{L}$ to $\mathcal{L}_{\boxdot}$.

Translating $\mathcal{V}$: For every $w \in W$, let $\mathcal{F}_w = \{\varphi \in \mathcal{L} \mid \sigma_0(\varphi) \in \mathit{dom}(\mathcal{V}_w)\}$. By Lemma 4.3.14 and the fact that $\mathcal{W}$ is $\circledcirc$-k-closed, we have that $\mathcal{F}_w$ is closed under $\circledcirc$-k-subformulas for every $w \in W$. Since σ_0 is a bijection, we also have that $\mathcal{F}_w$ is finite for every $w \in W$. Now, for every $w \in W$, let u_w be the $\mathcal{F}_w$-bivaluation given by $u_w = \lambda\varphi \in \mathcal{F}_w.\ \mathcal{V}_w(\sigma_0(\varphi))$. We show that u_w is $\mathbf{G}$-legal for every $w \in W$. Let $w \in W$, $s_1, \dots, s_n \,/\, s \in \mathbf{G}$, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, and σ a substitution such that $\sigma(\mathit{frm}(\{s'_1, \dots, s'_n, s\})) \subseteq \mathcal{F}_w$. We prove that $u_w(\sigma(\{s'_1, \dots, s'_n\})) \leq u_w(\sigma(s))$. Consider the substitution $\sigma' = \sigma_0 \circ \sigma$. It is easy to see that $\sigma'(\varphi) = \sigma_0(\sigma(\varphi))$ for every formula φ. Therefore, $\sigma'(\mathit{frm}(\{s'_1, \dots, s'_n, s\})) = \sigma_0(\sigma(\mathit{frm}(\{s'_1, \dots, s'_n, s\}))) \subseteq \sigma_0(\mathcal{F}_w) \subseteq \mathit{dom}(\mathcal{V}_w)$. Since $\mathcal{W}$ is $\mathbf{G}$-legal, we have

$$u_w(\sigma(\{s'_1, \dots, s'_n\})) = \mathcal{V}_w(\sigma'(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}_w(\sigma'(s)) = u_w(\sigma(s)).$$

Extending the translation: Let $w \in W$. Then, u_w is a $\mathbf{G}$-legal bivaluation whose domain $\mathcal{F}_w$ is a finite subset of $\mathcal{L}$ closed under $\circledcirc$-k-subformulas. Since $\mathbf{G}$ is $\circledcirc$-k-analytic, by Theorem 2.5.9, u_w can be extended to a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation u^*_w.

Defining $\mathcal{W}'$: For every $w \in W$, let $\mathcal{F}'_w$ be the following set:

$$\mathit{dom}(\mathcal{V}_w) \cup \circledcirc\mathit{dom}(\mathcal{W}) \cup \{\diamond(\varphi_1, \dots, \varphi_n) \mid \diamond \in \Diamond^n_{\mathcal{L}} \setminus \circledcirc, \circledcirc^{\leq k}\{\varphi_1, \dots, \varphi_n\} \subseteq \mathit{dom}(\mathcal{W})\}.$$

Let α be the inverse of σ_0. α is a bijection from $\mathcal{L}_{\boxdot}$ to $\mathcal{L}$. Let $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}' \rangle$, where $\mathcal{V}'$ is the function assigning to every $w \in W$, the $\mathcal{F}'_w$-bivaluation $\mathcal{V}'_w$ defined by:

$$\mathcal{V}'_w(\psi) = \begin{cases} \mathcal{V}_w(\psi) & \psi \in \mathit{dom}(\mathcal{V}_w) \\ u^*_w(\alpha(\psi)) & \psi \in \mathcal{F}'_w \setminus \mathit{dom}(\mathcal{V}_w) \end{cases}$$

First, we prove that $\mathcal{W}'$ is a biframe for M. Let $w \in W$ and $\psi, \Box\psi \in \mathcal{L}_{\boxdot}$. Suppose that $\Box\psi \in \mathit{dom}(\mathcal{V}'_w)$ and $\psi \in \mathit{dom}(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_{\Box}[w]$.

Then, since $\Box\psi \in \boxdot\mathcal{L}_{\boxdot}$, we have $\Box\psi \in dom(\mathcal{V}_w)$. Since $\mathcal{W}$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. Since $\mathcal{W}$ is a biframe, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$.

Clearly, $\mathcal{W}'$ extends $\mathcal{W}$, $\circledcirc dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$, and for every $\diamond \in \diamondsuit^n_{\mathcal{L}} \setminus \circledcirc$, $\diamond(\varphi_1, \dots, \varphi_n) \in dom(\mathcal{W}')$ whenever $\circledcirc^{\leq k}\{\varphi_1, \dots, \varphi_n\} \subseteq dom(\mathcal{W})$.

It remains to show that $\mathcal{W}'$ is $\circledcirc$-k-closed and **G**-legal.

(a) $\circledcirc$-k-closed: Let $w \in W$. First, $dom(\mathcal{V}'_w)$ is finite since $dom(\mathcal{W})$ and $\diamondsuit_{\mathcal{L}}$ are finite. Second, let $\circ\varphi \in dom(\mathcal{V}'_w)$ for some $\circ \in \circledcirc$. If $\circ\varphi \in dom(\mathcal{V}_w)$, then since $\mathcal{W}$ is $\circledcirc$-k-closed, $\varphi \in dom(\mathcal{V}_w) \subseteq dom(\mathcal{V}'_w)$. Otherwise, $\circ\varphi \in \mathcal{F}'_w \setminus dom(\mathcal{V}_w)$, which means that $\varphi \in dom(\mathcal{W}) \subseteq dom(\mathcal{V}_w) \subseteq dom(\mathcal{V}'_w)$. Third, let $\diamond(\psi_1, \dots, \psi_n) \in dom(\mathcal{V}'_w)$. We show that $\circledcirc^{\leq k}\psi_i \subseteq dom(\mathcal{V}'_w)$ for every $1 \leq i \leq n$. If $\diamond(\psi_1, \dots, \psi_n) \in dom(\mathcal{V}_w)$ then this holds since $\mathcal{W}$ is $\circledcirc$-k-closed. Otherwise, $\diamond(\psi_1, \dots, \psi_n) \in \mathcal{F}'_w \setminus dom(\mathcal{V}_w)$, which means that $\circledcirc^{\leq k}\psi_i \subseteq dom(\mathcal{W}) \subseteq dom(\mathcal{V}_w) \subseteq dom(\mathcal{V}'_w)$ for every $1 \leq i \leq n$. Finally, let $\Box\psi \in dom(\mathcal{V}'_w)$. Then, since $\Box\psi \in \boxdot\mathcal{L}_{\boxdot}$, $\Box\psi \in dom(\mathcal{V}_w)$. Since $\mathcal{W}$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_{w'}) \subseteq dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$.

(b) **G**-legal: Let $w \in W$, $s_1, \dots, s_n \,/\, s \in \mathbf{G}$, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, and σ a substitution such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}'_w)$. We prove that $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$. For that, we first prove that $\mathcal{V}'_w(\psi) = u^*_w(\alpha(\psi))$ for every $\psi \in dom(\mathcal{V}'_w)$. If $\psi \notin dom(\mathcal{V}_w)$, then this holds by definition. Suppose that $\psi \in dom(\mathcal{V}_w)$. Since $\sigma_0(\alpha(\psi)) = \psi$, $\alpha(\psi) \in \mathcal{F}_w$. Hence $u^*_w(\alpha(\psi)) = u_w(\alpha(\psi))$. By definition, $u_w(\alpha(\psi)) = \mathcal{V}_w(\sigma_0(\alpha(\psi))) = \mathcal{V}_w(\psi)$. Since $\psi \in dom(\mathcal{V}_w)$, $u^*_w(\alpha(\psi)) = \mathcal{V}'_w(\psi)$. Now, consider the substitution $\sigma' = \alpha \circ \sigma$. It is easy to see that $\sigma'(\psi) = \alpha(\sigma(\psi))$ for every $\psi \in frm(\{s'_1, \dots, s'_n, s\})$. Clearly, $\sigma'(frm(\{s'_1, \dots, s'_n, s\})) \subseteq \mathcal{L}$. Since u^*_w is **G**-legal, we have that $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) = u^*_w(\alpha(\sigma(\{s'_1, \dots, s'_n\}))) = u^*_w(\sigma'(\{s'_1, \dots, s'_n\}))$ is less than or equal to $u^*_w(\sigma'(s)) = u^*_w(\alpha(\sigma(s))) = \mathcal{V}'_w(\sigma(s))$. □

To complete the proof of Lemma 4.3.12, we use Lemma 4.3.15 repeatedly, and construct a full biframe from a partial one.

First, recursively construct an infinite sequence $\mathcal{W}^0 = \langle W, \mathcal{R}, \mathcal{V}^0 \rangle, \mathcal{W}^1 = \langle W, \mathcal{R}, \mathcal{V}^1 \rangle, \dots$, such that:

- $\mathcal{W}^0 = \mathcal{W}$.

- For every i, $\mathcal{W}^i$ is a **G**-legal $\circledcirc$-k-closed biframe for M.
- Each $\mathcal{W}^{i+1}$ extends $\mathcal{W}^i$.
- For every $\psi \in \mathcal{L}_{\boxdot}$, $\psi \in dom(\mathcal{W}^i)$ for some $i \geq 0$.

We begin with $\mathcal{W}^0 = \mathcal{W}$. Given $\mathcal{W}^i$, $\mathcal{W}^{i+1}$ is obtained as follows. By Lemma 4.3.15, $\mathcal{W}^i$ can be extended to a **G**-legal $\circledcirc$-k-closed biframe $\mathcal{W}^i_1$ for M such that $p_i \in dom(\mathcal{W}^i_1)$. In turn, $\mathcal{W}^i_1$ can be extended to a **G**-legal $\circledcirc$-k-closed biframe $\mathcal{W}^i_2$ for M such that $\boxdot dom(\mathcal{W}^i_1) \subseteq dom(\mathcal{W}^i_2)$. Finally, $\mathcal{W}^i_2$ can be extended to a **G**-legal $\circledcirc$-k-closed biframe $\mathcal{W}^i_3$ for M such that $\circledcirc dom(\mathcal{W}^i_2) \subseteq dom(\mathcal{W}^i_3)$, and for every $\diamond \in \diamondsuit^n_{\mathcal{L}} \setminus \circledcirc$, $\diamond(\varphi_1, \dots, \varphi_n) \in dom(\mathcal{W}^i_3)$ whenever $\circledcirc^{\leq k} \{\varphi_1, \dots, \varphi_n\} \subseteq dom(\mathcal{W}^i_2)$. We take $\mathcal{W}^{i+1} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V}^{i+1} \rangle$ to be $\mathcal{W}^i_3$.

Clearly, for every $i \geq 0$, $\mathcal{W}^{i+1}$ is a **G**-legal $\circledcirc$-k-closed biframe for M that extends $\mathcal{W}^i$. We prove that for every $\psi \in \mathcal{L}_{\boxdot}$ there exists some $i \geq 0$ such that $\psi \in dom(\mathcal{W}^i)$, by induction on the complexity of ψ:

1. If $\psi \in At$ then $\psi = p_i$ for some $i \geq 1$. By our construction, $p_i \in dom(\mathcal{W}^i_1)$ and hence $p_i \in dom(\mathcal{W}^{i+1})$.
2. If $\psi = \Box\varphi$ then by the induction hypothesis, $\varphi \in dom(\mathcal{W}^i)$ for some $i \geq 0$. By our construction, $\Box\varphi \in dom(\mathcal{W}^i_2)$ and hence $\psi \in dom(\mathcal{W}^{i+1})$.
3. If $\psi = \circ\varphi$, then by the induction hypothesis, there exists i such that $\varphi \in dom(\mathcal{W}^i)$. By our construction, $\circ\varphi \in dom(\mathcal{W}^i_3)$, and hence $\circ\varphi \in dom(\mathcal{W}^{i+1})$.
4. If $\psi = \diamond(\psi_1, \dots, \psi_n)$ then by the induction hypothesis, there exist $i_1, \dots, i_n$ such that $\psi_j \in dom(\mathcal{W}^{i_j})$ for every $1 \leq j \leq n$. Let $i = \max\{i_1, \dots, i_n\}$. By our construction, there exists $i_0 \geq i$ such that $\circledcirc^{\leq k}\psi_j \subseteq dom(\mathcal{W}^{i_0})$ for every $1 \leq j \leq n$ (in each step we add $\circ\varphi$ for every $\varphi \in dom(\mathcal{W}^i)$ and $\circ \in \circledcirc$. Since $\circledcirc^{\leq k}$ is finite, we exhaust it at some point). Hence $\diamond(\psi_1, \dots, \psi_n) \in dom(\mathcal{W}^{i_0}_3)$, which means that $\diamond(\psi_1, \dots, \psi_n) \in dom(\mathcal{W}^{i_0+1})$.

We now define $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}' \rangle$, a **G**-legal $\mathcal{L}_{\boxdot}$-biframe for M that extends $\mathcal{W}$. For every $\psi \in \mathcal{L}_{\boxdot}$, let i_ψ denote the first i such that $\psi \in dom(\mathcal{W}^i)$. For every $w \in W$, $\mathcal{V}'_w$ is defined by $\mathcal{V}'_w(\psi) = \mathcal{V}^{i_\psi}_w(\psi)$.

We prove that $\mathcal{W}'$ is a **G**-legal $\mathcal{L}_{\boxdot}$-biframe for M that extends $\mathcal{W}$. Clearly, $dom(\mathcal{W}') = \mathcal{L}_{\boxdot}$ and $\mathcal{W}'$ extends $\mathcal{W}$. We prove that $\mathcal{W}'$ is a biframe: Let $w \in W$ and $\psi, \Box\psi \in \mathcal{L}_{\boxdot}$. Let $k = \max\{i_\psi, i_{\Box\psi}\}$. Since $\mathcal{W}^i$ extends W^{i-1} for every i, we have $\mathcal{V}'_{w'}(\psi) = \mathcal{V}^k_{w'}(\psi)$ and $\mathcal{V}'_{w'}(\Box\psi) = \mathcal{V}^k_{w'}(\Box\psi)$ for every $w' \in W$. Since $\mathcal{W}^k$ is a biframe, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}^k_w(\Box\psi) = \mathcal{V}^k_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$. It remains to show that $\mathcal{W}'$ is **G**-legal. Let $w \in W$, $s_1, \dots, s_n / s \in \mathbf{G}$, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, and σ a substitution. We prove that $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$. Let $k = \max\{i_\psi \mid \psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))\}$. Since $\mathcal{W}^i$ extends W^{i-1} for every i, we have $\mathcal{V}'_w(\psi) = \mathcal{V}^k_w(\psi)$ for every $\psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$. Since $\mathcal{V}^k_w$ is **G**-legal, $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) = \mathcal{V}^k_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}^k_w(\sigma(s)) = \mathcal{V}'_w(\sigma(s))$. □

4.4 Extending The Decision Procedure

In this section we extend the reduction from Chapter 3 to standard pure calculi with modal operators, that are defined by (PF) and ($\mathrm{D_{PF}}$). We call such operators *Next operators*. These are often employed in temporal logics. In primal infon logic, they play the role of quotations (see Example 4.1.3). We start by defining a variant of the $\circledcirc$-k-subformula relation in Section 4.4.1. In Section 4.4.2 we make the necessary adjustments to the reduction from Chapter 3, and prove the correctness of the extended reduction. For the modal case, correctness is more challenging, and heavily relies on Lemma 4.3.12 above. In Section 4.4.3 we briefly describe how *Next*-operators are incorporated into our tool Gen2sat.

In what follows, we denote the specification function that assigns $\{(\mathrm{PF}), (\mathrm{D_{PF}})\}$ to every $\Box \in \boxdot$ by *Next*. In turn, we call biframes for *Next totally functional* (as their accessibility relations are functional and serial).

4.4.1 Local Formulas

While the reduction in Chapter 3 was based on $\circledcirc$-k-subformulas, the current generalization of it is based on *$\circledcirc$-k-local formulas*. This notion generalizes the *local formulas relation* from [58], that preserves prefixes of $\boxdot$-elements between formulas. A sequence $\bar{\Box} = \Box_1 \dots \Box_m$ ($m \geq 0$) of elements of $\boxdot$ is called a *$\boxdot$-prefix*. For any formula $\varphi \in \mathcal{L}$, $\bar{\Box}$ is a $\boxdot$-prefix of $\bar{\Box}\varphi$. The notation $\Box\mathcal{F}$ is naturally extended to prefixes $\bar{\Box}$.

Definition 4.4.1. Denote by $\bar{\Box}_\psi$ the longest (possibly, empty) $\boxdot$-prefix of ψ, and by b_ψ the formula for which $\psi = \bar{\Box}_\psi b_\psi$. A formula φ is *immediately $\circledcirc$-k-local* to a formula ψ if $\varphi = \bar{\Box}_\psi \varphi'$ for some immediate $\circledcirc$-k-subformula φ' of b_ψ. The $\circledcirc$-k-local formula relation is the reflexive transitive closure of the immediate $\circledcirc$-k-local formula relation. We denote the set of $\circledcirc$-k-local formulas of a formula ψ by $loc_k^{\circledcirc,\boxdot}(\psi)$. This notation is naturally extended to sequents, sets of sequents etc. When $\circledcirc = \emptyset$, we say that φ is *local* to ψ.

Note that for $\boxdot = \emptyset$, we have $loc_k^{\circledcirc,\boxdot}(\psi) = sub_k^{\circledcirc}(\psi)$ for every formula ψ.

Example 4.4.2. For $\boxdot = \{\Box, \boxtimes\}$, we have
$loc_1^{\{\neg\},\boxdot}(\Box(\boxtimes p_1 \supset p_2)) = \{\Box \boxtimes p_1, \Box\neg \boxtimes p_1, \Box p_2, \Box\neg p_2, \Box(\boxtimes p_1 \supset p_2)\}$.

The following lemma provides an alternative, inductive definition of the $\circledcirc$-k-local relation between formulas:

Lemma 4.4.3.

1. $loc_k^{\circledcirc,\boxdot}(p) = \{p\}$ for every $p \in At$.
2. $loc_k^{\circledcirc,\boxdot}(\circ\psi) = \{\circ\psi\} \cup loc_k^{\circledcirc,\boxdot}(\psi)$ for every $\circ \in \circledcirc$.
3. $loc_k^{\circledcirc,\boxdot}(\diamond(\psi_1, \dots, \psi_n)) = \{\diamond(\psi_1, \dots, \psi_n)\} \cup \bigcup_{1 \le i \le n} \circledcirc^{\le k}\psi_i \cup \bigcup_{1 \le i \le n} loc_k^{\circledcirc,\boxdot}(\psi_i)$ for every $\diamond \in \Diamond_{\mathcal{L}} \setminus \circledcirc$.
4. $loc_k^{\circledcirc,\boxdot}(\Box\psi) = \Box loc_k^{\circledcirc,\boxdot}(\psi)$.

Proof.

1. p does not have any immediate $\circledcirc$-k-local formulas.
2. ($\subseteq$): Let $\varphi \in loc_k^{\circledcirc,\boxdot}(\circ\psi)$. If $\varphi = \circ\psi$, then this trivially holds. Otherwise, by the definition of the $\circledcirc$-k-local relation as the reflexive transitive closure relation of the immediate one, either $\varphi = \psi$ or $\varphi \in loc_k^{\circledcirc,\boxdot}(\psi)$.
   ($\supseteq$): Obviously, $\circ\psi, \psi \in loc_k^{\circledcirc,\boxdot}(\circ\psi)$. From transitivity, $loc_k^{\circledcirc,\boxdot}(\psi) \subseteq loc_k^{\circledcirc,\boxdot}(\circ\psi)$.
3. ($\subseteq$): Let $\varphi \in loc_k^{\circledcirc,\boxdot}(\diamond(\psi_1, \dots, \psi_n))$. If $\varphi = \diamond(\psi_1, \dots, \psi_n)$, then we are done. If φ is immediately $\circledcirc$-k-local to $\diamond(\psi_1, \dots, \psi_n)$, then $\varphi \in \circledcirc^{\le k}\psi_i$ for some $1 \le i \le n$, and so we are also done. Otherwise, by the definition of the $\circledcirc$-k-local relation as the reflexive transitive closure of the immediate one, φ is $\circledcirc$-k-local to $\bar{\circ}\psi_i$ for some $\bar{\circ} \in \circledcirc^{\le k}$ and $1 \le i \le n$. So, either $\varphi = \bar{\circ}'\psi_i$ for some suffix $\bar{\circ}'$ of $\bar{\circ}$, or φ is $\circledcirc$-k-local to ψ_i. In the first case, $\varphi \in \bigcup_{1 \le i \le n} \circledcirc^{\le k}\psi_i$, and in the second, $\varphi \in loc_k^{\circledcirc,\boxdot}(\psi_i)$.
   ($\supseteq$): Obviously, $\diamond(\psi_1, \dots, \psi_n) \in loc_k^{\circledcirc,\boxdot}(\diamond(\psi_1, \dots, \psi_n))$. In addition, for every $\bar{\circ} \in \circledcirc^{\le k}$, $\bar{\circ}\psi_i$ is immediately $\circledcirc$-k-local to $\diamond(\psi_1, \dots, \psi_n)$, and hence it is in $loc_k^{\circledcirc,\boxdot}(\diamond(\psi_1, \dots, \psi_n))$. The same holds for ψ_i for every i, and from transitivity, it also holds for every element in $loc_k^{\circledcirc,\boxdot}(\psi_i)$.
4. ($\subseteq$): Let $\varphi \in loc_k^{\circledcirc,\boxdot}(\Box\psi)$. It is routine to show that $\varphi = \Box\varphi'$ for some φ', such that φ' is $\circledcirc$-k-local to ψ. Hence $\varphi' \in loc_k^{\circledcirc,\boxdot}(\psi)$ and therefore $\varphi = \Box\varphi' \in \Box loc_k^{\circledcirc,\boxdot}(\psi)$.
   ($\supseteq$): Similarly to the previous direction, let $\varphi \in \Box loc_k^{\circledcirc,\boxdot}(\psi)$. Then, $\varphi = \Box\varphi'$ for some φ' that is $\circledcirc$-k-local to ψ. Therefore, $\varphi = \Box\varphi'$ is $\circledcirc$-k-local to $\Box\psi$ and hence $\varphi \in loc_k^{\circledcirc,\boxdot}(\Box\psi)$. $\Box$

Similarly to the $\circledcirc$-k-subformula relation, since every formula has finitely many immediate $\circledcirc$-k-local formulas, it follows that $loc_k^{\circledcirc,\boxdot}(\psi)$ is finite for every $\psi \in \mathcal{L}_{\boxdot}$.

4.4.2 Extending The Reduction

For the case that the set S of assumptions is empty, it is possible to extend the reduction from Chapter 3 to pure calculi with *Next* operators. Like in the reduction for the non-modal case, we assume that the calculus $\mathbf{G}$ is axiomatic.

Definition 4.4.4. The SAT instance associated with a given axiomatic $\mathcal{L}$-calculus $\mathbf{G}$, a subset $\circledcirc$ of $\Diamond^1_{\mathcal{L}}$, a natural number $k \geq 0$, and an $\mathcal{L}_{\boxdot}$-sequent s, denoted $\mathtt{SAT}_k^{\circledcirc,\boxdot}(\mathbf{G}, s)$, is given by the union of the following SAT instances:

1. $\mathtt{SAT}^-(s)$
2. $\bigcup \left\{ \mathtt{SAT}^+(\bar{\Box}\sigma(s')) \mid \emptyset \,/\, s' \in \mathbf{G}, \bar{\Box}\sigma(\mathit{frm}(s')) \subseteq loc_k^{\circledcirc,\boxdot}(s), \bar{\Box} \text{ is an } \boxdot\text{-prefix} \right\}$

The following theorem establishes the correctness of the extended reduction.

Theorem 4.4.5. Let $\mathbf{G}$ be a standard axiomatic $\circledcirc$-k-analytic $\mathcal{L}$-calculus and s an $\mathcal{L}_{\boxdot}$-sequent. Then $\vdash_{\mathbf{G}_{\mathit{Next}}} s$ iff $\mathtt{SAT}_k^{\circledcirc,\boxdot}(\mathbf{G}, s)$ is unsatisfiable.

Proof. For a totally functional biframe $\langle W, \mathcal{R}, \mathcal{V} \rangle$ and a world $w \in W$, we denote by $\mathcal{R}_\Box(w)$ the world w' such that $\langle w, w' \rangle \in \mathcal{R}_\Box$ (w' always exists since $\langle W, \mathcal{R}, \mathcal{V} \rangle$ is totally functional). Then, we have $\mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box(w)}(\psi)$ whenever $\Box\psi \in \mathit{dom}(\mathcal{V}_w)$ and $\psi \in \mathit{dom}(\mathcal{V}_{\mathcal{R}_\Box(w)})$.
($\Rightarrow$): Suppose that $\not\vdash_{\mathbf{G}_{\mathit{Next}}} s$. By Theorem 4.2.7, we have $\mathcal{V}_w(s) = 0$ for some $\mathbf{G}$-legal $\mathcal{L}_{\boxdot}$-biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ for *Next* and $w \in W$. Consider the classical assignment u that assigns true to x_ψ iff $\mathcal{V}_w(\psi) = 1$. Since $\mathcal{V}_w(s) = 0$, u satisfies $\mathtt{SAT}^-(s)$. It remains to prove that $\mathcal{V}_w(\bar{\Box}\sigma(s')) = 1$ for every $\emptyset \,/\, s' \in \mathbf{G}$, substitution σ and $\boxdot$-prefix $\bar{\Box}$ such that $\bar{\Box}\sigma(\mathit{frm}(s')) \subseteq loc_k^{\circledcirc,\boxdot}(s)$. Suppose that $\bar{\Box} = \Box_1 \ldots \Box_n$, and let $w_0, w_1, \ldots, w_n$ be a sequence of worlds of $\mathcal{W}$ such that $w_0 = w$, and $\mathcal{R}_{\Box_i}(w_{i-1}) = w_i$ for every $1 \leq i \leq n$. Then $\mathcal{V}_{w_0}(\Box_1 \ldots \Box_n\psi) = \mathcal{V}_{w_1}(\Box_2 \ldots \Box_n\psi) = \ldots = \mathcal{V}_{w_n}(\psi)$ for every $\psi \in \mathcal{L}_{\boxdot}$. Since $\mathcal{W}$ is $\mathbf{G}$-legal, the bivaluation $\mathcal{V}_{w_n}$ is $\mathbf{G}$-legal, and therefore, $\mathcal{V}_w(\bar{\Box}\sigma(s')) = \mathcal{V}_{w_n}(\sigma(s')) = 1$.
($\Leftarrow$): Suppose that $\mathtt{SAT}_k^{\circledcirc,\boxdot}(\mathbf{G}, s)$ is satisfiable and let u be a satisfying assignment. Define the following biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$:

1. W is the set of all $\boxdot$-prefixes.
2. For every $\Box \in \boxdot$ and $\bar{\Box} \in W$, $\mathcal{R}_\Box(\bar{\Box}) = \bar{\Box}\Box$.

3. $\mathcal{V}_{\bar{\Box}}$ is defined by induction on the length of $\bar{\Box}$: $dom(\mathcal{V}_\epsilon) = loc_k^{\circledcirc,\boxdot}(s)$ and $\mathcal{V}_\epsilon(\psi) = 1$ iff u satisfies x_ψ; $dom(\mathcal{V}_{\Box_1 \dots \Box_n}) = \left\{\varphi \mid \Box_n\varphi \in dom(\mathcal{V}_{\Box_1 \dots \Box_{n-1}})\right\}$ and $\mathcal{V}_{\Box_1 \dots \Box_n}(\psi) = \mathcal{V}_{\Box_1 \dots \Box_{n-1}}(\Box_n\psi)$.

Clearly, $\mathcal{R}_\Box$ is a total function for every $\Box \in \boxdot$. Since u satisfies $\mathtt{SAT}^-(s)$, $\mathcal{V}_\epsilon(s) = 0$. We prove that $\mathcal{W}$ is a $\mathbf{G}$-legal $\circledcirc$-k-closed biframe for *Next* (see Definition 4.3.10).

1. biframe for *Next*: By the definition of $\mathcal{V}$.

2. $\mathbf{G}$-legal: We prove that $\mathcal{V}_{\Box_1 \dots \Box_n}$ is $\mathbf{G}$-legal for every $\Box_1 \dots \Box_n \in W$. Let $\emptyset \,/\, s' \in \mathbf{G}$ and σ a substitution such that $\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1 \dots \Box_n})$. We prove that $\mathcal{V}_{\Box_1 \dots \Box_n}(\sigma(s')) = 1$. We actually prove a stronger claim, namely that $\mathcal{V}_{\Box_1 \dots \Box_n}(\bar{\Box}\sigma(s')) = 1$ for every $\boxdot$-prefix $\bar{\Box}$ (including ϵ) such that $\bar{\Box}\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1 \dots \Box_n})$. We do so by induction on n. For $n = 0$ we have $\mathcal{V}_\epsilon(\bar{\Box}\sigma(s')) = 1$ because u satisfies $\mathtt{SAT}^+(\bar{\Box}\sigma(s'))$. Now, let $n \geq 1$. Since $\bar{\Box}\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1 \dots \Box_n})$, we have $\Box_n\bar{\Box}\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1 \dots \Box_{n-1}})$. By the induction hypothesis, $\mathcal{V}_{\Box_1 \dots \Box_{n-1}}(\Box_n\bar{\Box}\sigma(s')) = 1$. By $\mathcal{V}$'s definition, $\mathcal{V}_{\Box_1 \dots \Box_n}(\bar{\Box}\sigma(s')) = 1$.

3. $\circledcirc$-k-closed: $dom(\mathcal{V}_{\bar{\Box}})$ is finite for every $\bar{\Box}$ since $dom(\mathcal{V}_\epsilon) = loc_k^{\circledcirc,\boxdot}(s)$ is finite. In addition, if $\Box\psi \in dom(\mathcal{V}_{\bar{\Box}})$ then by our construction, $\psi \in dom(\mathcal{V}_{\bar{\Box}\Box})$. It remains to prove that for every $\bar{\Box} \in W$, $dom(\mathcal{V}_{\bar{\Box}})$ is $\circledcirc$-k-closed. First, note that every set which is closed under $\circledcirc$-k-local formulas is also $\circledcirc$-k-closed. This holds since ψ is $\circledcirc$-k-local to $\circ\psi$ for every $\circ \in \circledcirc$, and $\bar{\circ}\psi_i$ is $\circledcirc$-k-local to $\diamond(\psi_1, \dots, \psi_n)$ for every $1 \leq i \leq n$ and $\bar{\circ} \in \circledcirc^{\leq k}$. Therefore, it suffices to prove that $dom(\mathcal{V}_{\bar{\Box}})$ is closed under $\circledcirc$-k-local formulas for every $\bar{\Box} \in W$. We do so by induction on the length of $\bar{\Box}$. First, we have that $dom(\mathcal{V}_\epsilon) = loc_k^{\circledcirc,\boxdot}(s)$ is closed under $\circledcirc$-k-local formulas. Now, let $\Box_1 \dots \Box_n \in W$ ($n \geq 1$). We prove that $loc_k^{\circledcirc,\boxdot}(\psi) \subseteq dom(\mathcal{V}_{\Box_1 \dots \Box_n})$ for every $\psi \in dom(\mathcal{V}_{\Box_1 \dots \Box_n})$. Let $\psi \in dom(\mathcal{V}_{\Box_1 \dots \Box_n})$. Then, $\Box_n\psi \in dom(\mathcal{V}_{\Box_1 \dots \Box_{n-1}})$. By the induction hypothesis, $dom(\mathcal{V}_{\Box_1 \dots \Box_{n-1}})$ is closed under $\circledcirc$-k-local formulas. Therefore, $loc_k^{\circledcirc,\boxdot}(\Box_n\psi) \subseteq dom(\mathcal{V}_{\Box_1 \dots \Box_{n-1}})$. Now, let $\varphi \in loc_k^{\circledcirc,\boxdot}(\psi)$. Then, by Lemma 4.4.3, $\Box_n\varphi \in \Box_n loc_k^{\circledcirc,\boxdot}(\psi) = loc_k^{\circledcirc,\boxdot}(\Box_n\psi)$. Hence $\Box_n\varphi \in dom(\mathcal{V}_{\Box_1 \dots \Box_{n-1}})$. By $\mathcal{V}$'s definition, $\varphi \in dom(\mathcal{V}_{\Box_1 \dots \Box_n})$.

Now, since $\mathbf{G}$ is $\circledcirc$-k-analytic, By Lemma 4.3.12, $\mathcal{W}$ can be extended to a $\mathbf{G}$-legal $\mathcal{L}_\boxdot$-biframe $\langle W, \mathcal{R}, \mathcal{V}' \rangle$ for *Next*. By Theorem 4.2.6, since $\mathcal{V}'_\epsilon(s) = 0$, we have $\not\vdash_{\mathbf{G}_{Next}} s$. $\Box$

Note that Theorem 4.4.5 is restricted to derivability problems with an empty set of assumptions. The main difficulty with encoding a countermodel for the derivability of a sequent s from a set S of sequents, is that every element of S must hold in every world of the countermodel. This is in contrast to the rules of **G**, that are required to hold only in worlds whose domains include the instances of the rules. We leave the inclusion of sets of assumptions for future work.

For the case that $\circledcirc = \emptyset$ and $S = \emptyset$, the polynomial time algorithm from Theorem 3.1.6 can be modified to accommodate *Next* operators. We denote $\mathtt{SAT}_k^{\emptyset,\boxdot}$ by $\mathtt{SAT}^{\emptyset,\boxdot}$ for every k.

Theorem 4.4.6. Let **G** be an axiomatic $\mathcal{L}$-calculus. Given an $\mathcal{L}_{\boxdot}$-sequent s, the SAT instance $\mathtt{SAT}^{\emptyset,\boxdot}(\mathbf{G}, s)$ is computable in $O(n^m)$ time, where n is the length of the string representing s, and $m = c(\mathbf{G})$.

Proof. The algorithm from the proof of Theorem 3.1.6 is reused with several modifications. As in [44], an auxiliary trie (an ordered tree data structure commonly used for string processing) for $\boxdot$-prefixes is constructed in linear time, and every node in the input parse tree has a pointer to a node in this trie. Now, each node in the parse tree corresponds to an occurrence of a formula that is local to s. The tree is then compressed to a dag as in the proof of Theorem 3.1.6. The nodes of the dag one-to-one correspond to the local formulas of s. The rest of the algorithm is exactly as in the proof of Theorem 3.1.6. □

Following Section 3.2, we note that when **G** is a Horn calculus, $\mathtt{SAT}^{\emptyset,\boxdot}(\mathbf{G}, s)$ consists of Horn clauses. When, in addition, $c(\mathbf{G}) = 1$, a linear time decision procedure for the derivability problem for $\mathbf{G}_{Next}$ is obtained by applying a HORNSAT solver.

Example 4.4.7. Example 3.2.2 works also after adding *Next* operators: the derivability problem of $\mathbf{G}_{\mathbf{PIL}_{Next}}$ and $\mathbf{G}_{\mathbf{EPIL}_{Next}}$ can be solved in linear time using a HORNSAT solver.

4.4.3 Implementation

Our tool Gen2sat (see Section 3.3) supports $\emptyset$-analytic pure calculi that are augmented by a finite set of modal operators $\boxdot$, with the specification *Next*. In order to use the tool with such operators, an additional field should be added to the form described in Section 3.3.1: "Next Operators". This field contains a comma-separated list of unary connectives that do not occur in the field "connectives". Their arities are not mentioned, as they are assumed to be unary. Moreover, the rules (PF) and ($\mathrm{D_{PF}}$) should not be specified, as they are assumed by Gen2sat to be the rules that correspond to the operators in this list. In case the sequent is derivable, a sufficient subset of rules will be generated, excluding the modal rules (these are not encoded in any way to the SAT-solver, but only exist in the background of the reduction, and are the reason it is based on Kripke models). If the sequent is not derivable, Gen2sat outputs the truth values in the first world of the countermodel.

For example, adding quotations to the input file for primal infon logic from Figure 3.1, would result in the following input and output:

Input file

```
connectives: AND:2,OR:2,IMPLIES:2,TOP:0
nextOperators: q1 said, q2 said, q3 said
rule: =>p1; =>p2 / =>p1 AND p2
rule: p1,p2=> / p1 AND p2=>
rule: =>p1,p2 / =>p1 OR p2
rule: =>p2 / =>p1 IMPLIES p2
rule: =>p1; p2=> / p1 IMPLIES p2=>
rule: / => TOP
analyticity:
inputSequent: =>q1 said (p IMPLIES p)
```

Output

```
unprovable
Countermodel:
q1said p=false, q1said(p IMPLIES p)=false
```

4.5 Equivalence and Admissibility of Modal Rules

4.5.1 Functionality vs. Seriality

In this section, we show that the specifications $\{(\textsc{pf}), (\textsc{d}_{\textsc{pf}})\}$ and $\{(\textsc{k}), (\textsc{d}_{\textsc{k}})\}$ are equivalent when added to a pure Horn calculus (see Definition 2.7.4).

This is proven using a similar technique to the one used in the proof of Proposition 2.7.6.

Lemma 4.5.1. If $\mathbf{G}$ is a Horn calculus and $\mathsf{M}(\Box) = \{(\textsc{k}), (\textsc{d}_{\textsc{k}})\}$ for every $\Box \in \boxdot$, then for every set S of single-conclusion $\mathcal{L}_{\boxdot}$-sequents and $\mathcal{L}_{\boxdot}$-sequent $\Gamma \Rightarrow \Delta$, if $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma \Rightarrow \Delta$, then $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma \Rightarrow \mathcal{E}$ for some singleton or empty set $\mathcal{E} \subseteq \Delta$.

Proof. By induction on the length of the derivation of $\Gamma \Rightarrow \Delta$.

1. If $\Gamma \Rightarrow \Delta \in S$ or $\Gamma \Rightarrow \Delta$ is the conclusion of an application of $(\textsc{id})$ or $(\textsc{weak})$ then this is obvious.

2. If $\Gamma \Rightarrow \Delta$ is the conclusion of an application of $(\textsc{cut})$, then $\Gamma \Rightarrow \Delta = \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ and $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma_1 \Rightarrow \varphi, \Delta_1$ and $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma_2, \varphi \Rightarrow \Delta_2$ with shorter derivations. By the induction hypothesis, there are singletons or empty sets $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma_1 \Rightarrow \mathcal{E}_1$, $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma_2, \varphi \Rightarrow \mathcal{E}_2$, $\mathcal{E}_1 \subseteq \Delta_1 \cup \{\varphi\}$ and $\mathcal{E}_2 \subseteq \Delta_2$. If $\varphi \notin \mathcal{E}_1$ then using $(\textsc{weak})$ we obtain $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma \Rightarrow \mathcal{E}_1$. If $\mathcal{E}_1 = \{\varphi\}$, then using $(\textsc{cut})$, we get $S \vdash_{\mathbf{G}_{\mathsf{M}}} \Gamma \Rightarrow \mathcal{E}_2$.

3. If $\Gamma \Rightarrow \Delta$ is the conclusion of an application of some rule $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ of $\mathbf{G}$, then there is a substitution σ and sequents $\Gamma'_1 \Rightarrow \Delta'_1, \Gamma''_1 \Rightarrow \Delta''_1, \dots, \Gamma'_n \Rightarrow \Delta'_n, \Gamma''_n \Rightarrow \Delta''_n$ such that for every $1 \leq i \leq n$, $\Gamma'_i \Rightarrow \Delta'_i \subseteq \Gamma_i \Rightarrow \Delta_i$, $\Gamma \Rightarrow \Delta = \Gamma''_1, \dots, \Gamma''_n, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), \Delta''_1, \dots, \Delta''_n$, and $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i), \Delta''_i$ with shorter derivations for every $1 \leq i \leq n$. Since $\mathbf{G}$ is Horn, one of the following holds:

   (a) For every $1 \leq i \leq n$, $\Gamma'_i = \emptyset$. In this case, $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i \Rightarrow \sigma(\Delta'_i), \Delta''_i$ for every $1 \leq i \leq n$. By the induction hypothesis, for every $1 \leq i \leq n$, $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i \Rightarrow \mathcal{E}_i$ for some singleton or empty set $\mathcal{E}_i \subseteq \sigma(\Delta'_i) \cup \Delta''_i$. If $\mathcal{E}_i \subseteq \Delta''_i$ for some $1 \leq i \leq n$, then using (WEAK) we are done. Otherwise, for every $1 \leq i \leq n$, there exists $\varphi_i \in \sigma(\Delta'_i)$ such that $\mathcal{E}_i = \{\varphi_i\}$. Hence for every $1 \leq i \leq n$, $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i \Rightarrow \sigma(\varphi_i)$. Using (WEAK), we get that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i \Rightarrow \sigma(\Delta'_i)$. Now, we may apply the rule with context sequents $\Gamma''_i \Rightarrow$ and get that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_1, \dots, \Gamma''_n, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0)$, which means that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma \Rightarrow \sigma(\Delta_0)$. Since $\mathbf{G}$ is Horn, $|\sigma(\Delta_0)| \leq 1$.

   (b) There exists a single $1 \leq i \leq n$ such that $\Gamma'_i \neq \emptyset$. By the induction hypothesis, there exists a singleton or empty set $\mathcal{E}_i \subseteq \sigma(\Delta'_i) \cup \Delta''_i$ such that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i, \sigma(\Gamma'_i) \Rightarrow \mathcal{E}_i$. Also by the induction hypothesis, for every $j \neq i$, there exists a singleton or empty set $\mathcal{E}_j \subseteq \Delta''_j \cup \sigma(\Delta'_j)$ such that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_j \Rightarrow \mathcal{E}_j$ ($\Gamma_j = \emptyset$ since $i \neq j$, $\Gamma_i \neq \emptyset$ and $\mathbf{G}$ is Horn). If $\mathcal{E}_j \subseteq \Delta''_j$ for some $j \neq i$, then using (WEAK), we get that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma \Rightarrow \mathcal{E}_j$ (and $\mathcal{E}_j \subseteq \Delta$). Otherwise, for every $j \neq i$ there exists $\varphi_j \in \sigma(\Delta'_j)$ such that $\mathcal{E}_j = \{\varphi_j\}$. Using (WEAK) we get $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_j \Rightarrow \sigma(\Delta'_j)$ for every $j \neq i$. Moreover, using (WEAK) we get that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_i, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i), \mathcal{E}_i$. Apply the rule with context sequents $\Gamma''_i \Rightarrow \mathcal{E}_i$ and $\Gamma''_j \Rightarrow$ for every $j \neq i$ and get $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma''_1, \dots, \Gamma''_n, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), \mathcal{E}_i$. Since $\mathbf{G}$ is Horn and $\emptyset \neq \Gamma'_i \subseteq \Gamma_i$, $\Delta_0 = \emptyset$. Hence we actually get that $S \vdash_{\mathbf{G}_\mathsf{M}} \Gamma \Rightarrow \mathcal{E}_i$ (and $\mathcal{E}_i \subseteq \Delta$).

4. If $\Gamma \Rightarrow \Delta$ is the conclusion of an application of (K) or ($\mathrm{D_K}$) then it is clearly a single conclusion sequent. □

Corollary 4.5.2. Let $\mathbf{G}$ be a pure calculus, S a set of (strict) single-conclusion sequents and s a sequent. If $\mathbf{G}$ is Horn then $S \vdash_{\mathbf{G}_\mathsf{M}} s$ iff $S \vdash_{\mathbf{G}_{\mathsf{M}'}} s$, where $\mathsf{M}(\Box) = \{(\mathrm{PF}), (\mathrm{D_{PF}})\}$ for every $\Box$ and $\mathsf{M}'(\Box) = \{(\mathrm{K}), (\mathrm{D_K})\}$ for every $\Box$.

Proof. The right-to-left directions are trivial. Suppose that $S \vdash_{\mathbf{G}_\mathsf{M}} s$. We prove that $S \vdash_{\mathbf{G}_{\mathsf{M}'}} s$ by induction on the number of applications of ($\mathrm{D_{PF}}$) in the derivation of s from

S in $\mathbf{G}_{\mathsf{M}}$ (all applications of (PF) are also applications of (D_{PF})). If this number is 0 then the claim trivially holds. Otherwise, let $\langle \Gamma \Rightarrow \Delta, \Box\Gamma \Rightarrow \Box\Delta \rangle$ be the last application of (D_{PF}) in the derivation. By the induction hypothesis, $S \vdash_{\mathbf{G}_{\mathsf{M}'}} \Gamma \Rightarrow \Delta$. Since $\mathbf{G}$ is Horn, by Lemma 4.5.1, there exists a singleton or empty set $\mathcal{E} \subseteq \Delta$ such that $S \vdash_{\mathbf{G}_{\mathsf{M}'}} \Gamma \Rightarrow \mathcal{E}$. Apply either (K) or ($\text{D}_{\text{K}}$) and get $S \vdash_{\mathbf{G}_{\mathsf{M}'}} \Box\Gamma \Rightarrow \Box\mathcal{E}$. Now, use weakening and get $S \vdash_{\mathbf{G}_{\mathsf{M}'}} \Box\Gamma \Rightarrow \Box\Delta$. □

Example 4.5.3. Since $\mathbf{G}_{\mathbf{PIL}}$ (Example 2.2.10) is Horn, by Corollary 4.5.2, we may use either $\{(\text{K}), (\text{D}_{\text{K}})\}$ or $\{(\text{PF}), (\text{D}_{\text{PF}})\}$ as the rules for its *said* operators (see Example 4.1.3). The same holds for $\mathbf{G}_{\mathbf{EPIL}}$.

4.5.2 On the Admissibility of D-rules

Figure 4.1 associates each modal rule (X) with its own (D_{X})-rule, that essentially allows it to be applied with an empty right side. This is needed, for example, in the following derivation of the sequent $\Box\bot \Rightarrow$, which is valid in the modal logic KD, but not in K:

$$\frac{\bot \Rightarrow}{\Box\bot \Rightarrow}$$

For definite calculi (see Definition 2.7.2), however, we have that the addition of any of (D_{K}), (D_4), (D_{45}) and (D_{PF}), is redundant:

Proposition 4.5.4. Let $\mathbf{G}$ a definite calculus, S a set of sequents with non-empty right sides, and s a sequent such that $S \vdash_{\mathbf{G}_{\mathsf{M}}} s$. Let M' be obtained from M by removing (D_{X}) from $\mathsf{M}(\Box)$ whenever $X \in \{(\text{K}), (4), (45), (\text{PF})\}$ and $(\text{D}_{\text{X}}) \in \mathsf{M}(\Box)$. Then, $S \vdash_{\mathbf{G}_{\mathsf{M}'}} s$.

Proof. First, since $\mathbf{G}$ is definite, using induction on the length of the derivation of s from S in $\mathbf{G}_{\mathsf{M}}$, it can be shown that all sequents in this derivation have a non-empty right side. Next, we induct on the number of applications of (D_{X}). Let $\langle \Gamma \Rightarrow \Delta, s \rangle$ be the last application of (D_{X}). Then, $\Delta \neq \emptyset$, and in particular, X cannot be (K) or (4). By the induction hypothesis, $S \vdash_{\mathbf{G}_{\mathsf{M}'}} \Gamma \Rightarrow \Delta$. If $X = (45)$ then $\Gamma = \Box\Gamma_1 \cup \Gamma_2$ and $\Delta = \Box\Delta'$ for some $\Gamma_1, \Gamma_2, \Delta'$. Since $\emptyset \neq \Delta = \Box\Delta'$, we have $\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\psi, \Box\Delta'$ as the premise for some $\psi \in \Delta'$. Using (WEAK), we may obtain $\Box\Gamma_1, \Gamma_2 \Rightarrow \psi, \Box\psi, \Box\Delta'$ and now using (45), we get $\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi, \Box\Delta'$, which is s. If $X = (\text{PF})$ then since $\Delta \neq \emptyset$, $\langle \Gamma \Rightarrow \Delta, s \rangle$ is also an application of (PF). □

Note that (D_{B}) and (D_{B4}) are not admissible under the conditions above. Indeed, the sequent $\Box\Box p \Rightarrow p$ is derivable using either (D_{B}) or (D_{B4}), while it is not derivable using (B) or (B4).

Example 4.5.5. Following Example 4.5.3, when augmenting the definite calculus $\mathbf{G_{PIL}}$ with (K) ((PF)), one can either include (D_{K}) ((D_{PF})) or not, and the resulting calculi would be equivalent.

Chapter 5

Intuitionistic Calculi

We have seen that while pure sequent calculi form a useful and prominent framework for non-classical logics, some logics seem to require impure calculi. This was indeed the case in Chapter 4, where multimodal logics were studied. In this section we tackle another family of logics that seem to go beyond the scope of pure calculi, whose proof-theory resembles that of intuitionistic logic. The propositional fragment of Gentzen's original calculus for intuitionistic logic, LJ, is not pure, as it does not meet the requirement of allowing arbitrary context formulas in applications of rules: it manipulates only *single-conclusion sequents*. An equivalent cut-free sequent calculus, which we call $\mathbf{LJ'}$, was presented in [77, 97]. This calculus employs multiple-conclusion sequents, and restricts only the right introduction rules of implication and negation to apply on single-conclusion sequents. In other words, $\mathbf{LJ'}$ is obtained from $\mathbf{LK}$ by adding the requirement that in applications of $(\Rightarrow \supset)$ and $(\Rightarrow \neg)$, the context sequent has an empty right side. Such applications have the forms:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$$

Taking $\mathbf{LJ'}$ as a prototype, we consider a family of calculi that have similar structural properties. We call the calculi of this family *intuitionistic calculi*. The main result of this chapter is a variant of Theorem 2.6.20 for intuitionistic calculi, which identifies a sub-family in which cut-admissibility and analyticity coincide. Our proof has a similar general structure to the proof for pure calculi, but is more challenging, because valuation functions do not suffice to characterize the calculi of this family. Instead, we modify the Kripke semantics from Chapter 4 and accommodate it to intuitionistic calculi.

In Section 5.1 we explicitly define intuitionistic calculi, by restricting derivations in pure calculi. Section 5.2 provides a semantic counterpart for derivability in intuitionistic calculi. This semantics is then used in Section 5.3 in order to derive cut-admissibility from analyticity in a wide sub-family of intuitionistic calculi, as was shown in Section 2.6

for pure calculi.

Publications Related to this Chapter

Most of the results of this section appeared in [71, 74].

5.1 Intuitionistic Derivations

In Chapter 2 we studied the family of pure calculi, that can be seen as a generalization of **LK**. Just like **LK**, pure calculi employ all structural rules, and do not enforce any limitation on the context sequents. In this section we study a family of calculi that generalizes $\mathbf{LJ}'$. We first identify the structural characteristics of $\mathbf{LJ}'$: $\mathbf{LJ}'$ is obtained from **LK** by forbidding right context formulas in all premises with a non-empty left side of right-introduction rules. Another well-known calculus that follows this pattern, which we call $\mathbf{G}'_4$, is obtained by extending the positive fragment of $\mathbf{LJ}'$ with the rules for negation of $\mathbf{G}_4$ (see Example 2.2.8). $\mathbf{G}'_4$, investigated in [14, 100], is sound and complete for Nelson's paraconsistent constructive logic $N4$ [82].

We precisely define a wide family of impure calculi, that we call *intuitionistic calculi*, of which $\mathbf{LJ}'$ and $\mathbf{G}'_4$ are particular examples. Note that both $\mathbf{LJ}'$ and $\mathbf{G}'_4$ employ pure rules: $\mathbf{LJ}'$ is based on the rules of **LK**, and $\mathbf{G}'_4$ is based on the rules of $\mathbf{G}_4$. The difference lies in the allowed applications of the rules: both calculi forbid right context formulas in premises of the form $\Gamma \Rightarrow \Delta$ with $\Gamma \neq \emptyset$ of applications of rules that introduce some formula on the right side. This is formalized as follows:

Definition 5.1.1. A pure rule is called *positive* if its conclusion has the form $\Gamma \Rightarrow \Delta$ for some $\Delta \neq \emptyset$. A derivation in a pure calculus **G** is called *intuitionistic* if in every application $\langle \{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n\}, \sigma(s_0) \cup c_1 \cup \dots \cup c_n \rangle$ of a positive rule $s_1, \dots, s_n \,/\, s_0$, for every $1 \leq i \leq n$ we have that if s_i has a non-empty left side then c_i has an empty right side.

Example 5.1.2. Consider the rule $(\Rightarrow \supset)$ of **LK**: $p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$. $(\Rightarrow \supset)$ is positive, and its only premise has a non-empty left side. Applications of this rule in intuitionistic derivations will have a context sequent with an empty right side, and will therefore have one of the following forms:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \varphi \supset \psi}$$

Note that in the second application, only the subsequent $\Rightarrow p_2$ of the premise $p_1 \Rightarrow p_2$ is used, and therefore the premise of the *application* has an empty left side. Nevertheless, since the premise of the *rule* has a non-empty left side, the context sequent still must have an empty right side.

In contrast, $(\supset\Rightarrow)$ is not positive, and therefore applications of it in intuitionistic derivations have the same form as in ordinary derivations.

Remark 5.1.3. If **G** is Horn (see Definition 2.7.4), then every derivation in it is intuitionistic, simply because positive rules with a non-empty left side of a premise do not exist.

Derivability, cut-admissibility and analyticity are adopted to intuitionistic derivations as follows.

Definition 5.1.4. For a pure calculus **G**, we write $S \vdash_{\mathbf{G_{Int}}} s$ if there is an intuitionistic derivation of a sequent s from a set S of sequents in **G**. We write $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} s$ if there is such a derivation which is also cut-limited, and $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} s$ if there is such a derivation which contains only $\mathcal{F}$-formulas. We say that **G** enjoys *Int-cut-admissibility* if $\vdash_{\mathbf{G_{Int}}} = \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}}$, and is *Int-⊚-k-analytic* if $S \vdash_{\mathbf{G_{Int}}} s$ iff $S \vdash^{sub^{\circledcirc}_{k}(S\cup\{s\})}_{\mathbf{G_{Int}}} s$ for every set S of sequents and sequent s. If **G** is Int-$\emptyset$-k-analytic for some k, we call it *Int-$\emptyset$-analytic*.

The difference between pure and intuitionistic calculi is not in the *rules*, but rather in the allowed *applications*. Thus, any pure calculus has an intuitionistic counterpart, obtained by considering only intuitionistic derivations.

Example 5.1.5. Derivations in **LJ**′ are exactly the intuitionistic derivations of **LK**. Indeed, for a finite set Γ of formulas and a formula φ, φ follows from Γ in intuitionistic logic iff $\vdash_{\mathbf{LK_{Int}}} \Gamma \Rightarrow \varphi$. In contrast, φ follows from Γ in classical logic iff $\vdash_{\mathbf{LK}} \Gamma \Rightarrow \varphi$. Similarly, derivations in $\mathbf{G}'_4$ coincide with intuitionistic derivations of $\mathbf{G}_4$.

Our main theorem concerning intuitionistic calculi is presented next.

Theorem 5.1.6. Every Int-⊚-k-analytic ⊚-k-directed pure calculus enjoys Int-cut-admissibility.

Theorem 5.1.6, which we prove below, allows one to derive the fact that cut is admissible in **LJ**′ from the fact that **LJ**′ enjoys the subformula property. More precisely, Int-cut-admissibility of **LK** follows from its Int-$\emptyset$-analyticity. Such entailment also holds for the pure calculi presented in the examples of Chapter 2, as well as for the calculi of the next example.

Example 5.1.7 (Constructive Negations)**.** The paper [14] includes sequent calculi for logics that replace classical negation with several non-classical negations. One of the families investigated there consists of calculi that are obtained from the positive fragment of **LJ**′ by augmenting it with pure rules for negation. All calculi of this family, except those described in Example 5.1.8 below, allow only intuitionistic derivations, and are $\{\neg\}$-1-directed and Int-$\{\neg\}$-1-analytic. From these facts, Theorem 5.1.10 allows us to conclude that cut is admissible in them. These calculi include a calculus for Nelson's constructive logic N_3 [82], as well as the calculus $\mathbf{G}'_4$ presented above for its paraconsistent variant N_4.

Intuitionistic derivations disallow right context formulas in premises of positive rules, in which the left side is not empty. A natural question that arises regarding Theorem 5.1.6 is: Does it still hold if we allow right context formulas for certain premises of a right introduction rule with a non-empty left side, and forbid them in others? The answer is negative as the next example demonstrates.

Example 5.1.8 (Beyond Intuitionistic Derivations)**.** Following Example 5.1.7, we note that [14, 16] investigate also several calculi that include *both* the single-conclusion right-introduction rule of implication and the multiple-conclusion right-introduction rule of negation, namely:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$$

The first conforms with the restriction to intuitionistic derivations, as right context formulas are forbidden. The second allows for non-intuitionistic derivations, as it allows right context formulas in a premise of a positive rule, that has a non-empty left side. Such calculi are therefore left out from the scope of Theorems 2.6.20 and 5.1.10. And indeed, as was shown in [16], all of them are $\{\neg\}$-1-analytic, but none of them enjoys cut-admissibility.

Similarly to Section 2.6, we prove a stronger version of Theorem 5.1.6, for a more generalized notion of analyticity. Thus, in what follows, $\prec$ denotes an arbitrary safe and structural order relation (see Definition 2.6.16).

Definition 5.1.9. We say that a pure calculus $\mathbf{G}$ is Int-$\prec$-analytic if $S \vdash_{\mathbf{G_{Int}}} s$ iff $S \vdash_{\mathbf{G_{Int}}}^{\preceq[S\cup\{s\}]} s$ for every set S of sequents and sequent s.

Theorem 5.1.10. Every Int-$\prec$-analytic $\prec$-directed pure calculus enjoys Int-cut-admissibility.

Theorem 5.1.6 is obtained as a particular instance, as the $\circledcirc$-k-subformula relation is a safe and structural order relation for every $\circledcirc$ and k.

The rest of this chapter is devoted to the proof of Theorem 5.1.10. Similarly to the case of pure calculi, we go through a semantic interpretation of intuitionistic derivations, that is defined next.

5.2 Semantics

Similarly to the case of modal operators (see Chapter 4), the syntactic restrictions on context sequents that are enforced in intuitionistic derivations can be naturally captured by a Kripke-style semantics.

While many of the results in previous chapters relied on two-valued (partial) models, the models that were used in Section 2.6, whose results are generalized here to intuitionistic calculi, were three-valued. Therefore, in this chapter we use three-valued models from the beginning, without going through partial two-valued models. This way, we gain a more direct connection between the semantics of cut-limited derivations and derivations that include only (generalized) subformulas of the derived sequent.

We start by defining the three-valued Kripke-style models that will be used.

Definition 5.2.1. An $\mathcal{L}$-*triframe* is a tuple $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ where:

1. W is a set of elements called *worlds*.

2. $\mathcal{R}$ is a transitive and reflexive relation over W.

3. $\mathcal{V}$ is a function assigning a trivaluation $\mathcal{V}_w$ to every $w \in W$ (see Definition 2.6.2), such that $\mathcal{V}$ is *persistent*: for every $\varphi \in \mathcal{L}$ and $w \in W$, if $\mathcal{V}_w(\varphi) = 1$ then $\mathcal{V}_u(\varphi) = 1$ for every u such that $w\mathcal{R}u$.

We adopt the notations for biframes (see Definition 4.2.1 and notation 4.2.3) to triframes. For example, we denote the set $\{u \in W \mid w\mathcal{R}u\}$ by $\mathcal{R}[w]$ and $\min \{\mathcal{V}_w(s) \mid w \in W', s \in S\}$ by $\mathcal{V}_{W'}(S)$.

Remark 5.2.2. Aside from being based on three truth values, there are several other differences between triframes and biframes:

1. Triframes are totally defined – every formula is assigned a value in every world. In contrast, every world in a biframe has its own domain.

2. While biframes are defined over an extended language which includes modal operators, triframes are defined on the same language as trivaluations. The global semantics (used for modal operators in biframes) is used for a subset of the connectives of the language.

3. There is only one accessibility relation, and it must be transitive and reflexive.

4. The persistency condition of triframes is not enforced in biframes (although it may be seen as a counterpart of the semantics of $\Box$ in biframes).

We turn to the semantic reading of pure rules in triframes. Given a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ and a world w in it, the interpretation of a pure rule in w may involve not only w itself, but also the elements of $\mathcal{R}[w]$:

Definition 5.2.3. A triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ *respects* a pure rule $r = s_1, \dots, s_n \,/\, s$ if one of the following holds for every $w \in W$, substitution σ, and respective subsequents $s'_1, \dots, s'_n$ of $s_1, \dots, s_n$:

- r is positive, and $\mathcal{V}_w(\sigma(s)) = 1$ whenever both of the followings hold:
  - $\mathcal{V}_w(\sigma(s'_i)) = 1$ for every $1 \leq i \leq n$ such that s_i has an empty left side; and
  - $\mathcal{V}_{\mathcal{R}[w]}(\sigma(s'_i)) = 1$ for every $1 \leq i \leq n$ such that s_i has a non-empty left side.
- r is not positive, and $\mathcal{V}_w(\sigma(s))$ whenever $\mathcal{V}_w(\sigma(s'_i)) = 1$ for every $1 \leq i \leq n$.

Definition 5.2.4. For a pure calculus **G**, a triframe $\mathcal{W}$ is called **G**-*legal* if it respects the rules of **G**.

Finally, the notion of determined trivaluations (Definition 2.6.5) is generalized to apply on triframes in a natural way:

Definition 5.2.5. For a set $\mathcal{F} \subseteq \mathcal{L}$, a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ is called $\mathcal{F}$-*determined* if $\mathcal{V}_w$ is $\mathcal{F}$-determined for every $w \in W$. $\mathcal{W}$ is called *fully determined* if $\mathcal{V}_w$ is fully determined for every $w \in W$.

The semantic reading of derivation rules in Definition 5.2.3 conforms with the essence of Kripke's semantics for intuitionistic logic, according to which "constructive truth" is more demanding than "classical truth". Thus, when introducing a formula on the right side, the premises should be satisfied not only in the particular world in question, but also in all its accessible worlds.

Example 5.2.6 (Semantics of Intuitionistic Logic)**.** Recall that **LJ′** is obtained by considering only intuitionistic derivations in **LK**. Accordingly, Definition 5.2.4 associates **LJ′** with **LK**-legal *triframes*. It is easy to see that a fully determined triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ is **LK**-legal iff it respects the usual truth conditions of the intuitionistic connectives. For example, the truth condition for implication is obtained as follows: Let $w \in W$, and suppose that for every $u \in W$ such that $w\mathcal{R}u$, we have that if $\mathcal{V}_u(p_1) = 1$ then also $\mathcal{V}_u(p_2) = 1$. Then $\mathcal{V}_{\mathcal{R}[w]}(p_1 \Rightarrow p_2) = 1$. Since $\mathcal{W}$ is **LK**-legal, $\mathcal{V}_w(\ \Rightarrow p_1 \supset p_2) = 1$, which gives us $\mathcal{V}_w(p_1 \supset p_2) = 1$. For the converse, suppose there exists some $u \in W$ such that $w\mathcal{R}u$, $\mathcal{V}_u(p_1) = 1$ and $\mathcal{V}_u(p_2) = 0$. Then $\mathcal{V}_u(\ \Rightarrow p_1) = \mathcal{V}_u(p_2 \Rightarrow\ ) = 1$. Since $\mathcal{W}$ is **LK**-legal, $\mathcal{V}_u(p_1 \supset p_2) = 0$. By the persistency requirement, $\mathcal{V}_w(p_1 \supset p_2) = 0$ as well.

After setting all required adjustments in definitions, we prove soundness and completeness theorems for intuitionistic derivations.

Theorem 5.2.7.

1. $S \vdash_{\mathbf{G_{Int}}} s$ iff $\mathcal{V}_W(S) \leq \mathcal{V}_W(s)$ for every fully determined **G**-legal triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$.

2. $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} s$ iff $\mathcal{V}_W(S) \leq \mathcal{V}_W(s)$ for every *frm*(S)-determined **G**-legal triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$.

3. If $S \vdash^{\preceq[S \cup \{s\}]}_{\mathbf{G_{Int}}} s$ then $\mathcal{V}_W(S) \leq \mathcal{V}_W(s)$ for every $\preceq [S \cup \{s\}]$-determined **G**-legal triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$. The converse also holds, provided that **G** is $\prec$-ordered (see Definition 2.6.19).

Proof.

Soundness:

1. Suppose $S \vdash_{\mathbf{G_{Int}}} s$, and let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ be a fully determined **G**-legal triframe. Assume $\mathcal{V}_W(S) = 1$. We prove that $\mathcal{V}_w(s) = 1$ for every $w \in W$, by induction on the length of the intuitionistic derivation of s from S in **G**. The cases where $s \in S$ or s is the conclusion of an application of (ID) or (WEAK) are trivial.

   If s is the conclusion of (CUT), then there are $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ and ψ such that $s = \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$, and both $S \vdash_{\mathbf{G_{Int}}} \Gamma_1 \Rightarrow \psi, \Delta_1$ and $S \vdash_{\mathbf{G_{Int}}} \Gamma_2, \psi \Rightarrow \Delta_2$ with shorter derivations. Let $w \in W$. By the induction hypothesis, $\mathcal{V}_w(\Gamma_1 \Rightarrow \psi, \Delta_1) = \mathcal{V}_w(\Gamma_2, \psi \Rightarrow \Delta_2) = 1$. Since $\mathcal{W}$ is fully determined, $\mathcal{V}_w(\psi) \neq 1/2$, and so $\mathcal{V}_w(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) = 1$.

   Now suppose s is the conclusion of an application $\dfrac{\sigma(s'_1) \cup c_1 \ldots \sigma(s'_n) \cup c_n}{\sigma(s_0) \cup c_1 \cup \ldots \cup c_n}$ of a rule $r = s_1, \ldots, s_n \,/\, s_0$ of **G**. Then $s = \sigma(s_0) \cup c_1 \cup \ldots \cup c_n$, and $S \vdash_{\mathbf{G_{Int}}} \sigma(s'_i) \cup c_i$ for every $1 \leq i \leq n$, with shorter derivations. By the induction hypothesis, $\mathcal{V}_w(\sigma(s'_i) \cup c_i) = 1$ for every $1 \leq i \leq n$ and $w \in W$. Let $w \in W$. Assume for contradiction that $\mathcal{V}_w(\sigma(s_0) \cup c_1 \cup \ldots \cup c_n) = 0$. In particular, $\mathcal{V}_w(\sigma(s_0)) = 0$, and $\mathcal{V}_w(\sigma(c_i)) = 0$ for every $1 \leq i \leq n$.

   Assume first that r is positive. Then, since $\mathcal{W}$ is **G**-legal, we have some $1 \leq i \leq n$ such that either s_i has an empty left side and $\mathcal{V}_w(\sigma(s'_i)) = 0$, or s_i has a non-empty left side and $\mathcal{V}_{w'}(\sigma(s'_i)) = 0$ for some $w' \in \mathcal{R}[w]$. In the first case, $\mathcal{V}_w(\sigma(s'_i) \cup c_i) = 0$, contradicting the induction hypothesis. In the second, c_i must have the form $\Omega'_i \Rightarrow$. Hence, for every $\psi \in \Omega_i$ we have $\mathcal{V}_w(\psi) = 1$. Since $\mathcal{W}$ is a triframe, we also have

$\mathcal{V}_{w'}(\psi) = 1$ for every such ψ. Hence, $\mathcal{V}_{w'}(\sigma(s_i') \cup c_i) = \mathcal{V}_{w'}(\sigma(s_i') \cup (\Omega_i \Rightarrow\ )) = 0$, contradicting the induction hypothesis.

Now assume that r is not positive. Then since $\mathcal{W}$ is **G**-legal, we have some $1 \leq i \leq n$ such that $\mathcal{V}_w(\sigma(s_i')) = 0$, and thus $\mathcal{V}_w(\sigma(s_i') \cup c_i) = 0$ for that i, which contradicts the induction hypothesis.

2. For $\vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}}$, the proof is similar. The only difference is in the case where s is the conclusion of (CUT). In such a case, the cut formula ψ is guaranteed to be an element of $frm(S)$. Since $\mathcal{W}$ is assumed to be $frm(S)$-determined, we have $\mathcal{V}_w(\psi) \neq 1/2$. The proof then carries as in the previous item, also for the case that s is the conclusion of a rule of **G**.

3. Denote the set $\preceq[S \cup \{s\}]$ by $\mathcal{F}$. Suppose $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} s$, and let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ be an $\mathcal{F}$-determined **G**-legal triframe. Assume $\mathcal{V}_W(S) = 1$. We prove that $\mathcal{V}_w(s) = 1$ for every $w \in W$, by induction on the length of the intuitionistic derivation of s from S in **G**, that only includes $\mathcal{F}$-formulas. The proof is similar to that of $\vdash_{\mathbf{G_{Int}}}$, with the only difference being the cut rule. If s is the conclusion of (CUT), then the cut formula ψ, as a formula that occurs in the derivation, is guaranteed to be an element of $\mathcal{F}$. Since $\mathcal{W}$ is assumed to be $\mathcal{F}$-determined, $\mathcal{V}_w(\psi) \neq 1/2$. The proof carries as before, also for the case that s is the conclusion of a rule of **G**.

Completeness:

1. Recall the definitions concerning ω-sequents from the proof of Theorem 2.3.5, and adapt them to $\vdash_{\mathbf{G_{Int}}}$. We say that an ω-sequent $L \Rightarrow R$ is maximal unprovable if the following holds:

   - $S \nvdash_{\mathbf{G_{Int}}} L \Rightarrow R$
   - $S \vdash_{\mathbf{G_{Int}}} L, \psi \Rightarrow R$ for every $\psi \notin L$ and $S \vdash_{\mathbf{G_{Int}}} L \Rightarrow \psi, R$ for every $\psi \notin R$.

   It is routine to extend any ω-sequent $L \Rightarrow R$ such that $S \nvdash_{\mathbf{G_{Int}}} L \Rightarrow R$ to a maximal unprovable ω-sequent. In particular, s can be extended to such an ω-sequent $L_s \Rightarrow R_s$.

   Define a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ as follows: W is the set of all maximal unprovable ω-sequents; $(L \Rightarrow R)\mathcal{R}(L' \Rightarrow R')$ iff $L \subseteq L'$; and for every $L \Rightarrow R \in W$ and $\varphi \in \mathcal{L}$, $\mathcal{V}_{L\Rightarrow R}(\varphi) = 1$ if $\varphi \in L$, $\mathcal{V}_{L\Rightarrow R}(\varphi) = 0$ if $\varphi \in R$, and $\mathcal{V}_{L\Rightarrow R}(\varphi) = 1/2$ otherwise.

   $\mathcal{W}$ is indeed a triframe, as $\mathcal{R}$ is transitive and reflexive, and if $\varphi \in L$ and $(L \Rightarrow R)\mathcal{R}(L' \Rightarrow R')$ then $\varphi \in L'$. The fact that $\mathcal{V}_W(S) > \mathcal{V}_W(s)$ is shown similarly to the proof of Theorem 2.3.5, using the fact that $\mathcal{V}_{L_s \Rightarrow R_s}(s) = 0$. We show that $\mathcal{W}$ is fully determined. Let $L \Rightarrow R \in W$ and assume for contradiction that there exists a formula $\varphi \notin L \cup R$. Then $S \vdash_{\mathbf{G_{Int}}} L \Rightarrow \varphi, R$ and $S \vdash_{\mathbf{G_{Int}}} L, \varphi \Rightarrow R$. Using (CUT), we get that $S \vdash_{\mathbf{G_{Int}}} L \Rightarrow R$, which is a contradiction.

It is left to prove that $\mathcal{W}$ is $\mathbf{G}$-legal. Let $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a positive rule of $\mathbf{G}$, σ a substitution, $\Gamma'_1 \Rightarrow \Delta'_1, \ldots, \Gamma'_n \Rightarrow \Delta'_n$ respective subsequents of $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$, and $L \Rightarrow R \in W$. Suppose $\mathcal{V}_{L \Rightarrow R}(\sigma(\Gamma_0 \Rightarrow \Delta_0)) = 0$. We prove that there exists $1 \leq i \leq n$ such that either $\Gamma_i = \emptyset$ and $\mathcal{V}_{L \Rightarrow R}(\sigma(\ \Rightarrow \Delta'_i)) = 0$ or $\Gamma_i \neq \emptyset$ and $\mathcal{V}_{L' \Rightarrow R'}(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 0$ for some $L' \Rightarrow R' \in W$ such that $(L \Rightarrow R)\mathcal{R}(L' \Rightarrow R')$. By our assumption, $\mathcal{V}_{L \Rightarrow R}(\sigma(\varphi)) = 1$ for every $\varphi \in \Gamma_0$, and $\mathcal{V}_{L \Rightarrow R}(\sigma(\psi)) = 0$ for every $\psi \in \Delta_0$. This means that $S \nvdash_{\mathbf{G_{Int}}} L, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), R$. Thus, we must either have $S \nvdash_{\mathbf{G_{Int}}} L, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i)$ for some $1 \leq i \leq n$ with $\Gamma_i \neq \emptyset$, or $S \nvdash_{\mathbf{G_{Int}}} L \Rightarrow \sigma(\Delta'_i), R$ for some $1 \leq i \leq n$ with $\Gamma_i = \emptyset$. In the first case, extend $L, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i)$ to a maximal unprovable ω-sequent $L' \Rightarrow R'$. Clearly, $L \subseteq L'$, and $\mathcal{V}_{L' \Rightarrow R'}(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 0$. In the second, the fact that $L \Rightarrow R$ is maximal unprovable ensures that $\sigma(\Delta'_i) \subseteq R$, which means that $\mathcal{V}_{L \Rightarrow R}(\sigma(\ \Rightarrow \Delta'_i)) = 0$.

Next, let $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow$ be a non-positive rule of $\mathbf{G}$, σ a substitution, $\Gamma'_1 \Rightarrow \Delta'_1, \ldots, \Gamma'_n \Rightarrow \Delta'_n$ respective subsequents of $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$, and $L \Rightarrow R \in W$. Suppose $\mathcal{V}_{L \Rightarrow R}(\sigma(\Gamma_0) \Rightarrow\ ) = 0$. We prove that $\mathcal{V}_{L \Rightarrow R}(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 0$ for some $1 \leq i \leq n$. By our assumption, $\mathcal{V}_{L \Rightarrow R}(\sigma(\varphi)) = 1$ for every $\varphi \in \Gamma_0$. This means that $\sigma(\Gamma_0) \subseteq L$, and so $S \nvdash_{\mathbf{G_{Int}}} L, \sigma(\Gamma_0) \Rightarrow R$. In particular, we have $S \nvdash_{\mathbf{G_{Int}}} L, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i), R$ for some $1 \leq i \leq n$. By the fact that $L \Rightarrow R$ is maximal unprovable, we have that $\sigma(\Gamma'_i) \subseteq L$ and $\sigma(\Delta'_i) \subseteq R$, which means that $\mathcal{V}_{L \Rightarrow R}(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 0$.

2. We say that an ω-sequent is *cf*-maximal unprovable if $S \nvdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L \Rightarrow R$, $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L, \psi \Rightarrow R$ for every $\psi \notin L$, and $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L \Rightarrow \psi, R$ for every $\psi \notin R$. It is routine to extend any ω-sequent $L \Rightarrow R$ such that $S \nvdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L \Rightarrow R$ to a *cf*-maximal unprovable ω-sequent. Define a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ such that W is the set of all *cf*-maximal unprovable ω-sequents, and $\mathcal{R}$ and $\mathcal{V}$ are defined as in the proof for $\vdash_{\mathbf{G_{Int}}}$. The proofs that $\mathcal{W}$ is a triframe, $\mathcal{V}_W(S) > \mathcal{V}_W(s)$, and that $\mathcal{W}$ is $\mathbf{G}$-legal are similar. However, $\mathcal{W}$ need not be fully determined, but only *frm*(S)-determined. Indeed, let $L \Rightarrow R \in W$, and assume for contradiction that there exists $\varphi \in \mathit{frm}(S) \setminus (L \cup R)$. Then $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L \Rightarrow \varphi, R$ and $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L, \varphi \Rightarrow R$. Since $\varphi \in \mathit{frm}(S)$, it is allowed to be a cut formula in cut-limited derivations from S. Using (CUT), we get that $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} L \Rightarrow R$, which is a contradiction.

3. Denote the set $\preceq[S \cup \{s\}]$ by $\mathcal{F}$. We say that an ω-sequent is $\mathcal{F}$-maximal unprov-

able if $L \cup R \subseteq \mathcal{F}$, $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow R$, $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \psi \Rightarrow R$ for every $\psi \in \mathcal{F} \setminus L$, and $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow \psi, R$ for every $\psi \in \mathcal{F} \setminus R$. It is routine to show that every ω-sequent $L \Rightarrow R$ such that $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow R$ can be extended to an $\mathcal{F}$-maximal unprovable ω-sequent, provided that $frm(L \Rightarrow R) \subseteq \mathcal{F}$ (including s). Define a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ such that W is the set of all $\mathcal{F}$-maximal unprovable ω-sequents, and $\mathcal{R}$ and $\mathcal{V}$ are defined as before. The proof that $\mathcal{W}$ is a triframe and $\mathcal{V}_W(S) > \mathcal{V}_W(s)$ stays the same. We prove that $\text{supp}(\mathcal{V}_{L\Rightarrow R}) = \mathcal{F}$ for every $L \Rightarrow R \in W$, that is, $L \cup R = \mathcal{F}$. In particular, this would mean that $\mathcal{W}$ is $\mathcal{F}$-determined. Let $L \Rightarrow R \in W$. By definition, $L \cup R \subseteq \mathcal{F}$. Now assume for contradiction that there exists $\varphi \in \mathcal{F} \setminus (L \cup R)$. Since $L \Rightarrow R$ is $\mathcal{F}$-maximal unprovable, we have that $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow \varphi, R$ and $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \varphi \Rightarrow R$. Since $\varphi \in \mathcal{F}$, it is allowed to appear in proofs that are restricted to $\mathcal{F}$-formulas. Using cut with φ, we have $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow R$, which is a contradiction.

We show that $\mathcal{W}$ is $\mathbf{G}$-legal. Let $r = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a rule of $\mathbf{G}$, σ a substitution, $\Gamma'_1 \Rightarrow \Delta'_1, \dots, \Gamma'_n \Rightarrow \Delta'_n$ respective subsequents of $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$, and $L \Rightarrow R \in W$. Suppose $\mathcal{V}_{L\Rightarrow R}(\sigma(\Gamma_0 \Rightarrow \Delta_0)) = 0$. Then $\sigma(\Gamma_0) \subseteq L$ and $\sigma(\Delta_0) \subseteq R$. $L \Rightarrow R$ is $\mathcal{F}$-maximal unprovable, and therefore $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), R$. Also, $\sigma(frm(\Gamma_0 \Rightarrow \Delta_0)) \subseteq \mathcal{F}$, and since $\mathbf{G}$ is $\prec$-ordered and $\preceq[\mathcal{F}] \subseteq \mathcal{F}$, we also have that $\sigma(frm(\Gamma'_i \Rightarrow \Delta'_i)) \subseteq \mathcal{F}$ for every $1 \le i \le n$.

First suppose r is a positive rule. Then, we must either have $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i)$ for some $1 \le i \le n$ with $\Gamma_i \neq \emptyset$, or $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow \sigma(\Delta'_i), R$ for some $1 \le i \le n$ with $\Gamma_i = \emptyset$. In the first case, extend $L, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i)$ to an $\mathcal{F}$-maximal unprovable ω-sequent $L' \Rightarrow R'$. Clearly, $L \subseteq L'$ and $\mathcal{V}_{L'\Rightarrow R'}(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 0$. In the second case, the fact that $L \Rightarrow R$ is $\mathcal{F}$-maximal unprovable and $\sigma(frm(\Gamma'_i \Rightarrow \Delta'_i)) \subseteq \mathcal{F}$, ensures that $\sigma(\Delta'_i) \subseteq R$, which means that $\mathcal{V}_{L\Rightarrow R}(\sigma(\Rightarrow \Delta'_i)) = 0$.

Next, suppose r is not positive. Then $\Delta_0 = \emptyset$, and we must have $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma'_i) \Rightarrow \sigma(\Delta'_i), R$ for some $1 \le i \le n$. Since $L \Rightarrow R$ is $\mathcal{F}$-maximal unprovable and $\sigma(frm(\Gamma'_i \Rightarrow \Delta'_i)) \subseteq \mathcal{F}$, we have that $\sigma(\Gamma'_i) \subseteq L$ and $\sigma(\Delta'_i) \subseteq R$, and thus $\mathcal{V}_{L\Rightarrow R}(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 0$. □

Remark 5.2.8. The completeness part of the third item in Theorem 5.2.7 holds for $\prec$-ordered calculi, but not necessarily for all calculi. To see this, we consider intuitionistic (and also ordinary) derivations in the calculus $\mathbf{G_{C_1}}$ (see Example 2.2.9), which is not

$\emptyset$-ordered. Consider the following sequent s, that was used in [69] in order to prove that $\mathbf{G_{C_1}}$ is not $\emptyset$-analytic:

$$\neg p_1, \neg p_2, \neg p_1 \wedge \neg p_2, \neg(\neg p_1 \wedge \neg p_2) \Rightarrow p_1, p_2, \neg\neg p_1$$

$frm(s)$ is closed under subformulas, and thus $sub(s) = frm(s)$. Denote this set by $\mathcal{F}$. Consider the following $\mathcal{F}$-bivaluation v (see Definition 2.3.1), defined by $v(\neg p_1) = v(\neg p_2) = v(\neg p_1 \wedge \neg p_2) = v(\neg(\neg p_1 \wedge \neg p_2)) = 1$, and $v(p_1) = v(p_2) = v(\neg\neg p_1) = 0$. It is routine to verify that v is $\mathbf{G_{C_1}}$-legal, and that $v(s) = 0$. By Theorem 2.3.5, we have that $\not\vdash^{\mathcal{F}}_{\mathbf{G_{C_1}}} s$, which means that there is no derivation of s in $\mathbf{G_{C_1}}$ that consists solely of $\mathcal{F}$-formulas. In particular, there is no such intuitionistic derivation, and thus $\not\vdash^{\mathcal{F}}_{\mathbf{G_{C_1 Int}}} s$. However, we show that $\mathcal{V}_w(s) = 1$ for every $\mathcal{F}$-determined $\mathbf{G_{C_1}}$-legal triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ and $w \in W$. Assume otherwise. Then there exists an $\mathcal{F}$-determined $\mathbf{G_{C_1}}$-legal triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ and $w \in W$ such that $\mathcal{V}_w(s) = 0$. In particular, $\mathcal{V}_w(\psi) = v(\psi)$ for every $\psi \in \mathcal{F}$. Since $\mathcal{W}$ is $\mathbf{G_{C_1}}$-legal, it respects the following non-positive rules:

$$\neg p_1 \Rightarrow \ ; \ \neg p_2 \Rightarrow \ / \ \neg(p_1 \wedge p_2) \Rightarrow \qquad \text{and} \qquad p_1 \Rightarrow \ / \ \neg\neg p_1 \Rightarrow$$

According to the first, since $\mathcal{V}_w(\neg(\neg p_1 \wedge \neg p_2)) = 1$ and $\mathcal{V}_w(\neg\neg p_1) = 0$, we must have that $\mathcal{V}_w(\neg\neg p_2) = 1$. However, according to the second, $\mathcal{V}_w(\neg\neg p_2) = 0$, as $\mathcal{V}_w(p_2) = 0$, which is a contradiction. Therefore, such $\mathcal{W}$ cannot exist, and so s is satisfied by every $\mathcal{F}$-determined $\mathbf{G_{C_1}}$-legal triframe, although $\not\vdash^{\mathcal{F}}_{\mathbf{G_{C_1 Int}}} s$.

Note, however, that $\mathbf{G_{C_1}}$ is $\{\neg\}$-1-ordered, and indeed, if we set $\mathcal{F}' = \mathcal{F} \cup \{\neg\neg p_2\} = sub_1^{\{\neg\}}(s)$, we do have $\vdash^{\mathcal{F}'}_{\mathbf{G_{C_1 Int}}} s$.

Theorem 5.2.7 gives rise to a sufficient semantic criterion for Int-cut-admissibility, that is a variant of Corollary 2.6.11:

Definition 5.2.9. We say that a triframe $\mathcal{W}' = \langle W', \mathcal{R}', \mathcal{V}'\rangle$ is a *determination* of a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ (or that $\mathcal{W}'$ *determines* $\mathcal{W}$) if $W = W'$, $\mathcal{R} = \mathcal{R}'$, and $\mathcal{V}'_w$ is a determination of $\mathcal{V}_w$ for every $w \in W$. $\mathcal{W}'$ is called an *$\mathcal{F}$-determination* of $\mathcal{W}$ if, in addition, it is $\mathcal{F}$-determined. If $\mathcal{W}'$ is fully determined, we call it a *full determination* of $\mathcal{W}$.

Corollary 5.2.10. If every $\mathbf{G}$-legal triframe has a $\mathbf{G}$-legal full determination, then $\mathbf{G}$ enjoys Int-cut-admissibility.

Proof. Suppose $S \not\vdash^{\text{cf}}_{\mathbf{G_{Int}}} s$. By Part 2 of Theorem 5.2.7, there exists some $frm(S)$-determined $\mathbf{G}$-legal triframe $\mathcal{W}$ such that $\mathcal{V}_W(S) > \mathcal{V}_W(s)$. $\mathcal{W}$ has a $\mathbf{G}$-legal full determination $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$. For $\mathcal{W}'$ we also have $\mathcal{V}'_W(S) > \mathcal{V}'_W(s)$. By part 1 of Theorem 5.2.7, $S \not\vdash_{\mathbf{G_{Int}}} s$. $\square$

5.3 Proof of Theorem 5.1.10

The semantic view of intuitionistic derivations via Kripke-style semantics enables us to provide proper variants of Lemmas 2.6.21 and 2.6.22, and then, together with Corollary 5.2.10, to derive Theorem 5.1.10. We start by showing that a single formula can be added to the support of a given triframe.

Definition 5.3.1. The support of a triframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$, denoted $\mathrm{supp}(\mathcal{W})$ is the set $\bigcap\{\mathrm{supp}(\mathcal{V}_w) \mid w \in W\}$.

Lemma 5.3.2. Let $\mathbf{G}$ be an Int-$\prec$-analytic $\prec$-directed calculus, $\mathcal{W}$ a $\mathbf{G}$-legal triframe, and ψ a formula such that $\prec[\psi] \subseteq \mathrm{supp}(\mathcal{W})$. Then $\mathcal{W}$ has a $\mathbf{G}$-legal $\mathrm{supp}(\mathcal{W}) \cup \{\psi\}$-determination.

Proof. Suppose $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$. For every $w \in W$, define the following sets:

- $\Gamma_w = \{\varphi \in \prec[\psi] \mid \mathcal{V}_w(\varphi) = 1\}$;
- $\Delta_w = \{\varphi \in \prec[\psi] \mid \mathcal{V}_w(\varphi) = 0\}$; and
- $S_w = \left\{s \mid \mathit{frm}(s) \subseteq \prec[\psi], \mathcal{V}_{\mathcal{R}[w]}(s) = 1\right\}$.

We start by proving that for every $w \in W$, $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w \Rightarrow \Delta_w$. Let $w \in W$. Define the following triframe $\mathcal{W}^* = \langle W^*, \mathcal{R}^*, \mathcal{V}^*\rangle$: $W^* = \mathcal{R}[w]$, $\mathcal{R}^* = \mathcal{R} \cap (W^* \times W^*)$, and $\mathcal{V}^*_u = \mathcal{V}_u$ for every $u \in W^*$. $\mathcal{W}^*$ is clearly a $\mathbf{G}$-legal triframe, $\mathcal{V}^*_{W^*}(S_w) = 1$ and $\mathcal{V}^*_w(\Gamma_w \Rightarrow \Delta_w) = 0$ (and thus $\mathcal{V}^*_{W^*}(S_w) > \mathcal{V}^*_{W^*}(\Gamma_w \Rightarrow \Delta_w)$). $\mathcal{W}^*$ is also $\preceq[S_w \cup \{\Gamma_w \Rightarrow \Delta_w\}]$-determined, as $\preceq[S_w \cup \{\Gamma_w \Rightarrow \Delta_w\}] \subseteq \prec[\psi] \subseteq \mathrm{supp}(\mathcal{W}^*)$. By part 3 of Theorem 5.2.7, $S_w \nvdash^{\preceq[S_w \cup \{\Gamma_w \Rightarrow \Delta_w\}]}_{\mathbf{G_{Int}}} \Gamma_w \Rightarrow \Delta_w$, and since $\mathbf{G}$ is Int-$\prec$-analytic, $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w \Rightarrow \Delta_w$.

Define a $\mathbf{G}$-legal $\mathrm{supp}(\mathcal{W}) \cup \{\psi\}$-determination $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$ of $\mathcal{W}$ as follows. For every $w \in W$ and $\varphi \in \mathcal{L}$, $\mathcal{V}'_w(\varphi)$ is defined by:

- if $\varphi \neq \psi$ or $\varphi \in \mathrm{supp}(\mathcal{V}_w)$, then $\mathcal{V}'_w(\varphi) = \mathcal{V}_w(\varphi)$;
- if $\varphi = \psi$, $\varphi \notin \mathrm{supp}(\mathcal{V}_w)$, and either $\mathcal{V}_u(\varphi) = 0$ or $S_u \vdash_{\mathbf{G_{Int}}} \Gamma_u, \varphi \Rightarrow \Delta_u$ for some $u \in \mathcal{R}[w]$, then $\mathcal{V}'_w(\varphi) = 0$;
- otherwise, $\mathcal{V}'_w(\varphi) = 1$.

Clearly, $\mathcal{W}'$ is a $\mathrm{supp}(\mathcal{W}) \cup \{\psi\}$-determination of $\mathcal{W}$. Let us show that $\mathcal{W}'$ is a triframe. Suppose $\mathcal{V}'_w(\varphi) = 1$ and let $u \in \mathcal{R}[w]$. We prove that $\mathcal{V}'_u(\varphi) = 1$. If $\varphi \neq \psi$ or $\varphi \in \mathrm{supp}(\mathcal{V}_w)$, then $\mathcal{V}_w(\varphi) = 1$ as well. Since $\mathcal{W}$ is a triframe, $\mathcal{V}_u(\varphi) = 1$. Thus, $\varphi \in \mathrm{supp}(\mathcal{V}_u)$ and hence $\mathcal{V}'_u(\varphi) = 1$. Otherwise, $\varphi = \psi$ and $\psi \notin \mathrm{supp}(\mathcal{V}_w)$. Since

$\mathcal{V}'_w(\psi) = 1$, by the definition of $\mathcal{V}'$, we have that $\mathcal{V}_z(\psi) \neq 0$ and $S_z \nvdash_{\mathbf{G_{Int}}} \Gamma_z, \psi \Rightarrow \Delta_z$ for every $z \in \mathcal{R}[w]$ (including u). $\mathcal{R}$ is transitive, and therefore the same holds for every $z \in \mathcal{R}[u]$. Since $\mathcal{V}_u(\psi) \neq 0$, we are left with two options: either $\mathcal{V}_u(\psi) = 1$ or $\mathcal{V}_u(\psi) = 1/2$. In the first, $\psi \in \text{supp}(\mathcal{V}_u)$, and thus $\mathcal{V}'_u(\psi) = \mathcal{V}_u(\psi) = 1$. In the second, the definition of $\mathcal{V}'$ ensures that $\mathcal{V}'_u(\psi) = 1$

It is left to prove that $\mathcal{W}'$ is $\mathbf{G}$-legal. Let $s_1, \dots, s_n \,/\, s$ be a rule of $\mathbf{G}$, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, σ a substitution and $w \in W$. Suppose $\mathcal{V}'_w(\sigma(s)) = 0$. Then $\mathit{frm}(\sigma(s)) \subseteq \text{supp}(\mathcal{V}_w) \cup \{\psi\}$. First assume that $\mathit{frm}(\sigma(s)) \subseteq \text{supp}(\mathcal{V}_w)$. In this case, $\mathcal{V}_w(\sigma(s)) = 0$ as well. If $s_1, \dots, s_n \,/\, s$ is positive, then since $\mathcal{W}$ is $\mathbf{G}$-legal, there exists $1 \leq i \leq n$ such that either s_i has an empty left side and $\mathcal{V}_w(\sigma(s'_i)) = 0$, or s_i has a non-empty left side, and $\mathcal{V}_u(\sigma(s'_i)) = 0$ for some $u \in \mathcal{R}[w]$. Now, $\mathcal{W}'$ determines $\mathcal{W}$. Thus, in the first case we have $\mathcal{V}'_w(\sigma(s'_i)) = 0$, and in the second, $\mathcal{V}'_u(\sigma(s'_i)) = 0$ for the same u. If $s_1, \dots, s_n \,/\, s$ is not positive, then similarly, $\mathcal{V}'_w(\sigma(s'_i)) = 0$ for some $1 \leq i \leq n$.

We therefore assume that $\mathit{frm}(\sigma(s)) \nsubseteq \text{supp}(\mathcal{V}_w)$, which means that $\psi \in \mathit{frm}(\sigma(s))$ and $\psi \notin \text{supp}(\mathcal{V}_w)$. By the fact that $\mathbf{G}$ is $\prec$-directed, $\mathit{frm}(\sigma(s)) = \{\psi\}$, and $\mathit{frm}(\sigma(s'_i)) \subseteq \prec[\psi]$ for every $1 \leq i \leq n$.

First suppose that $s_1, \dots, s_n \,/\, s$ is not positive. Then $\sigma(s) = \psi \Rightarrow$. We prove that $\mathcal{V}'_w(\sigma(s'_i)) = 0$ for some $1 \leq i \leq n$. By our assumption, $\mathcal{V}'_w(\psi) = 1$. This in particular means that $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w, \psi \Rightarrow \Delta_w$, and so there exists $1 \leq i \leq n$ such that $S_w \nvdash_{\mathbf{G_{Int}}} (\Gamma_w \Rightarrow \Delta_w) \cup \sigma(s'_i)$. Since $\mathit{frm}(\sigma(s'_i)) \subseteq \prec[\psi] = \mathit{frm}(\Gamma_w \Rightarrow \Delta_w)$, we must have $\sigma(s'_i) \subseteq (\Gamma_w \Rightarrow \Delta_w)$, which means that $\mathcal{V}_w(\sigma(s'_i)) = 0$. $\mathcal{V}'_w$ determines $\mathcal{V}_w$, and so $\mathcal{V}'_w(\sigma(s'_i)) = 0$.

Next, suppose that $s_1, \dots, s_n \,/\, s$ is positive. We prove that there exists $1 \leq i \leq n$ such that either s_i has an empty left side and $\mathcal{V}'_w(\sigma(s'_i)) = 0$, or s_i has a non-empty left side and $\mathcal{V}'_z(\sigma(s'_i)) = 0$ for some $z \in \mathcal{R}[w]$. By our assumption, $\sigma(s) = \;\Rightarrow \psi$, and $\mathcal{V}'_w(\psi) = 0$. According to the definition of $\mathcal{V}'$, there exists some $u \in \mathcal{R}[w]$ such that either $\mathcal{V}_u(\psi) = 0$, or $S_u \vdash_{\mathbf{G_{Int}}} \Gamma_u, \psi \Rightarrow \Delta_u$. We first show that there exists $1 \leq i \leq n$ and $z \in \mathcal{R}[w]$ such that $\mathcal{V}_z(\sigma(s'_i)) = 0$, by considering each of these two cases:

1. $\mathcal{V}_u(\psi) = 0$: Then $\mathcal{V}_u(\;\Rightarrow \psi) = 0$. Since $\mathcal{W}$ is $\mathbf{G}$-legal, we must have some $1 \leq i \leq n$ such that either s_i has an empty left side and $\mathcal{V}_u(\sigma(s'_i)) = 0$, or s_i has a non-empty left side and $\mathcal{V}_z(\sigma(s'_i)) = 0$ for some $z \in \mathcal{R}[u] \subseteq \mathcal{R}[w]$.

2. $S_u \vdash_{\mathbf{G_{Int}}} \Gamma_u, \psi \Rightarrow \Delta_u$: Using cut, since $S_u \nvdash_{\mathbf{G_{Int}}} \Gamma_u \Rightarrow \Delta_u$, we must have that $S_u \nvdash_{\mathbf{G_{Int}}} \Gamma_u \Rightarrow \psi, \Delta_u$, and thus there exists some $1 \leq i \leq n$ such that either s_i has an empty left side and $S_u \nvdash_{\mathbf{G_{Int}}} (\Gamma_u \Rightarrow \Delta_u) \cup \sigma(s'_i)$, or s_i has a non-empty left side and $S_u \nvdash_{\mathbf{G_{Int}}} (\Gamma_u \Rightarrow\;) \cup \sigma(s'_i)$. Either way, $\sigma(s'_i) \notin S_u$. Since $\mathit{frm}(\sigma(s'_i)) \subseteq \prec[\psi]$, we have that $\mathcal{V}_z(\sigma(s'_i)) = 0$ for some $z \in \mathcal{R}[u] \subseteq \mathcal{R}[w]$.

Now, fix some $1 \leq i \leq n$ and $z \in \mathcal{R}[w]$ such that $\mathcal{V}_z(\sigma(s'_i)) = 0$. Since $\mathcal{W}'$ determines $\mathcal{W}$, we also have $\mathcal{V}'_z(\sigma(s'_i)) = 0$. If s_i has a non-empty left side, then we are done, as we have found some $1 \leq i \leq n$ such that s_i has a non-empty left side and $\mathcal{V}'_z(\sigma(s'_i)) = 0$ for some $z \in \mathcal{R}[w]$. Otherwise, we show that $\mathcal{V}'_w(\sigma(s'_i)) = 0$. Let φ be a formula in the right side of s'_i. Then $\mathcal{V}'_z(\sigma(\varphi)) = 0$. Since $\mathcal{W}'$ is a triframe, we have $\mathcal{V}'_w(\sigma(\varphi)) \neq 1$. Since $\sigma(\varphi) \in \mathit{frm}(\sigma(s'_i)) \subseteq \prec[\psi] \subseteq \mathrm{supp}(\mathcal{W}) \subseteq \mathrm{supp}(\mathcal{W}')$, we also have $\mathcal{V}'_w(\sigma(\varphi)) \neq 1/2$. Thus $\mathcal{V}'_w(\sigma(\varphi)) = 0$ for every such φ, which means that $\mathcal{V}'_w(\sigma(s'_i)) = 0$. □

Next, an iterated application of Lemma 5.3.2 allows us to fully determine triframes, in a similar manner to the determination of trivaluations:

Lemma 5.3.3. Let **G** be an Int-$\prec$-analytic $\prec$-directed calculus and $\mathcal{W}$ a **G**-legal triframe. Then there exists a **G**-legal full determination of $\mathcal{W}$.

Proof. For every triframe $\mathcal{U}$ and formula ψ, $\mathcal{U}_\psi$ denotes an arbitrary **G**-legal $\mathrm{supp}(\mathcal{U}) \cup \{\psi\}$-determination of $\mathcal{U}$, if such exists. Otherwise, $\mathcal{U}_\psi$ is undefined. Whenever $\mathcal{U}$ is **G**-legal and $\prec[\psi] \subseteq \mathrm{supp}(\mathcal{U})$, Lemma 5.3.2 provides us with such a triframe, in which case $\mathcal{U}_\psi$ is defined. Let $\psi_1, \psi_2, \ldots$ be an enumeration of $\mathcal{L}$ satisfying $i < j$ whenever $\psi_i \prec \psi_j$. Such an enumeration exists since $\prec$ is safe. For every i, denote the set $\{\psi_1, \ldots, \psi_i\}$ by $\Phi_{\leq i}$ ($\Phi_{\leq 0} = \emptyset$).

Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ be a **G**-legal triframe. We show that it has a **G**-legal full determination. Define a sequence $\mathcal{W}^0, \mathcal{W}^1, \ldots$ of triframes as follows: $\mathcal{W}^0 = \mathcal{W}$ and for every $i > 0$, $\mathcal{W}^i = \mathcal{W}^{i-1}_{\psi_i}$. We prove by induction on i that $\mathcal{W}^i$ is defined, and is a **G**-legal $\Phi_{\leq i}$-determination of $\mathcal{W}$, and also of $\mathcal{W}^{i-1}$ (if $\mathcal{W}^{i-1}$ exists).

For $i = 0$, this holds by our assumption on $\mathcal{W}$. Now let $i > 0$. By the induction hypothesis, $\mathcal{W}^{i-1}$ is **G**-legal and is a $\Phi_{\leq i-1}$-determination of $\mathcal{W}$. By the enumeration, $\prec[\psi_i] \subseteq \Phi_{\leq i-1} \subseteq \mathrm{supp}(\mathcal{W}^{i-1})$. Thus, $\mathcal{W}^i = \mathcal{W}^{i-1}_{\psi_i}$ is defined, and is a **G**-legal $\mathrm{supp}(\mathcal{W}^{i-1}) \cup \{\psi_i\}$-determination of $\mathcal{W}^{i-1}$, and therefore also of $\mathcal{W}$. Also, $\Phi_{\leq i} \subseteq \mathrm{supp}(\mathcal{W}^i)$, and thus $\mathcal{W}^i$ is actually a $\Phi_{\leq i}$-determination of $\mathcal{W}^{i-1}$ and of $\mathcal{W}$.

We now define $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$, a **G**-legal full determination of $\mathcal{W}$. Suppose $\mathcal{W}^i = \langle W^i, \mathcal{R}^i, \mathcal{V}^i\rangle$ for every i. Then $W^i = W$ and $\mathcal{R}^i = \mathcal{R}$, as $\mathcal{W}^i$ determines $\mathcal{W}$. For every $\varphi \in \mathcal{L}$, let i_φ be the index of φ in the enumeration. For every $w \in W$, define $\mathcal{V}'_w = \lambda\varphi \in \mathcal{L}.\mathcal{V}^{i_\varphi}_w(\varphi)$.

First, we show that $\mathcal{W}'$ is a triframe. Suppose $\mathcal{V}'_w(\varphi) = 1$ and let $u \in \mathcal{R}[w]$. Then $\mathcal{V}^{i_\varphi}_w(\varphi) = 1$, and since $\mathcal{W}^{i_\varphi}$ is a triframe, $\mathcal{V}^{i_\varphi}_u(\varphi) = 1$. Therefore, $\mathcal{V}'_u(\varphi) = 1$. Next, we show that $\mathcal{W}'$ fully determines $\mathcal{W}$. For every $\varphi \in \mathcal{L}$ and $w \in W$, $\mathcal{V}'_w(\varphi) = \mathcal{V}^{i_\varphi}_w(\varphi) \neq 1/2$, as $\mathcal{W}^{i_\varphi}$ is $\Phi_{\leq i_\varphi}$-determined. Also, for every $\varphi \in \mathrm{supp}(\mathcal{W})$ and $w \in W$, $\mathcal{V}'_w(\varphi) = \mathcal{V}^{i_\varphi}_w(\varphi) = \mathcal{V}_w(\varphi)$, as $\mathcal{W}^{i_\varphi}$ determines $\mathcal{W}$. Finally, let

$s_1, \dots, s_n \,/\, s$ be a rule of **G**, $s'_1, \dots, s'_n$ respective subsequents of $s_1, \dots, s_n$, and σ a substitution. Let $k = \min\{i \mid \mathit{frm}(\sigma(\{s'_1, \dots, s'_n, s\})) \subseteq \Phi_{\leq i}\}$. Then $\mathcal{V}'_w(\varphi) = \mathcal{V}^k_w(\varphi)$ for every $\varphi \in \mathit{frm}(\sigma(\{s'_1, \dots, s'_n, s\}))$ and $w \in W$. Also, $\mathcal{W}^k$ is **G**-legal. Suppose $\mathcal{V}'_w(\sigma(s)) = 0$ for some $w \in W$. Then we must also have $\mathcal{V}^k_w(\sigma(s)) = 0$. If $S \,/\, s$ is positive, then there exists $1 \leq i \leq n$ such that either s_i has an empty left side and $\mathcal{V}'_w(\sigma(s'_i)) = \mathcal{V}^k_w(\sigma(s'_i)) = 0$, or s_i has a non-empty left side and $\mathcal{V}'_u(\sigma(s'_i)) = \mathcal{V}^k_u(\sigma(s'_i)) = 0$ for some $u \in \mathcal{R}[w]$. If it is not positive, $\mathcal{V}'_w(\sigma(s'_i)) = \mathcal{V}^k_w(\sigma(s'_i)) = 0$ for some $1 \leq i \leq n$. $\square$

As in the case of pure calculi, Theorem 5.1.10 is now obtained as a direct corollary of Lemma 5.3.3 and corollary 5.2.10. $\square$

Chapter 6

Rexpansions of Non-deterministic Matrices

As evident from the preceding chapters, our main tool for investigating sequent calculi is a semantic interpretation of derivation rules, that is based on various semantic frameworks. So far, the frameworks that we have used are very restrictive with regard to the number of truth values that are allowed to be used (either two or three). However, they are very permissive in the minimal conditions that are imposed on models. As an extreme example, even the set of all bivaluations is a legitimate set of models, when considering the empty pure calculus, and a similar situation exists with the other frameworks as well.

In contrast, the framework of many-valued matrices [99], which is widely used in non-classical logics, does not restrict the number of truth values, which may even be infinite. Matrices consist of a set of truth values, a subset of designated truth values (that are considered "true"), and an interpretation function, that can be thought of (in the finite case) as a set of truth tables. This framework is completely deterministic: the truth value of a compound formula is uniquely determined by the truth values of its immediate subformulas. This makes the framework of many-valued matrices less modular, and therefore inadequate for studying families of proof systems.

A semantic framework that sits in between the determinism of many-valued matrices and the unrestricted non-determinism of bivaluations and trivaluations is the framework of non-deterministic matrices (Nmatrices) [21, 22]. These are obtained from matrices by allowing a set of possible values inside the entries of the truth tables. While this framework allows for a non-deterministic choice of truth values, it is much more restrictive than the frameworks used in previous chapters of the thesis, as the semantic constraints on the truth value that is assigned to a compound formula must be constrained solely by the truth values of its immediate subformulas.

The modular nature of the use of Nmatrices makes them an important ingredient in the construction of proof systems for various non-classical logics. In fact, when searching for a Gentzen-type system for a logic that is defined in some other form (e.g. by a deterministic semantics or by a Hilbert-type calculus), it suffices to find a finite Nmatrix that is characteristic for the logic. If such an Nmatrix is found, the algorithm of [25] can be employed in order to transform it into an equivalent analytic (generalized) Gentzen-type system, that also admits cut-admissibility. The resulting system might employ a more general notion of a sequent, in which more than two sides are used. However, when some additional expressivity conditions are met, the outcome is ensured to be an ordinary pure sequent calculus, which is still cut-free.

To apply the method of [25] for a systematic study of *families* of logics, the Nmatrix that is found is sometimes not enough, and an equivalent (usually less deterministic) Nmatrix is constructed from it. Then, characteristic Nmatrices for the various logics of the studied family are obtained by performing very simple modifications of the truth tables of the new Nmatrix (see, e.g., [14] for an example of such a process). It is thus evident from several constructions of sequent calculi for non-classical logics from the literature that the manipulation of Nmatrices is tightly related to the construction of analytic sequent calculi. Are these manipulations systematic in any way? Can they be formally defined as Nmatrices operations and become an object of independent study?

In this section we provide an affirmative answer, and introduce a useful operation on Nmatrices, called *rexpansion*, which is obtained by the composition of two previously studied operations: expansion and refinement (see [5, 15]). Properties of this combined operation are presented, along with its effects on the consequence and derivability relations which are induced by the operated Nmatrices. We then show that many transformations on Nmatrices from the literature, especially in the context of constructing analytic proof systems, are actually particular instances of rexpansion. This places rexpansion as a fundamental ingredient in the already established usefulness of Nmatrices for the construction of analytic sequent calculi for non-classical logics. In many cases the calculi that are obtained from the algorithm of [25] are pure, and admit the generalized notion of analyticity employed throughout this thesis. As such, they are subject to the reduction to SAT from Chapter 3, as well as to the analyticity-preserving addition of modal operators from Chapter 4. Thus, while previous chapters of this thesis provided tools and techniques that are applicable after a calculus is constructed, and also a method to construct new analytic calculi that induce new logics (with some desirable properties), the current chapter focuses on the construction of analytic calculi for logics that were originally introduced in some other manner.

Our investigation also leads to a general method for conservatively extending a given logic (or calculus) with new connectives which have some desirable properties. An important specific usage of this method provides a new solution to the problem of constructing *paraconsistent fuzzy logics* and proof systems for them. These are logics that are useful for modeling vague propositions, while avoiding the explosion principle. A first solution to this problem was given in [52], using a completely different approach. However, we show that this solution has some drawbacks, which are overcome in the solution proposed here.

The rest of this chapter is organized as follows. In Section 6.1 we review existing definitions and results in the theory of Nmatrices, and provide examples. In Section 6.2 we combine expansion and refinement into a single operation that is called *rexpansion*, and prove several results regarding this operation and its effects on consequence relations. Section 6.3 includes basic examples of rexpansions in well-known logics from the literature. Section 6.4 shows how rexpansions are (implicitely) used in the construction of sequent calculi for many non-classical logics. In Section 6.5 we introduce paraconsistent conservative extensions of Gödel fuzzy logic that are obtained by performing various rexpansions on the Gödel matrix.

Publications Related to this Chapter

This chapter is mainly based on [23, 24].

6.1 (N)matrices, Expansions, and Refinements

In this section we review the definitions of matrices and Nmatrices, and provide some examples of logics that are defined using them. We then review the two fundamental operations that can be performed on Nmatrices: expansions and refinements, and also provide some examples of these operations.

6.1.1 Logical Matrices

The most standard way of defining logics semantically is by using *matrices* [99]:

Definition 6.1.1.

1. A *matrix* for $\mathcal{L}$ is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that:

    (a) $\mathcal{V}$ is a non-empty set (of *truth values*).

    (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (of *designated* truth values).

(c) $\mathcal{O} : \diamondsuit_{\mathcal{L}} \to \bigcup_{i=0}^{\infty}(\mathcal{V}^i \to \mathcal{V})$ such that for every $i \in \mathbb{N}$ and $\diamond \in \diamondsuit_{\mathcal{L}}^i$, $\mathcal{O}(\diamond) : \mathcal{V}^i \to \mathcal{V}$.[1]

2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$. An $\mathcal{M}$-valuation is a function v from $\mathcal{L}$ to $\mathcal{V}$ such that for every $\diamond \in \diamondsuit_{\mathcal{L}}^n$ and $\psi_1, \dots, \psi_n \in \mathcal{L}$, $v(\diamond(\psi_1, \dots, \psi_n)) = \mathcal{O}(\diamond)(v(\psi_1), \dots, v(\psi_n))$. An $\mathcal{M}$-valuation v is an $\mathcal{M}$-*model* of a formula ψ (in symbols: $v \vDash^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. It is an $\mathcal{M}$-*model* of a set $\mathcal{T}$ of formulas (in symbols: $v \vDash^{\mathcal{M}} \mathcal{T}$) if $v \vDash^{\mathcal{M}} \psi$ for every $\psi \in \mathcal{T}$. A formula ψ is an $\mathcal{M}$-*consequence* of a set $\mathcal{T}$ of formulas (in symbols: $\mathcal{T} \vdash_{\mathcal{M}} \psi$) if every $\mathcal{M}$-model of $\mathcal{T}$ is an $\mathcal{M}$-model of ψ. We say that $\mathcal{M}$ *induces a logic* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ (or that $\mathcal{M}$ is *characteristic for* $\mathbf{L}$) if $\vdash_{\mathcal{M}} = \vdash_{\mathbf{L}}$.

Many well-known non-classical logics are characterized using matrices:

Example 6.1.2. Asenjo–Priest's three-valued logic of paradox **LP** [6, 86] and Kleene's three-valued logic **KL** [65] are both defined by matrices that differ only in the set of designated values. Consider the set $\mathcal{V}_3 = \{t, f, i\}$, and the interpretation function $\mathcal{O}_3$ that is defined by the following tables:

$\mathcal{O}_3(\wedge)$	t	f	i
t	t	f	i
f	f	f	f
i	i	f	i

$\mathcal{O}_3(\vee)$	t	f	i
t	t	t	t
f	t	f	i
i	t	i	i

	$\mathcal{O}_3(\neg)$
t	f
f	t
i	i

LP is characterized by the matrix $\mathcal{M}_{\mathbf{LP}} = \langle \mathcal{V}_3, \{t, i\}, \mathcal{O}_3 \rangle$, and **KL** by the matrix $\mathcal{M}_{\mathbf{KL}} = \langle \mathcal{V}_3, \{t\}, \mathcal{O}_3 \rangle$. A matrix for Łukasiewicz three-valued logic Ł$_3$ (see Example 2.2.8) is obtained from $\mathcal{M}_{\mathbf{KL}}$ by the extension of $\mathcal{O}_3$ with the following table:

$\mathcal{O}_3(\supset)$	t	f	i
t	t	f	i
f	t	t	t
i	t	i	t

Example 6.1.3. Gödel fuzzy logic **G** [48] is characterized by the following matrix $\mathcal{M}_{\mathbf{G}} = \langle \mathcal{V}_{\mathbf{G}}, \mathcal{D}_{\mathbf{G}}, \mathcal{O}_{\mathbf{G}} \rangle$ for $\{\wedge, \vee, \supset, \bot\}$:

1. $\mathcal{V}_{\mathbf{G}} = [0, 1]$

2. $\mathcal{D}_{\mathbf{G}} = \{1\}$

[1] $\mathcal{O}(\diamond)$ is often regarded as the "truth table" of $\diamond$.

3. $\mathcal{O}_{\mathbf{G}}(\bot) = 0$, $\mathcal{O}_{\mathbf{G}}(\vee)(a,b) = \max\{a,b\}$, $\mathcal{O}_{\mathbf{G}}(\wedge)(a,b) = \min\{a,b\}$, and $\mathcal{O}_{\mathbf{G}}(\supset)(a,b) = \begin{cases} 1 & a \leq b \\ b & a \not\leq b \end{cases}$.

Łukasiewicz fuzzy logic $Ł_\infty$ [76] is characterized by a matrix that differs from $\mathcal{M}_{\mathbf{G}}$ solely in the interpretation of $\supset$, that is changed to: $\mathcal{O}(\supset)(a,b) \begin{cases} 1 & a \leq b \\ 1-a+b & a \not\leq b \end{cases}$.

6.1.2 Non-deterministic Matrices

Matrices are *truth-functional*, that is, the truth value of a compound formula is uniquely determined by the truth values of its immediate subformulas. In [22], matrices are generalized to allow non-deterministic assignments of truth values to compound formulas.

Definition 6.1.4.

1. A *non-deterministic matrix (Nmatrix)* for $\mathcal{L}$ is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that:

   (a) $\mathcal{V}$ is a non-empty set (of *truth values*).

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (of *designated* truth values).

   (c) $\mathcal{O} : \diamondsuit_{\mathcal{L}} \to \bigcup_{i=0}^{\infty}(\mathcal{V}^i \to P^+(\mathcal{V}))$ such that for every $i \in \mathbb{N}$ and $\diamond \in \diamondsuit^i_{\mathcal{L}}$, $\mathcal{O}(\diamond) : \mathcal{V}^i \to P^+(\mathcal{V})$ (where $P^+(\mathcal{V}) = P(\mathcal{V}) \setminus \{\emptyset\}$).

2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$-valuation is a function v from $\mathcal{L}$ to $\mathcal{V}$ such that for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $\psi_1, \dots, \psi_n \in \mathcal{L}$, $v(\diamond(\psi_1, \dots, \psi_n)) \in \mathcal{O}(\diamond)(v(\psi_1), \dots, v(\psi_n))$. The definitions of $\mathcal{M}$-models and $\mathcal{M}$-consequences are as in Definition 6.1.1, using the non-deterministic notion of an $\mathcal{M}$-valuation.

To be considered as a particular instance of Nmatrices, we take matrices to be Nmatrices in which $\mathcal{O}(\diamond)(x_1, \dots, x_n)$ is a singleton for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $x_1, \dots, x_n \in \mathcal{V}$. In matrices mentioned above and below we freely interchange truth values with their singletons, whenever there is no danger of confusion. Unless stated otherwise, or clear from the context, all Nmatrices below are for $\mathcal{L}$.

Example 6.1.5. Recall the pure calculus $\mathbf{G_{PIL}}$ for quotations-free primal infon logic (Example 2.2.10). It is easy to verify that a bivaluation is $\mathbf{G_{PIL}}$-legal iff it is an $\mathcal{M}$-valuation, for $\mathcal{M} = \langle \{0,1\}, \{1\}, \mathcal{O} \rangle$, where $\mathcal{O}(\vee)(0,0) = \mathcal{O}(\supset)(0,0) = \{0,1\}$, and all other entries are defined exactly like in the truth tables for classical logic in the language $\{\wedge, \vee, \supset \bot, \top\}$. Note that this Nmatrix is two-valued, and is a proper Nmatrix (meaning it is an Nmatrix which is not a matrix). Using Theorem 3.4 of [21], we conclude that primal infon logic **PIL** does not have a finite characteristic matrix.

Like matrices, Nmatrices provide an *analytic* semantic framework, in the sense that for every Nmatrix $\mathcal{M}$, every partial $\mathcal{M}$-valuation can be extended to a full $\mathcal{M}$-valuation.[2] [3] A useful consequence of this property is the modular character that the framework of Nmatrices exhibits:

Definition 6.1.6. A logic $\mathbf{L}_2 = \langle \mathcal{L}_2, \vdash_{\mathbf{L}_2} \rangle$ is *conservative over* a logic $\mathbf{L}_1 = \langle \mathcal{L}_1, \vdash_{\mathbf{L}_1} \rangle$ (or: $\mathbf{L}_2$ is a *conservative extension* of $\mathbf{L}_1$) if $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and for every $\mathcal{T} \subseteq \mathcal{L}_1$ and $\varphi \in \mathcal{L}_1$ it holds that $\mathcal{T} \vdash_{\mathbf{L}_1} \varphi$ iff $\mathcal{T} \vdash_{\mathbf{L}_2} \varphi$.

Definition 6.1.7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. $\mathcal{M}_2$ is an *extension of* $\mathcal{M}_1$ *to* $\mathcal{L}_2$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$, and $\mathcal{O}_1(\diamond) = \mathcal{O}_2(\diamond)$ for every $\diamond \in \Diamond_{\mathcal{L}_1}$.

Proposition 6.1.8. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ be Nmatrices for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. If $\mathcal{M}_1$ is an extension of $\mathcal{M}_2$ to $\mathcal{L}_2$ then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$.

6.1.3 Expansions and Refinements

Next we present two basic operations from [5] and [15], that can be performed on Nmatrices: *expansions* and *refinements*. Loosely speaking, an expansion of an Nmatrix is obtained by making several distinct copies of each truth value, so that the new designated values are the copies of the original ones, and each value in the interpretation of the connectives is replaced by all of its copies. This is formally defined as follows:

Definition 6.1.9.

1. A function F is called an *expansion function* if for every $x \in dom(F)$, $F(x)$ is a non-empty set, and $F(x) \cap F(y) = \emptyset$ whenever $x \neq y$. We say that F is an *expansion function for an Nmatrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\mathcal{L}$ if it is an expansion function and $dom(F) = \mathcal{V}$.

[2] Following [22], we use the term *analytic* for this property. Our Theorem 2.5.9 also justifies the use of this term.

[3] Note, however, that if $\mathcal{M}$ is an ordinary matrix, then every partial $\mathcal{M}$-valuation can be extended to a single full $\mathcal{M}$-valuation, while there can be several distinct full $\mathcal{M}$-valuations that extend a partial $\mathcal{M}$-valuation in case $\mathcal{M}$ is an Nmatrix.

2. For every expansion function F and $y \in \bigcup Im(F)$, we denote by $\widetilde{F}[y]$ the unique element $x \in dom(F)$ such that $y \in F(x)$.

3. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and F an expansion function for $\mathcal{M}$. The *F-expansion of* $\mathcal{M}$ is the Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$, where:

   (a) $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F(x)$

   (b) $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F(x)$

   (c) $\mathcal{O}_F(\diamond)(y_1, \ldots, y_n) = \bigcup_{z \in \mathcal{O}(\diamond)(\widetilde{F}[y_1], \ldots, \widetilde{F}[y_n])} F(z)$ for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $y_1, \ldots, y_n \in \mathcal{V}_F$.

   $\mathcal{M}_2$ is an *expansion* of $\mathcal{M}_1$ if it is the F-expansion of it for some F.

Nothing but uniformly duplicating all truth values is done in expansions, and hence the consequence relation remains the same, as was shown in [5]:

Proposition 6.1.10. Let $\mathcal{M}_2$ be an expansion of $\mathcal{M}_1$. Then $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$.

Example 6.1.11.

1. Two Nmatrices are isomorphic to one another if and only if one is the F-expansion of the other for some expansion function F (in which $F(x)$ is always a singleton).

2. Consider the usual matrix which is characteristic for classical logic, where the truth values are t and f. By assigning $\{t, \top\}$ to t and $\{f\}$ to f, we obtain an expansion function. The outcome of this expansion would be a non-deterministic matrix for classical logic, in which, for example, the interpretation of negation is $\mathcal{O}(\neg)(t) = \mathcal{O}(\neg)(\top) = \{f\}$ and $\mathcal{O}(\neg)(f) = \{t, \top\}$.

3. The classical matrix can be further expanded by assigning $[0, 1/2)$ to f and $[1/2, 1]$ to t. The outcome would be another non-deterministic matrix which is characteristic for classical logic. The interpretation of negation would then be $\mathcal{O}(\neg)(x) = [0, 1/2)$ whenever $x \geq 1/2$ and $\mathcal{O}(\neg)(x) = [1/2, 1]$ whenever $x < 1/2$.

Next, we define the *refinement* operation on Nmatrices. Loosely speaking, refining an Nmatrix means deleting some of its truth values, and then reducing the amount of non-determinism (each of these steps is optional). This is formally defined as follows:

Definition 6.1.12. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$. $\mathcal{M}_2$ is a *refinement* of $\mathcal{M}_1$ if:

1. $\mathcal{V}_2 \subseteq \mathcal{V}_1$.

2. $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$.

3. $\mathcal{O}_2(\diamond)(x_1, \dots, x_n) \subseteq \mathcal{O}_1(\diamond)(x_1, \dots, x_n)$ for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $x_1, \dots, x_n \in \mathcal{V}_2$.

$\mathcal{M}_2$ is a *simple refinement* of $\mathcal{M}_1$ if in addition, $\mathcal{V}_2 = \mathcal{V}_1$.

Example 6.1.13. The infinite characteristic Nmatrix for classical logic from Example 6.1.11 can be (simply) refined by e.g. redefining $\mathcal{O}(\neg)$ in the following way: $\mathcal{O}(\neg)(x) = \{0\}$ whenever $x \geq 1/2$ and $\mathcal{O}(\neg)(x) = \{1\}$ whenever $x < 1/2$.

Refining an Nmatrix $\mathcal{M}$ can only reduce the set of $\mathcal{M}$-valuations. Consequently, we have the following proposition from [15]:

Proposition 6.1.14. Let $\mathcal{M}_2$ be a refinement of $\mathcal{M}_1$. Then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$.

6.2 Refined Expansions

In this section we combine the two basic operations defined above and obtain *refined expansions* (in short: *rexpansions*).

6.2.1 Combining Expansions and Refinements

We start by explicitly defining the combined operation and exploring its properties.

Definition 6.2.1. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices and F an expansion function for $\mathcal{M}_1$. We say that $\mathcal{M}_2$ is an *F-rexpansion of* $\mathcal{M}_1$ if it is a refinement of the F-expansion of $\mathcal{M}_1$. It is called:

1. *simple* if it is a simple refinement of the F-expansion of $\mathcal{M}_1$.

2. *preserving* if $F(x) \cap \mathcal{V}_2 \neq \emptyset$ for every $x \in \mathcal{V}_1$.

3. *strongly preserving* if it is preserving, and for every $x_1, \dots, x_n \in \mathcal{V}_2$, $\diamond \in \diamondsuit^n_{\mathcal{L}}$, and $y \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1], \dots, \widetilde{F}[x_n])$, it holds that the set $F(y) \cap \mathcal{O}_2(\diamond)(x_1, \dots, x_n)$ is not empty.

$\mathcal{M}_2$ is called a *rexpansion of* $\mathcal{M}_1$ if it is an F-rexpansion of it for some expansion function F for $\mathcal{M}_1$. If $\mathcal{M}_2$ is a rexpansion of $\mathcal{M}_1$, then we may call $\mathcal{M}_2$ "simple", "preserving", or "strongly preserving" (without the suffix "rexpansion of $\mathcal{M}_1$") whenever that is clear from the context.

Loosely speaking, being a preserving rexpansion amounts to keeping at least one "copy" of every original truth value. Being strongly preserving means that this property holds not only for the set of truth values, but also for the interpretation of the connectives.

Example 6.2.2. The Nmatrix from Example 6.1.13 is a rexpansion of the classical matrix, which is simple and strongly preserving.

Let us elaborate on the connections between the different properties of rexpansions:

Lemma 6.2.3. Every simple rexpansion is preserving, every expansion is a strongly preserving rexpansion, and every preserving rexpansion of a matrix is strongly preserving.

Proof. We prove that every preserving rexpansion of a matrix is strongly preserving. (The other statements are trivial.) Let $\mathcal{M}_1$ be a matrix, $\mathcal{M}_2$ an Nmatrix, and F an expansion function such that $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$. Let $x_1, \dots, x_n \in \mathcal{V}_2$, $\diamond \in \Diamond^n_{\mathcal{L}}$, and $y \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1], \dots, \widetilde{F}[x_1])$. We prove that $F(y) \cap \mathcal{O}_2(\diamond)(x_1, \dots, x_n) \neq \emptyset$. $\mathcal{O}_2(\diamond)(x_1, \dots, x_n) \subseteq \bigcup_{z \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1], \dots, \widetilde{F}[x_1])} F(z)$ and $\mathcal{M}_1$ is a matrix. Therefore, $\mathcal{O}_2(\diamond)(x_1, \dots, x_n) \subseteq F(y)$, which means that $F(y) \cap \mathcal{O}_2(\diamond)(x_1, \dots, x_n) = \mathcal{O}_2(\diamond)(x_1, \dots, x_n)$. This set cannot be empty, as $\mathcal{M}_2$ is an Nmatrix. □

Next we provide a necessary and sufficient condition for an Nmatrix to be a rexpansion of another Nmatrix.

Proposition 6.2.4. $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is a rexpansion of $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ iff there is a function $f : \mathcal{V}_2 \to \mathcal{V}_1$ such that:

1. For every $x \in \mathcal{V}_2$, $x \in \mathcal{D}_2$ iff $f(x) \in \mathcal{D}_1$.
2. For every $x_1, \dots, x_n \in \mathcal{V}_2$ and $y \in \mathcal{O}_2(\diamond)(x_1, \dots, x_n)$, it holds that $f(y) \in \mathcal{O}_1(\diamond)(f(x_1), \dots, f(x_n))$.

Proof.
($\Leftarrow$): Suppose such a function f exists. For every subset Y of $\mathcal{V}_1$, denote the set $\{x \in \mathcal{V}_2 \mid f(x) \in Y\}$ by $f^{-1}[Y]$. Let $\mathcal{V}$ be some set such that $\mathcal{V} \cap \mathcal{V}_2 = \emptyset$ and $|\mathcal{V}| = |\mathcal{V}_1|$, and let $g : \mathcal{V}_1 \to \mathcal{V}$ be a bijection. We show that $\mathcal{M}_2$ is an F-rexpansion of $\mathcal{M}_1$ for

$$F = \lambda x \in \mathcal{V}_1. \begin{cases} f^{-1}[\{x\}] & x \in Im(f) \\ \{g(x)\} & otherwise \end{cases}$$

F is clearly an expansion function for $\mathcal{M}_1$. Now, $\mathcal{V}_2$ is the domain of f, and thus it is contained in $(\mathcal{V}_1)_F$. Next, by property (1) of f, $\mathcal{D}_2 = f^{-1}[\mathcal{D}_1]$, which, by the definition of F is equal to $(\mathcal{D}_1)_F \cap \mathcal{V}_2$. Finally, by property (2) of f,

$$\mathcal{O}_2(\diamond)(x_1, \dots, x_n) \subseteq f^{-1}\left[\mathcal{O}_1(\diamond)(f(x_1), \dots, f(x_n))\right] \subseteq$$

$$\bigcup_{z \in \mathcal{O}_1(\diamond)(f(x_1),\dots,f(x_n))} F(z) = \bigcup_{z \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1],\dots,\widetilde{F}[x_n])} F(z) = (\mathcal{O}_1)_F(\diamond)(x_1, \dots, x_n)$$

($\Rightarrow$): If $\mathcal{M}_2$ is an F-rexpansion of $\mathcal{M}_1$ for some F, then the function $\lambda x \in \mathcal{V}_2.\widetilde{F}[x]$ satisfies the required conditions. □

Remark 6.2.5. In [22], the term 'simple refinement' was reserved for what is called here 'refinement', while the term 'refinement' was related to the functions from Proposition 6.2.4.

Another useful property of the rexpansion operation is that it induces some forms of transitivity:

Theorem 6.2.6.

1. If $\mathcal{M}_2$ is a preserving rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a (preserving) rexpansion of $\mathcal{M}_2$, then $\mathcal{M}_3$ is a (preserving) rexpansion of $\mathcal{M}_1$.

2. If $\mathcal{M}_2$ is a strongly preserving rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a strongly preserving rexpansion of $\mathcal{M}_2$, then $\mathcal{M}_3$ is a strongly preserving rexpansion of $\mathcal{M}_1$.

Proof. Let F and G be expansion functions such that $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a G-rexpansion of $\mathcal{M}_2$. For every $1 \leq i \leq 3$, assume that $\mathcal{M}_i = \langle \mathcal{V}_i, \mathcal{D}_i, \mathcal{O}_i \rangle$. Define $H = \lambda x \in \mathcal{V}_1. \bigcup_{y \in F(x) \cap \mathcal{V}_2} G(y)$. Using the fact that $\mathcal{M}_2$ is preserving, it can easily be shown that H is an expansion function for $\mathcal{M}_1$. We first prove that $\mathcal{M}_3$ is a H-rexpansion of $\mathcal{M}_1$:

$$\mathcal{V}_3 \subseteq (\mathcal{V}_2)_G = \bigcup_{y \in \mathcal{V}_2} G(y) = \bigcup_{y \in (\mathcal{V}_1)_F \cap \mathcal{V}_2} G(y) = \bigcup_{y \in \left(\bigcup_{x \in \mathcal{V}_1} F(x)\right) \cap \mathcal{V}_2} G(y) =$$

$$\bigcup_{y \in \bigcup_{x \in \mathcal{V}_1} (F(x) \cap \mathcal{V}_2)} G(y) = \bigcup_{x \in \mathcal{V}_1} \left(\bigcup_{y \in F(x) \cap \mathcal{V}_2} G(y) \right) = \bigcup_{x \in \mathcal{V}_1} H(x) = (\mathcal{V}_1)_H$$

and

$$(\mathcal{D}_2)_G = \left(\bigcup_{y\in\mathcal{D}_2} G(y)\right) = \left(\bigcup_{y\in(\mathcal{D}_1)_F\cap\mathcal{V}_2} G(y)\right) = \left(\bigcup_{y\in\left(\bigcup_{x\in\mathcal{D}_1} F(x)\right)\cap\mathcal{V}_2} G(y)\right) =$$
$$\left(\bigcup_{x\in\mathcal{D}_1}\bigcup_{y\in F(x)\cap\mathcal{V}_2} G(y)\right) = \left(\bigcup_{x\in\mathcal{D}_1} H(x)\right) = (\mathcal{D}_1)_H$$

which means that $\mathcal{D}_3 = (\mathcal{D}_2)_G \cap \mathcal{V}_3 = (\mathcal{D}_1)_H \cap \mathcal{V}_3$.

As for $\mathcal{O}_3$, let $\diamond \in \Diamond^n_{\mathcal{L}}$, $x_1, \dots, x_n \in \mathcal{V}_3$, and $w \in \mathcal{O}_3(\diamond)(x_1, \dots, x_n)$. We show that $w \in (\mathcal{O}_1)_H(\diamond)(x_1, \dots, x_n)$. $\mathcal{M}_3$ is a refinement of $(\mathcal{M}_2)_G$, and hence $w \in (\mathcal{O}_2)_G(\diamond)(x_1, \dots, x_n)$. Now, there must exists $z \in \mathcal{O}_2(\diamond)(\widetilde{G}[x_1], \dots, \widetilde{G}[x_n])$ such that $w \in G(z)$, and since $\mathcal{M}_2$ is a refinement of $(\mathcal{M}_1)_F$, we have $z \in (\mathcal{O}_1)_F(\diamond)(\widetilde{G}[x_1], \dots, \widetilde{G}[x_n])$. Similarly, there exists $u \in \mathcal{O}_1(\diamond)(\widetilde{F}\left[\widetilde{G}[x_1]\right], \dots, \widetilde{F}\left[\widetilde{G}[x_1]\right])$ such that $z \in F(u)$. So we have that there exists $z \in F(u) \cap \mathcal{V}_2$ such that $w \in G(z)$. Hence $w \in H(u)$. To prove that $w \in (\mathcal{O}_1)_H(\diamond)(x_1, \dots, x_n)$, we show that $u \in \mathcal{O}_1(\diamond)(\widetilde{H}[x_1], \dots, \widetilde{H}[x_n])$. That is, we show that for every $1 \le i \le n$, $\widetilde{H}[x_i] = \widetilde{F}\left[\widetilde{G}[x_i]\right]$. For every $1 \le i \le n$ let $y_i = \widetilde{H}[x_i]$, $z_i = \widetilde{G}[x_i]$, and $w_i = \widetilde{F}[z_i]$. We prove that $y_i = w_i$: $x_i \in H(y_i)$, and hence there exists $y \in F(y_i) \cap \mathcal{V}_2$ such that $x_i \in G(y)$. Since G is an expansion function, $y = z_i$. Hence $y \in F(w_i)$. Similarly, F is an expansion function, and hence $y_i = w_i$.

Next, we show that if $\mathcal{M}_3$ is a preserving G-rexpansion of $\mathcal{M}_2$ then it is a preserving H-rexpansion of $\mathcal{M}_1$, that is, $H(x) \cap \mathcal{V}_3 \neq \emptyset$ for every $x \in \mathcal{V}_1$. Since $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$, there exists $y \in F(x) \cap \mathcal{V}_2$. And since $\mathcal{M}_3$ is a preserving G-rexpansion of $\mathcal{M}_2$, there exists $z \in G(y) \cap \mathcal{V}_3 \subseteq H(x) \cap \mathcal{V}_3$.

Finally, we show that if $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a strongly preserving G-rexpansion of $\mathcal{M}_2$, then $\mathcal{M}_3$ is a strongly preserving H-rexpansion of $\mathcal{M}_1$. Let $z_1, \dots, z_n \in \mathcal{V}_3$ and $\diamond \in \Diamond^n_{\mathcal{L}}$. We show that $H(x) \cap \mathcal{O}_3(\diamond)(z_1, \dots, z_n) \neq \emptyset$ for every $x \in \mathcal{O}_1(\diamond)(\widetilde{H}[z_1], \dots, \widetilde{H}[z_n])$. Let $x \in \mathcal{O}_1(\diamond)(\widetilde{H}[z_1], \dots, \widetilde{H}[z_n])$. For every $1 \le i \le n$ let $x_i = \widetilde{H}[z_i]$. Then there exists $y_i \in F(x_i) \cap \mathcal{V}_2$ such that $z_i \in G(y_i)$. Since $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$, there exists $y_0 \in F(x) \cap \mathcal{O}_2(\diamond)(y_1, \dots, y_n)$. Since $\mathcal{M}_3$ is a strongly preserving G-rexpansion of $\mathcal{M}_2$, there also exists $z_0 \in G(y_0) \cap \mathcal{O}_3(\diamond)(z_1, \dots, z_n) \subseteq H(x) \cap \mathcal{O}_3(\diamond)(z_1, \dots, z_n)$. □

6.2.2 Consequence Relations

In this section we investigate the effect rexpansions induce on semantically defined consequence relations. Our main theorem is the following:

Theorem 6.2.7. If $\mathcal{M}_2$ is a rexpansion of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Moreover, if $\mathcal{M}_2$ is strongly preserving then $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$.

Proof. The first part follows directly from Propositions 6.1.10 and 6.1.14 above. Suppose $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$. We prove that $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$. For this, it suffices to prove that for every $\mathcal{M}_1$-valuation v there exists an $\mathcal{M}_2$-valuation v' such that $v \models^{\mathcal{M}_1} \psi$ iff $v' \models^{\mathcal{M}_2} \psi$ for every $\psi \in \mathcal{L}$. Let $c : P(\mathcal{V}_2) \setminus \{\emptyset\} \to \mathcal{V}_2$ and suppose that for every $X \in P(\mathcal{V}_2) \setminus \{\emptyset\}$, $c(X) \in X$.[4] Let $\psi_1, \psi_2, ...$ be an enumeration of $\mathcal{L}$ such that if ψ_i is a subformula of ψ_j then $i < j$. Now let v be an $\mathcal{M}_1$-valuation. For the construction of v', we first define a sequence $v_0, v_1, ...$ of *partial functions* from $\mathcal{L}$ to $\mathcal{V}_2$: v_0 is the empty function, and for every $i > 0$, v_i is defined as follows. For every $\psi \in dom(v_{i-1})$, $v_i(\psi) = v_{i-1}(\psi)$. If $\psi_i \notin dom(v_{i-1})$, then:

- If ψ_i is atomic and $F(v(\psi_i)) \cap \mathcal{V}_2$ is not empty, $v_i(\psi_i) = c(F(v(\psi_i)) \cap \mathcal{V}_2)$.
- If ψ_i has the form $\diamond(\varphi_1, \dots, \varphi_n)$ for $\varphi_1, \dots, \varphi_n \in dom(v_{i-1})$ and $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \dots, v_{i-1}(\varphi_n))$ is not empty, $v_i(\psi_i) = c(F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \dots, v_{i-1}(\varphi_n)))$.

We prove by induction on i that:

1. $v_i(\psi) \in F(v(\psi))$ for every $\psi \in dom(v_i)$;
2. $dom(v_i) = \{\psi_1, \dots, \psi_i\}$; and
3. v_i satisfies the conditions induced by $\mathcal{M}_2$, that is: $v_i(\diamond(\varphi_1, \dots, \varphi_n)) \in \mathcal{O}_2(\diamond)(v_i(\varphi_1), \dots, v_i(\varphi_n))$ whenever $\varphi_1, \dots, \varphi_n$ and $\diamond(\varphi_1, \dots, \varphi_n)$ are in $dom(v_i)$.

For $i = 0$, this trivially holds. Let $i > 0$.

1. Let $\psi \in dom(v_i)$. If $\psi \in dom(v_{i-1})$ then this holds by the induction hypothesis. Otherwise, $\psi = \psi_i$, and then this holds by definition.
2. By the induction hypothesis, $dom(v_{i-1}) = \{\psi_1, \dots, \psi_{i-1}\}$, and therefore we have to prove that $\psi_i \in dom(v_i)$. If ψ_i is atomic, this amounts to showing that the set $F(v(\psi_i)) \cap \mathcal{V}_2$ is not empty, which holds as $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$. Otherwise, ψ_i has the form $\diamond(\varphi_1, \dots, \varphi_n)$. By our enumeration and the induction hypothesis, $\varphi_1, \dots, \varphi_n \in dom(v_{i-1})$, and therefore this amounts to showing that $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \dots, v_{i-1}(\varphi_n))$ is not empty. By the induction

[4] The existence of such a function relies on the axiom of choice in case $\mathcal{V}_2$ is infinite.

hypothesis, we have that $v_{i-1}(\varphi_j) \in F(v(\varphi_j))$ for every $1 \leq j \leq n$. In other words, $v(\varphi_j) = \widetilde{F}[v_{i-1}(\varphi_j)]$ for every $1 \leq j \leq n$. By the fact that v is an $\mathcal{M}_1$-valuation, $v(\psi_i) \in \mathcal{O}_1(\diamond)(v(\varphi_1), \dots, v(\varphi_n)) = \mathcal{O}_1(\diamond)(\widetilde{F}[v_{i-1}(\varphi_1)], \dots, \widetilde{F}[v_{i-1}(\varphi_n)])$, and hence $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1), \dots, v_{i-1}(\varphi_n)) \neq \emptyset$, as $\mathcal{M}_2$ is strongly preserving.

3. Let $\diamond(\varphi_1, \dots, \varphi_n), \varphi_1, \dots, \varphi_n \in dom(v_i)$. We prove that $v_i(\diamond(\varphi_1, \dots, \varphi_n)) \in \mathcal{O}_2(\diamond)(v_i(\varphi_1), \dots, v_i(\varphi_n))$. If $\diamond(\varphi_1, \dots, \varphi_n) \in dom(v_{i-1})$, then this holds by the induction hypothesis and our enumeration. Otherwise, $\psi_i = \diamond(\varphi_1, \dots, \varphi_n)$, and then this holds by the induction hypothesis and the definition of v_i.

For every $\psi \in \mathcal{L}$, let i_ψ be the index of ψ in the enumeration. v' is defined by $v'(\psi) = v_{i_\psi}(\psi)$. First, we show that v' is an $\mathcal{M}_2$-valuation. Let $\diamond \in \diamondsuit^n_{\mathcal{L}}$, $\varphi_1, \dots, \varphi_n \in \mathcal{L}$, and $k = i_{\diamond(\varphi_1,\dots,\varphi_n)}$. $v'(\diamond(\varphi_1, \dots, \varphi_n)) = v_k(\diamond(\varphi_1, \dots, \varphi_n))$, which belongs to $\mathcal{O}_2(\diamond)(v_k(\varphi_1), \dots, v_k(\varphi_n)) = \mathcal{O}_2(\diamond)(v'(\varphi_1), \dots, v'(\varphi_n))$. Second, we show that $v \vDash^{\mathcal{M}_1} \psi$ iff $v' \vDash^{\mathcal{M}_2} \psi$. Suppose $v \vDash^{\mathcal{M}_1} \psi$. Then $v(\psi) \in \mathcal{D}_1$. Now, by the construction of v', $v'(\psi) \in F(v(\psi)) \subseteq (\bigcup_{x \in \mathcal{D}_1} F(x)) \cap \mathcal{V}_2 = \mathcal{D}_2$, which means that $v' \vDash^{\mathcal{M}_2} \psi$. For the converse, suppose $v' \vDash^{\mathcal{M}_2} \psi$. Then $v'(\psi) \in \mathcal{D}_2 \subseteq (\bigcup_{x \in \mathcal{D}_1} F(x))$. Hence there exists $x \in \mathcal{D}_1$ such that $v'(\psi) \in F(x)$. Now, by the construction of v', $v'(\psi) \in F(v(\psi))$. Since F is an expansion function, $v(\psi) = x \in \mathcal{D}_1$, which means that $v \vDash^{\mathcal{M}_1} \psi$. □

The following corollary immediately follows as a consequence of Lemma 6.2.3 and Theorem 6.2.7:

Corollary 6.2.8. Let $\mathcal{M}_2$ be a preserving rexpansion of $\mathcal{M}_1$. If $\mathcal{M}_1$ is a matrix then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.

An important consequence of Corollary 6.2.8 and Proposition 6.1.8 (the usefulness of which is demonstrated in Sections 6.3 and 6.5) is a general method for providing a given logic with an alternative new semantics, and then use it for conservatively augmenting it with new connectives. This is established in the following corollary:

Corollary 6.2.9. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, $\mathcal{M}_1$ a (N)matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some (strongly) preserving rexpansion of $\mathcal{M}_1$. Then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$.

We conclude this section with a stronger instance of Corollary 6.2.9, that applies only for two-valued matrices:

Corollary 6.2.10. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, $\mathcal{M}_1 = \langle \{t, f\}, \{t\}, \mathcal{O}_1 \rangle$ a matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some rexpansion of $\mathcal{M}_1$. Then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$.

Proof. By Definition 6.1.4, for every F-rexpansion $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$ of $\mathcal{M}_1$ we must have $F(t) \cap \mathcal{V}' \neq \emptyset$ (as otherwise $\mathcal{D}' = \emptyset$) and $F(f) \cap \mathcal{V}' \neq \emptyset$ (as otherwise $\mathcal{D}' = \mathcal{V}'$). The result then follows from Corollary 6.2.9. □

6.3 Some Examples

In this section we present some examples for the usefulness of rexpansion in non-classical logics. This is done by performing it on fragments of well-known matrices and Nmatrices, and thus obtaining conservative extensions of their induced logics.

Denote the (propositional) language whose set of connectives is $\{\wedge, \vee, \supset, \neg, \bot\}$ by $\mathcal{CL}_\bot$, classical logic (over $\mathcal{CL}_\bot$) by **CL**, and the classical two-valued matrix for $\mathcal{CL}_\bot$ by $\mathcal{M}_{\mathbf{CL}}$. For any $C \subseteq \{\wedge, \vee, \supset, \neg, \bot\}$, we denote the C-fragment of $\mathcal{CL}_\bot$ by ${\mathcal{CL}_\bot}^C$, the C-fragment of classical logic by $\mathbf{CL}^C$, and the C-fragment of the classical matrix by $\mathcal{M}^C_{\mathbf{CL}}$. For example, the language $\mathcal{CL}$ used in previous examples is ${\mathcal{CL}_\bot}^{\{\neg,\wedge,\vee,\supset\}}$. We start with a direct consequence of Corollary 6.2.10:

Lemma 6.3.1. Let $C \subseteq \{\wedge, \vee, \supset, \neg, \bot\}$. If $\mathcal{L}$ is a language such that ${\mathcal{CL}_\bot}^C \subseteq \mathcal{L}$, and $\mathcal{M}$ is an extension to $\mathcal{L}$ of some rexpansion of $\mathcal{M}^C_{\mathbf{CL}}$, then $\langle \mathcal{L}, \vdash_\mathcal{M} \rangle$ is conservative over $\mathbf{CL}^C$.

We shall use Lemma 6.3.1 to present conservative extensions of fragments of classical logic. Some of the resulting logics are *paraconsistent*, that is: unlike classical logic, they tolerate contradictions. Here is a formal definition, based on properties that were investigated in [4, 5, 38, 78]:[5]

Definition 6.3.2. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_\mathbf{L} \rangle$ be a logic such that $\neg$ is included in $\mathcal{L}$. $\neg$ is a *weak negation in* **L** if $p \nvdash_\mathbf{L} \neg p$ and $\neg p \nvdash_\mathbf{L} p$ for every $p \in At$. **L** is *paraconsistent* if $\neg$ is a weak negation in **L** and $\neg p, p \nvdash_\mathbf{L} q$ for every two distinct atoms p and q. It is *boldly paraconsistent* if it is paraconsistent, and $\neg p, p \nvdash_\mathbf{L} \varphi$ whenever $\nvdash_\mathbf{L} \varphi$ and $p \in At$ does not occur in φ. Further, $\neg$ is a *negation in* **L** if $\mathcal{CL} \subseteq \mathcal{L}$ ($\mathcal{CL}_\bot \subseteq \mathcal{L}$) and the $\mathcal{CL}$-fragment ($\mathcal{CL}_\bot$-fragment) of **L** is *subclassical*, that is, $\mathcal{T} \vdash_\mathbf{L} \varphi$ only if $\mathcal{T} \vdash_\mathbf{CL} \varphi$ for any subset $\mathcal{T} \cup \{\varphi\}$ of $\mathcal{CL}$ ($\mathcal{CL}_\bot$). **L** is *strictly paraconsistent* if it is paraconsistent and $\neg$ is a negation in **L**.

Remark 6.3.3. The requirement for being a weak negation is regarded in [5] as a minimal condition that is expected from a unary connective to be called a negation (see also [78]).

[5]The term "paraconsistent logic" already appears in previous chapters of this thesis, that included sequent calculi for logics that are recognized in the literature as paraconsistent. In this chapter, however, we provide a more concrete analysis of this term, that requires a formal definition.

This is hardly enough, though, to characterize negation. Therefore, [4] generalizes it to the requirement of subclassicality, leading to what is called here "strict paraconsistency". (In practice, almost all non-classical logics in $\mathcal{CL}_\perp$ ever studied are subclassical.) Finally, the requirement of bold paraconsistency connects paraconsistency to (and justifies it by) the broader principle of *relevance*: the inconsistency of p should not be a reason for inferring a formula that is completely irrelevant to p.

We start with (deterministic) finite-valued conservative extensions of classical logic.

Example 6.3.4. The $\{\wedge, \vee\}$-fragments of $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{KL}}$ (see Example 6.1.2) are simple rexpansions of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee\}}$, as can be witnessed by the expansion functions $\lambda x \in \{t, f\} . \begin{cases} \{t, i\} & x = t \\ \{f\} & x = f \end{cases}$ and $\lambda x \in \{t, f\} . \begin{cases} \{t\} & x = t \\ \{f, i\} & x = f \end{cases}$, respectively. By Lemma 6.3.1, **LP** and **KL** are both conservative over $\mathbf{CL}^{\{\wedge,\vee\}}$. Note that neither of the matrices is a preserving rexpansion of the other: suppose for contradiction that $\mathcal{M}_{\mathbf{LP}}$ is a preserving F-rexpansion of $\mathcal{M}_{\mathbf{KL}}$. Then we must have that $\{t, i\} \subseteq F(t)$, and so $f \in F(i) \cup F(f)$. If $f \in F(i)$ then $F(f) \cap \{t, f, i\} = \emptyset$, and if $f \in F(f)$ then $F(i) \cap \{t, f, i\} = \emptyset$. Either way, $\mathcal{M}_{\mathbf{LP}}$ is not preserving. Clearly, $\mathcal{M}_{\mathbf{KL}}$ cannot be a preserving F-rexpansion of $\mathcal{M}_{\mathbf{LP}}$, as if this were the case, it would have two designated values. **LP** is one of the strict paraconsistent logics studied in [4] (see Section 6.4.1).

The next example concerns Gödel fuzzy logic **G** and its relation to classical logic.

Example 6.3.5. It is routine to verify that $\mathcal{M}_{\mathbf{G}}$ (Example 6.1.3) is an extension to $\{\wedge, \vee, \supset, \perp\}$ of a simple refinement of the F-expansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\perp\}}$, for $F(f) = [0, 1)$ and $F(t) = \{1\}$. Consider $\wedge$, for example, and denote its classical interpretation by $\mathcal{O}(\wedge)$. Let $x, y \in [0, 1]$ and $z = \min(x, y)$, and suppose $x' = \widetilde{F}[x]$ and $y' = \widetilde{F}[y]$. We show that $z \in F(\mathcal{O}(\wedge)(x', y'))$ (recall that in matrices we identify singletons with their unique elements). If $z < 1$, then either $x < 1$ or $y < 1$, and so either $x' = f$ or $y' = f$, which means that $\mathcal{O}(\wedge)(x', y') = f$. In this case, we get $z \in [0, 1) = F(f) = F(\mathcal{O}(\wedge)(x', y'))$. Otherwise, $z = 1$, which means that $x = y = 1$, and so $x' = y' = t$. In this case, $\mathcal{O}(x', y') = t$, and so we have $z \in \{1\} = F(t) = F(\mathcal{O}(\wedge)(x', y'))$. By Lemma 6.3.1, **G** is conservative over $\mathbf{CL}^{\{\wedge,\vee,\perp\}}$.

The process described in the above examples need not start with classical logic, as can be seen by the following example:

Example 6.3.6. Consider the following matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, defined by $\mathcal{V} = \{t, f, \top, \perp\}$, $\mathcal{D} = \{t\}$, and $\mathcal{O}$ is given by:

$\mathcal{O}(\wedge)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	$\bot$
$\bot$	$\bot$	f	$\bot$	$\bot$

$\mathcal{O}(\vee)$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	$\top$
$\bot$	t	$\bot$	$\top$	$\bot$

	$\mathcal{O}_4(\neg)$
t	f
f	t
$\top$	t
$\bot$	t

Its conjunction and disjunction are interpreted as minimum and maximum (respectively) over the ordering $f \leq \bot \leq \top \leq t$. Its $\{\wedge, \vee\}$-fragment is a simple F-rexpansion of the $\{\wedge, \vee\}$-fragment of $\mathcal{M}_{\mathbf{KL}}$ (Example 6.1.2), for $F(t) = \{t\}$, $F(f) = \{f, \bot\}$ and $F(i) = \{\top\}$. By Corollary 6.2.9, the logic it induces is conservative over the $\{\wedge, \vee\}$-fragment of **KL**. It is a different logic than **KL**, as it has tautologies (e.g. $p \vee \neg p$).

6.4 Applications to Sequent Calculi

In this section we show how Nmatrices, and rexpansions of them in particular, are used in order to construct analytic sequent calculi for a given (family of) logic(s). We do so by uncovering the underlying process in the construction of several sequent calculi for non-classical logics from the literature (see, e.g., [14, 27]), and identifying it as an implicit sequence of rexpansions. Roughly speaking, the process begins with some basic logic, whose various extensions form a family of non-classical logics. For this basic logic, an Nmatrix is found, and then the algorithm of [25] is used to translate it to a cut-free Gentzen-type system, that admits the generalized notion of analyticity employed in this thesis. In some cases, this construction can be done directly, without relying on [25]. In order to do the same for the various extensions of the logic, the original Nmatrix sometimes needs to be transformed into another Nmatrix, using a rexpansion. Then, the various extensions of the logic are translated into refinements of the new Nmatrix, and the algorithm of [25] is again used for each of these refinements. We provide detailed examples of this process for three and four valued logics, as well as for the logics of formal inconsistency from [38].

6.4.1 Three-valued Paraconsistent Logics

Consider the rules $(\neg \Rightarrow)$ and $(\Rightarrow \neg)$ of **LK**. Their axiomatic counterparts (see Section 2.4) are $p, \neg p \Rightarrow$ and $\Rightarrow p, \neg p$, respectively. Now, having in mind the intuitive reading of sequents, that interprets a comma on the left as a conjunction, and a comma on the right as a disjunction, we see that $(\neg \Rightarrow)$ corresponds to the law of contradiction, and that $(\Rightarrow \neg)$ corresponds to the law of excluded middle. Accordingly, these two rules represent the two basic principles that characterize classical negation. For the construction of sequent calculi for paraconsistent logics, we must give up $(\neg \Rightarrow)$. Therefore, in order for $\neg$ to still have some properties of a *negation*, we should at least keep intact the law of excluded middle. Thus we are left with $\mathbf{LK} \setminus \{(\neg \Rightarrow)\}$ as the calculus for the basic paraconsistent extension of positive classical logic. The induced logic is known to be identical to the logic **CLuN** from [32] (see, e.g., [14]). We denote this calculus by $\mathbf{G_{CLuN}}$. It is routine to verify that a $\mathcal{CL}$-bivaluation v is $\mathbf{G_{CLuN}}$-legal iff it is an $\mathcal{M}^2_{\mathbf{CLuN}}$-bivaluation, where $\mathcal{M}^2_{\mathbf{CLuN}} = \langle \{t, f\}, \{t\}, \mathcal{O}^2_{\mathbf{CLuN}} \rangle$ is defined like the classical matrix for $\wedge$, $\vee$ and $\supset$, while $\mathcal{O}^2_{\mathbf{CLuN}}(\neg)$ is given by:

	$\mathcal{O}^2_{\mathbf{CLuN}}(\neg)$
t	$\{t, f\}$
f	$\{t\}$

Many well-known paraconsistent logics from the literature are characterized as axiomatic extensions of **CLuN**, by augmenting it with classically valid axioms that are not valid in **CLuN**. Examples include the following axioms for the double negation principle and for connections between negation and the binary connectives (e.g. De Morgan laws):

(**c**) $\neg\neg\varphi \supset \varphi$

($\mathbf{n}^{\mathbf{l}}_{\wedge}$) $\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

($\mathbf{n}^{\mathbf{l}}_{\vee}$) $\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$

($\mathbf{n}^{\mathbf{l}}_{\supset}$) $\neg(\varphi \supset \psi) \supset (\varphi \wedge \neg\psi)$

(**e**) $\varphi \supset \neg\neg\varphi$

($\mathbf{n}^{\mathbf{r}}_{\wedge}$) $(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$

($\mathbf{n}^{\mathbf{r}}_{\vee}$) $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

($\mathbf{n}^{\mathbf{r}}_{\supset}$) $(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$

For example, the logic $\mathbf{C_{min}}$ (see, e.g., [38, 39]) is the axiomatic extension of **CLuN** with (**c**). In order to obtain corresponding analytic sequent calculi for these various extensions, appropriate finite Nmatrices should be constructed. Let us examine, for example, the effect of incorporating (**c**) into $\mathcal{M}^2_{\mathbf{CLuN}}$. This would mean that if p is assigned f, then $\neg p$ must be assigned t, and in order for the axiom to be satisfied, we must have that $\neg\neg p$ is assigned f. Hence we must set the negation of t to f, which brings us back to the classical matrix.[6] However, the addition of the rules of excluded middle and of double negation elimination to any Hilbert-type system for positive classical logic does not result in classical logic. The problem here is that $\mathcal{M}^2_{\mathbf{CLuN}}$ is almost deterministic: it has a single non-deterministic entry that allows for exactly two refinements. One results in the classical matrix, and the other is equivalent to positive classical logic. Hence $\mathcal{M}^2_{\mathbf{CLuN}}$ cannot serve as a basis for extensions of **CLuN** with more rules that concern

[6] Obviously, this would not be the case if we were working in the more general framework of bivaluations (see Definition 2.3.1). In such a case, (**c**) would simply be translated to a semantic constraint, namely $v(\varphi) = 0$ implies $v(\neg\neg\varphi) = 0$ for any formula φ. However, for the purpose of constructing analytic sequent calculi, we would like to generate only Nmatrices, and then the inclusion of (**c**) indeed leads to the classical matrix.

negation. A solution to this problem is to perform a rexpansion on $\mathcal{M}^2_{\mathbf{CLuN}}$, and obtain a less deterministic Nmatrix. By the above argument, we need more flexibility with the negation of f, which is t. Thus we shall make two copies of t, using an expansion function F that is defined by $F(t) = \{t, \top\}$ and $F(f) = \{f\}$. The resulting expansion $(\mathcal{M}^2_{\mathbf{CLuN}})_F = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O}_F \rangle$ is given by:

$\mathcal{O}_F(\wedge)$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\mathcal{O}_F(\vee)$	t	f	$\top$
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\mathcal{O}_F(\supset)$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

	$\mathcal{O}_F(\neg)$
t	$\{t, \top, f\}$
f	$\{t, \top\}$
$\top$	$\{t, \top, f\}$

Now, in order to construct an ordinary analytic sequent calculus from an Nmatrix, the Nmatrix should be *sufficiently expressive*, in the sense that we must be able to distinguish between the different truth values using the connectives themselves and the disjoint sets of designated and undesignated values (see [25] for a precise definition). This is not the case for $(\mathcal{M}^2_{\mathbf{CLuN}})_F$, as we cannot distinguish this way between t and $\top$. To be able to do that, we need to perform a refinement. The minimal refinement that would allow for this distinction between truth values,[7] that we denote by $\mathcal{M}^3_{\mathbf{CLuN}} = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O}^3_{\mathbf{CLuN}} \rangle$, is given by $\mathcal{O}^3_{\mathbf{CLuN}}(\diamond) = \mathcal{O}_F(\diamond)$ for every $\diamond \in \{\wedge, \vee, \supset\}$, while $\mathcal{O}^3_{\mathbf{CLuN}}(\neg)$ is given by:

	$\mathcal{O}^3_{\mathbf{CLuN}}(\neg)$
t	$\{f\}$
f	$\{t, \top\}$
$\top$	$\{t, \top\}$

Now, for every $\mathcal{M}^3_{\mathbf{CLuN}}$-valuation v, we have the following:

- $v(p) = t$ iff $v(p)$ is designated and $v(\neg p)$ is not
- $v(p) = \top$ iff both $v(p)$ and $v(\neg p)$ are designated
- $v(p) = f$ iff $v(p)$ is not designated

[7]There is a second minimal refinement that is dual to this one. It will be described and used in Section 6.4.2.

As we shall see, this makes $\mathcal{M}^3_{\mathbf{CLuN}}$ a good basis for the addition of the most useful axioms in paraconsistent logics (including the above axioms). Each axiom is first translated into a semantic condition, that in turn induces a refinement of $\mathcal{M}^3_{\mathbf{CLuN}}$. The refinement is then translated to a Gentzen-type rule. By collecting all the resulting rules that correspond to a given axiomatic extension of **CLuN**, we obtain a cut-free $\{\neg\}$-1-analytic pure calculus for it.

Note that although $\mathcal{M}^3_{\mathbf{CLuN}}$ is a non-strongly preserving rexpansion of $\mathcal{M}^2_{\mathbf{CLuN}}$, it can be shown that the two Nmatrices are equivalent, and both induce **CLuN** (see, e.g., [14]).

Let us now examine the effect of (**c**) on $\mathcal{M}^3_{\mathbf{CLuN}}$, rather than on $\mathcal{M}^2_{\mathbf{CLuN}}$. If p is assigned f, then in order for (**c**) to be satisfied, we must have that $\neg\neg p$ is also assigned f, which is only possible if $\neg p$ is assigned t. This simply means that the negation of f should be $\{t\}$ and not $\{t, \top\}$. We thus obtain a simple refinement of $\mathcal{M}^3_{\mathbf{CLuN}}$, that we denote by $\mathcal{M}_c = \langle \{t, f, \top\}, \{t, \top\}, \mathcal{O}_c \rangle$, and is obtained by replacing $\mathcal{O}^3_{\mathbf{CLuN}}(\neg)$ with:

	$\mathcal{O}_c(\neg)$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t, \top\}$

By applying the algorithm from [25], we get a calculus $\mathbf{G}_c$ for $\vdash_{\mathcal{M}_c}$, which is the extension of $\mathbf{G}_{\mathbf{CLuN}}$ with the rule $(\neg\neg \Rightarrow)$ of $\mathbf{G}_{\mathbf{L}_3}$ (see Example 2.2.8). Let us see how we obtain this rule directly from the semantics:[8] The only change that was made from $\mathcal{M}^3_{\mathbf{CLuN}}$ is setting the negation of f to be t. This means that if $v(p) = f$ then $v(\neg p) = t$. Going back to the characterization above using designated and undesignated values, we get that if $v(p)$ is undesignated, then $v(\neg p)$ is designated and $v(\neg\neg p)$ is not. Translating "designated" to being on the right side of a sequent, and "undesignated" to being on the left, we obtain $p \Rightarrow\ /\ \Rightarrow \neg p$ and $p \Rightarrow\ /\ \neg\neg p \Rightarrow$. The first is already included in $\mathbf{G}_{\mathbf{CLuN}}$, and the second is the new rule $(\neg\neg \Rightarrow)$. This is how the rule $(\neg\neg \Rightarrow)$ is constructed from (**c**). By using similar arguments, we get that the refinement associated with $(\mathbf{n}^{\mathbf{l}}_{\wedge})$ is obtained from $\mathcal{M}^3_{\mathbf{CLuN}}$ by setting $\mathcal{O}(\wedge)(t, t)$ to $\{t\}$ (instead of $\{t, \top\}$), which is similarly translated to the rule $(\neg\wedge \Rightarrow)$ of $\mathbf{G}_{\mathbf{L}_3}$.

A particularly interesting simple refinement $\mathcal{M}_{ca} = \langle \{t, f, \top\}, \{t, \top\}, \mathcal{O}_{ca} \rangle$ of $\mathcal{M}_c$ is obtained by $\mathcal{O}_{ca}(\neg) = \mathcal{O}_c(\neg)$, and for the rest of the connectives $\mathcal{O}_{ca}$ is given by:

[8]Actually, $(\neg\neg \Rightarrow)$ could have been "guessed" from (**c**), without rexpansions. This is not always the case. Some axioms do not naturally translate into sequent rules, and then the method of [25], that goes through Nmatrices, is essential. We shall see an example for this in the next section.

$\mathcal{O}_{ca}(\wedge)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t,\top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$

$\mathcal{O}_{ca}(\vee)$	t	f	$\top$
t	$\{t\}$	$\{t\}$	$\{t,\top\}$
f	$\{t\}$	$\{f\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

$\mathcal{O}_{ca}(\supset)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t,\top\}$
f	$\{t\}$	$\{t\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$

The Nmatrix $\mathcal{M}_{ca}$ has a very important role in the family of paraconsistent logics that are characterized by three-valued *matrices*. Theorem 42 of [4] characterizes all three-valued strictly paraconsistent logics in the language $\mathcal{CL}$ (see Definition 6.3.2) that admit some natural properties. These logics coincide with the $\mathcal{CL}$-fragments of the family of $8K$ conservative extensions of positive classical logic studied in [38] and [39]. The three-valued matrices that induce these logics are exactly the deterministic simple refinements of $\mathcal{M}_{ca}$. When applying the same rexpansion function F from above on $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$, rather than on $\mathcal{M}_{\mathbf{CLuN}}^{2}$, we see that $\mathcal{M}_{ca}$ (and also $\mathcal{M}_{c}$) is an extension to $\mathcal{CL}$ of an F-rexpansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$. As a consequence of Lemma 6.3.1, all these logics are conservative over positive classical logic.

Each of the simple refinements of $\mathcal{M}_{ca}$, including all the deterministic ones studied in [4], can be given a cut-free $\{\neg\}$-1-analytic pure calculus, using the method of [25], or by translating the semantic conditions that correspond to each axiom into pure rules. Calculi for the most important such refinements are explicitly given in [4]. One of these logics is called **PAC**, and is the axiomatic extension of **CLuN** with all the axioms (**c**), (**e**) and ($\mathbf{b}_{\diamond}^{\mathbf{x}}$) such that $\diamond \in \{\wedge,\vee,\supset\}$ and $x \in \{l,r\}$ above. The characteristic matrix $\mathcal{M}_{\mathbf{PAC}}$ for **PAC** is the extension to $\mathcal{CL}$ of $\mathcal{M}_{\mathbf{LP}}$, obtained by setting:

$\mathcal{O}(\supset)$	t	f	$\top$
t	t	f	$\top$
f	t	t	t
$\top$	t	f	$\top$

A calculus $\mathbf{G}_{\mathbf{PAC}}$ for this logic is obtained from the calculus $\mathbf{G}_{\text{Ł}_3}$ (see Example 2.2.8) by replacing the rules $(\neg\Rightarrow)$, $(\supset\Rightarrow)_{\text{Ł}_3}$ and $(\Rightarrow\supset)_{\text{Ł}_3}$ with the rule $(\Rightarrow\neg)$.

6.4.2 Logics of Formal (In)consistency

Let us revisit the construction of $\mathbf{G}_{\mathbf{C_1}}$ (Example 2.2.9) in [27]. We shall relate $\mathbf{G}_{\mathbf{C_1}}$ to a calculus that we call $\mathbf{G}_{BKca}$ (for reasons to be clarified below), obtained from $\mathbf{G}_{\mathbf{C_1}}$ by dismissing the rule

$$(\neg\wedge\Rightarrow)_{(\mathbf{1})} \;\; \Rightarrow p_1;\; \Rightarrow \neg p_1 \,/\, \neg(p_1 \wedge \neg p_1) \Rightarrow$$

$\mathbf{G}_{\mathbf{C_1}}$ and $\mathbf{G}_{BKca}$ are just two of various sequent calculi that were introduced in [27] for paraconsistent logics of the family of *Logics of Formal Inconsistency (LFIs)* [38, 39] (see also [28, 29]). While [27] did not explicitly use rexpansions, we show that they were actually being implicitly used, and had a very important role in the various constructions of [27] (and in particular, that of $\mathbf{G}_{\mathbf{C_1}}$ and of $\mathbf{G}_{BKca}$). As sketched at the beginning of this chapter, the calculi of [27] are constructed uniformly, by first finding a characteristic Nmatrix, and then extracting a sequent calculus from it. The underlying language is $\mathcal{CL}^\circ = \mathcal{CL} \cup \{\circ\}$, where $\circ$ is a unary connective which is intended to classify a given proposition as *consistent* (that is, $\circ\varphi$ should be read as "φ is consistent"). The most basic properties of the consistency connective $\circ$ are represented by the following two axioms:

$$(\mathbf{b})\ \circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi) \qquad\qquad (\mathbf{k})\ \circ\varphi \vee (\varphi \wedge \neg\varphi)$$

The first means that the law of contradiction is valid for consistent formulas. The second means that every formula is either consistent or both it and its negation hold. Accordingly, the most basic logic that is investigated in [27] is called BK, and is obtained from **CLuN** by the addition of these two axioms. It is proven there to be characterized by the Nmatrix $\mathcal{M}_{BK}$, which is the extension to $\mathcal{CL}^\circ$ of the Nmatrix $\mathcal{M}^3_{\mathbf{CLuN}}$ from Section 6.4.1, given by:

	$\mathcal{O}(\circ)$
t	$\{t, \top\}$
f	$\{t, \top\}$
$\top$	$\{f\}$

Note that if we replace the symbol $\neg$ in $\mathcal{M}^2_{\mathbf{CLuN}}$ from Section 6.4.1, we again obtain an F-rexpansion of $\mathcal{M}^2_{\mathbf{CLuN}}$ (for the same F that was used there, that is $F(t) = \{t, \top\}$ and $F(f) = \{f\}$), where in the performed refinement, the roles of t and $\top$ are exchanged.

BK serves as a basis for the modular construction of more powerful paraconsistent logics, that are obtained by including the axioms from Section 6.4.1, as well as basic properties of the consistency operator, such as:

- Inconsistency: $(\mathbf{i})\ \neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$
- Propagation laws: $(\mathbf{a})\ (\circ\varphi \sharp \circ\psi) \supset \circ(\varphi \sharp \psi)$ for $\sharp \in \{\wedge, \vee, \supset\}$

Accordingly, a set A_0 of well-known axioms for LFIs is considered (that includes, among others, those just mentioned, as well as the axioms from Section 6.4.1), and is modularly incorporated into this Nmatrix: each subset of A_0 induces a simple refinement of $\mathcal{M}_{BK}$. As we have already seen in Section 6.4.1, the addition of the axiom $(\mathbf{c})$ amounts to setting $\mathcal{O}(\neg)(f)$ to $\{t\}$ (instead of $\{t, \top\}$). The result is the extension to $\mathcal{CL}^\circ$ of the Nmatrix $\mathcal{M}_c$ from Section 6.4.1 with the above table for $\circ$. Further, the addition of $(\mathbf{a})$ amounts to ensuring that $\varphi\sharp\psi$ is given a value from $\{t, f\}$ whenever both φ and ψ are given values from $\{t, f\}$, for each $\sharp \in \{\wedge, \vee, \supset\}$. Accordingly, the Nmatrix that corresponds to the logic $BKca$, obtained by the addition of (**c**) and (**a**) to BK, is the extension to $\mathcal{CL}^\circ$ of the Nmatrix $\mathcal{M}_{ca}$ from Section 6.4.1, obtained by including the same truth table for $\circ$ above. We denote the resulting Nmatrix by $\mathcal{M}_{BKca} = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O}_{BKca} \rangle$:

$\mathcal{O}_{BKca}(\wedge)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\mathcal{O}_{BKca}(\vee)$	t	f	$\top$
t	$\{t\}$	$\{t\}$	$\{t, \top\}$
f	$\{t\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\mathcal{O}_{BKca}(\supset)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t, \top\}$
f	$\{t\}$	$\{t\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

	$\mathcal{O}_{BKca}(\neg)$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t, \top\}$

	$\mathcal{O}_{BKca}(\circ)$
t	$\{t, \top\}$
f	$\{t, \top\}$
$\top$	$\{f\}$

Using the algorithm of [25], a cut-free pure calculus for $BKca$ is obtained from the calculus $\mathbf{G}_c$ (see Section 6.4.1) by the addition of several rules for the binary connectives. For example, the additional rule for $\wedge$ is ($\neg\wedge \Rightarrow$) from $\mathbf{G}_{\text{Ł}_3}$ (see Example 2.2.8). Unlike the case of (**c**), the rules that correspond to (**a**) are not natural translations of it. In fact, the connective $\circ$ does not even occur in the resulting rules. To obtain them, the method of [25], or the translation of axioms to semantic conditions and then to rules, is essential.

Things become more complicated when the following two well-known axioms of LFIs are added to A_0:

$$(\mathbf{l})\ \neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \qquad \text{and} \qquad (\mathbf{d})\ \neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$$

It was shown in [15] that most of the systems in the family induced by $A_0 \cup \{(\mathbf{l}), (\mathbf{d})\}$ that include at least one of $\{(\mathbf{l}), (\mathbf{d})\}$ cannot be characterized by a finite Nmatrix. This means that they go beyond the reach of $\mathcal{M}_{BK}$ and its refinements. For this reason, another rexpansion is performed on $\mathcal{M}_{BK}$ (whose $\circ$-free fragment is itself a rexpansion of $\mathcal{M}^2_{\mathbf{CLuN}}$), that incorporates infinitely many copies of each of its two designated truth values. This is done by employing the following three (disjoint) sets: $\mathcal{T} = \left\{t^j_i \mid i \geq 0, j \geq 0\right\}$, $\mathcal{I} = \left\{\top^j_i \mid i \geq 0, j \geq 0\right\}$, and $\mathcal{F} = \{f\}$. An expansion function G is then defined by

$$G = \lambda x \in \{t, f, \top\} . \begin{cases} \mathcal{T} & x = t \\ \mathcal{F} & x = f \\ \mathcal{I} & x = \top \end{cases}$$

The $\mathbf{G}$-expansion $(\mathcal{M}_{BK})_G = \langle \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}, \mathcal{T} \cup \mathcal{I}, \mathcal{O}_G \rangle$ of $\mathcal{M}_{BK}$ is given by:

$$\mathcal{O}_G(\wedge)(a,b) = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D}^{\infty}_{BK} & otherwise \end{cases} \quad \mathcal{O}_G(\vee)(a,b) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{D}^{\infty}_{BK} \text{ or } b \in \mathcal{D}^{\infty}_{BK} \\ \mathcal{F} & otherwise \end{cases}$$

$$\mathcal{O}_G(\neg)(a) = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \cup \mathcal{I} \end{cases} \quad \mathcal{O}_G(\circ)(a) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & otherwise \end{cases}$$

$$\mathcal{O}_G(\supset)(a,b) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \text{ or } b \in \mathcal{D}^{\infty}_{BK} \\ \mathcal{F} & otherwise \end{cases}$$

Now, (**l**) and (**d**) deal with the conjunction of a formula with its own negation. We thus need to reflect the difference between an atom and its own negation (and not only negations of formulas that were assigned the same value) in the truth tables. For that, we do not treat the negation of every copy of $\top$ the same way, but distinguish between the different copies. One way to do this is to set the negation of $\top^j_i$ to $\top^{j+1}_i$. However, since we would like to modularly add (**l**) and (**d**) to all the logics considered above, our starting point should be an Nmatrix for the most basic logic BK. We would therefore want to have a strongly preserving rexpansion of $\mathcal{M}_{BK}$. The above choice for the negation of $\top^j_i$ is not enough, as then there is no copy of t in the negation of $\top^j_i$, although t is one of the values in the negation of $\top$ in $\mathcal{M}_{BK}$. We therefore add also the corresponding copy of t, namely t^{j+1}_i. The result, which is now a *strongly preserving* G-rexpansion of $\mathcal{M}_{BK}$, is denoted by $\mathcal{M}^{\infty}_{BK} = \langle \mathcal{V}^{\infty}_{BK}, \mathcal{D}^{\infty}_{BK}, \mathcal{O}^{\infty}_{BK} \rangle$, and is defined by $\mathcal{V}^{\infty}_{BK} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, $\mathcal{D}^{\infty}_{BK} = \mathcal{T} \cup \mathcal{I}$, and $\mathcal{O}^{\infty}_{BK}$ is obtained from O_G by setting:

$$\mathcal{O}^{\infty}_{BK}(\neg)(a) = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \\ \{\top^{j+1}_i, t^{j+1}_i\} & a = \top^j_i \end{cases}$$

We stress that while this is the actual Nmatrix that was built in [27], it was built there without explicitly taking into account the arguments concerning strongly preserving rexpansions. As we can see, these turn out to be essential. Now, not only $\mathcal{M}^{\infty}_{BK}$ is a strongly preserving G-rexpansion of M_{BK}, but also every subset A of A_0 induces a simple refinement of $\mathcal{M}^{\infty}_{BK}$, that is a strongly preserving G-rexpansion of the simple refinement of $\mathcal{M}_{BK}$ that is associated with A. In particular, without (**l**) and (**d**), we obtain an infinite characteristic Nmatrix for each system, equivalent to the three-valued one. Moreover, by applying the corresponding semantic arguments for the construction of the Gentzen-type systems on the infinite Nmatrices, we obtain the same rules. For example, going back to $BKca$, we obtain the Nmatrix $\mathcal{M}^{\infty}_{BKca} = \langle \mathcal{V}^{\infty}_{BK}, \mathcal{D}^{\infty}_{BK}, \mathcal{O}^{\infty}_{BKca} \rangle$, where $\mathcal{O}^{\infty}_{BKca}$ is obtained from $\mathcal{O}^{\infty}_{BK}$ by setting $\mathcal{O}^{\infty}_{BKca}(\neg)(f) = \mathcal{T}$, and ensuring that $\varphi\sharp\psi$ is given a value from $\mathcal{T} \cup \mathcal{F}$, whenever both φ and ψ are given values from $\mathcal{T} \cup \mathcal{F}$, for every $\sharp \in \{\wedge, \vee, \supset\}$:

$$\mathcal{O}^{\infty}_{BKca}(\wedge)(a,b) = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & a, b \in \mathcal{T} \\ \mathcal{D}^{\infty}_{BK} & \textit{otherwise} \end{cases} \qquad \mathcal{O}^{\infty}_{BKca}(\vee)(a,b) = \begin{cases} \mathcal{T} & a \in \mathcal{T} \text{ and } b \in \mathcal{T} \cup \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T} \cup \mathcal{F} \text{ and } b \in \mathcal{T} \\ \mathcal{F} & a, b \in \mathcal{F} \\ \mathcal{D}^{\infty}_{BK} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}^{\infty}_{BKca}(\neg)(a) = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{T} & a \in \mathcal{F} \\ \left\{\top^{j+1}_i, t^{j+1}_i\right\} & a = \top^j_i \end{cases} \qquad \mathcal{O}^{\infty}_{BKca}(\circ)(a) = \begin{cases} \mathcal{D}^{\infty}_{BK} & a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}^{\infty}_{BKca}(\supset)(a,b) = \begin{cases} \mathcal{T} & a \in \mathcal{F} \text{ and } b \in \mathcal{T} \cup \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T} \cup \mathcal{F} \text{ and } b \in \mathcal{T} \\ \mathcal{F} & a \in \mathcal{D}^{\infty}_{BK} \text{ and } b \in \mathcal{F} \\ \mathcal{D}^{\infty}_{BK} & \textit{otherwise} \end{cases}$$

It is routine to verify that $\mathcal{M}^{\infty}_{BKca}$ is a strongly preserving G-rexpansion $\mathcal{M}_{BKca}$. Also, the calculus for the induced logic can be similarly reconstructed, based on the semantic conditions of $\mathcal{M}^{\infty}_{BKca}$, that are natural extensions of those of $\mathcal{M}_{BKca}$.

When either (**l**) or (**d**) are included, however, what is obtained is again a preserving rexpansion of the corresponding three-valued Nmatrix, but not a strongly preserving one. For example, the refinement that is associated with (**l**) amounts to the requirement that $\varphi \wedge \psi$ is assigned a value from $\mathcal{T}$ whenever φ is assigned $\top^j_i$ and ψ is assigned either $\top^{j+1}_i$ or t^{j+1}_i. Thus, the logic $BKcal$ (obtained from $BKca$ by the addition of (**l**)) is characterized by the Nmatrix $\mathcal{M}^{\infty}_{BKcal} = \langle \mathcal{V}^{\infty}_{BK}, \mathcal{D}^{\infty}_{BK}, \mathcal{O}^{\infty}_{BKcal} \rangle$, which is obtained from $\mathcal{M}^{\infty}_{BKca}$ by setting

$$\mathcal{O}^{\infty}_{BKcal}(\wedge) = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & a, b \in \mathcal{T} \\ \mathcal{T} & a = \top^j_i \text{ and } b \in \left\{\top^{j+1}_i, t^{j+1}_i\right\} \\ \mathcal{D}^{\infty}_{BK} & \textit{otherwise} \end{cases}$$

$\mathcal{M}^{\infty}_{BKcal}$ is indeed a preserving G-rexpansion of $\mathcal{M}_{BKca}$, but not a strongly preserving one. For example, $G(\top) \cap \mathcal{O}^{\infty}_{BKcal}(\wedge)(\top^j_i, \top^{j+1}_i) = \emptyset$, although $\top \in \mathcal{O}_{BKca}(\wedge)(\top, \top)$. Since $\mathcal{M}^{\infty}_{BKcal}$ is infinite, it falls out of the method of [25]. However, [27] employs a similar technique and translates this semantic requirement to the rule $(\neg\wedge \Rightarrow)_{(\mathbf{l})}$ above, in a similar manner to the reconstruction of the rules in Section 6.4.1. This is how a calculus $\mathbf{G}_{BKcal}$ for $BKcal$ has been constructed in [27]. The calculus $\mathbf{G}_{\mathbf{C_1}}$ was obtained by simply dismissing the rules for $\circ$, as C_1 is proven in [27] to be characterized by the $\circ$-free fragment of $\mathcal{M}_{BKcal}$.

Note that the route that we have taken here started with the Nmatrix $\mathcal{M}_{BK}$ for BK. Then, a strongly preserving rexpansion was performed to obtain $\mathcal{M}^{\infty}_{BK}$, that was refined according to (**c**), (**a**), and (**l**), resulting with $\mathcal{M}^{\infty}_{BKcal}$. The calculus $\mathbf{G}_{BKcal}$ was built according to the semantic conditions of the resulting Nmatrix. An alternative route would be to first refine BK according to (**c**) and (**a**), and then perform a strongly preserving rexpansion to obtain $\mathcal{M}^{\infty}_{BKca}$. Only then, (**l**) would be added, and the corresponding refinement would result with exactly the same Nmatrix $\mathcal{M}_{BKcal}$, and consequently, with the same sequent calculus.

6.4.3 Conservative Extensions of Sequent Calculi

Suppose that we are looking for a sequent calculus for a logic $\mathbf{L}_2 = \langle \mathcal{L}_2, \vdash_{\mathbf{L}_2} \rangle$, which is a conservative extension of a logic $\mathbf{L}_1 = \langle \mathcal{L}_1, \vdash_{\mathbf{L}_1} \rangle$, and that we already have a calculus $\mathbf{G}_1$ for $\mathbf{L}_1$. When this is the case, we can keep all the rules of $\mathbf{G}_1$, and be sure that they are sound and complete for the $\mathcal{L}_1$-fragment of $\mathbf{L}_2$. The task of finding a calculus for $\mathbf{L}_2$ then reduces to finding rules that involve connectives from $\mathcal{L}_2 \setminus \mathcal{L}_1$. In order to make this intuition more formal, we extend Definitions 6.1.4 and 6.1.6 to sequents.

Definition 6.4.1. Let $\mathcal{M}$ be an Nmatrix. An $\mathcal{M}$-valuation v is an $\mathcal{M}$-*model* of a sequent $\Gamma \Rightarrow \Delta$ (in symbols: $v \vDash^{\mathcal{M}} \Gamma \Rightarrow \Delta$) if $v \nvDash^{\mathcal{M}} \varphi$ for some $\varphi \in \Gamma$ or $v \vDash^{\mathcal{M}} \varphi$ for some $\varphi \in \Delta$. It is an $\mathcal{M}$-*model* of a set $\mathcal{S}$ of sequents (in symbols: $v \vDash^{\mathcal{M}} S$) if $v \vDash^{\mathcal{M}} s$ for every $s \in S$. A sequent s is an $\mathcal{M}$-*consequence* of a set S of sequents (in symbols: $S \vdash_{\mathcal{M}} s$) if every $\mathcal{M}$-model of S is an $\mathcal{M}$-model of s. We say that a sequent calculus $\mathbf{G}$ is *sound* for an Nmatrix $\mathcal{M}$ if $S \vdash_{\mathbf{G}} s$ implies $S \vdash_{\mathcal{M}} s$ for every sequent s and set S of sequents. We say that $\mathbf{G}$ is *complete* for $\mathcal{M}$ if the converse holds. A pure calculus $\mathbf{G}_2$ for $\mathcal{L}_2$ is *conservative over* a pure calculus $\mathbf{G}_1$ for $\mathcal{L}_1$ if $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and for every set S of $\mathcal{L}_1$-sequents and $\mathcal{L}_1$-sequent s it holds that $S \vdash_{\mathbf{G}_1} s$ iff $S \vdash_{\mathbf{G}_2} s$.

Using these definitions, we obtain the following variant of Corollary 6.2.9 for sequents, rather than formulas:

Corollary 6.4.2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, $\mathcal{M}_1$ a (N)matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some (strongly) preserving rexpansion of $\mathcal{M}_1$. Then $S \vdash_{\mathcal{M}_1} s$ iff $S \vdash_{\mathcal{M}_2} s$ for every set S of $\mathcal{L}_1$-sequents and $\mathcal{L}_1$-sequent s.[9]

Proof. It suffices to verify that Proposition 6.1.8 and theorem 6.2.7 are correct also for sequents. Proposition 6.1.8 still follows from the structure of Nmatrices also when

[9]The left-to-right direction holds for all rexpansions.

considering sequents. As for Theorem 6.2.7, we have that also for sequents it suffices to prove that for every $\mathcal{M}_1$-valuation v there exists an $\mathcal{M}_2$-valuation v' such that $v \vDash^{\mathcal{M}_1} s$ iff $v' \vDash^{\mathcal{M}_2} s$ for every $\mathcal{L}$-sequent s. This follows from the corresponding claim for formulas. From here the proof carries as before. □

We are now able to lift the sufficient criterion for conservativity of Corollary 6.2.9 from Nmatrices to sequent calculi:

Proposition 6.4.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, $\mathbf{G}_1$ a pure $\mathcal{L}_1$-calculus and $\mathbf{G}_2$ a pure $\mathcal{L}_2$-calculus. Suppose $\mathbf{G}_1$ is sound and complete for some (N)matrix $\mathcal{M}_1$ for $\mathcal{L}_1$, and that $\mathbf{G}_2$ is sound for some Nmatrix $\mathcal{M}_2$ which is an extension to $\mathcal{L}_2$ of some (strongly) preserving rexpansion of $\mathcal{M}_1$. Then $\mathbf{G}_1 \cup \mathbf{G}_2$ is conservative over $\mathbf{G}_1$.

Proof. Let S be a set of $\mathcal{L}_1$-sequents and s an $\mathcal{L}_1$-sequent. Clearly, if $S \vdash_{\mathbf{G}_1} s$ then $S \vdash_{\mathbf{G}_1 \cup \mathbf{G}_2} s$. Now suppose $S \vdash_{\mathbf{G}_1 \cup \mathbf{G}_2} s$. We prove that $S \vdash_{\mathbf{G}_1} s$.

We start by proving that $S \vdash_{\mathcal{M}_2} s$. Let v be an $\mathcal{M}_2$-model of S. We prove that $v \vDash^{\mathcal{M}_2} s$ by induction on the derivation of s from S in $\mathbf{G}_1 \cup \mathbf{G}_2$. The cases where $s \in S$, or s is the conclusion of an application of (ID), (WEAK) or (CUT) are trivial. Suppose s is the conclusion of an application $\dfrac{\sigma(s'_1) \cup c_1 \dots \sigma(s'_n) \cup c_n}{\sigma(s_0) \cup c_1 \cup \dots \cup c_n}$ of some rule $s_1, \dots, s_n \,/\, s_0$ of $\mathbf{G}_1$. Clearly, $s'_1, \dots, s'_n \vdash_{\mathbf{G}_1} s_0$. Since $\mathbf{G}_1$ is sound for $\mathcal{M}_1$, we have $s'_1, \dots, s'_n \vdash_{\mathcal{M}_1} s_0$. By Corollary 6.4.2, we have $s'_1, \dots, s'_n \vdash_{\mathcal{M}_2} s_0$, and hence $\sigma(s'_1), \dots, \sigma(s'_n) \vdash_{\mathcal{M}_2} \sigma(s_0)$. By the induction hypothesis, $v \vDash^{\mathcal{M}_2} \sigma(s'_i) \cup c_i$ for every $1 \leq i \leq n$. Then, either $v \vDash^{\mathcal{M}_2} c_i$ for some $1 \leq i \leq n$ or $v \vDash^{\mathcal{M}_2} \sigma(s'_i)$ for every $1 \leq i \leq n$. Either way, $v \vDash^{\mathcal{M}_2} s$. It is left to consider the case where s is the conclusion of an application $\dfrac{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n}{\sigma(s_0) \cup c_1, \dots, c_n}$ of some rule $s_1, \dots, s_n \,/\, s_0$ of $\mathbf{G}_2$. By the induction hypothesis, $v \vDash^{\mathcal{M}_2} \sigma(s'_i) \cup c_i$ for every $1 \leq i \leq n$. Since $\mathbf{G}_2$ is sound for $\mathcal{M}_2$, we must have $v \vDash^{\mathcal{M}_2} s$.

Since $S \vdash_{\mathcal{M}_2} s$ and $S \cup \{s\}$ consists of $\mathcal{L}_1$-sequents, we use Corollary 6.4.2 again, and get that $S \vdash_{\mathcal{M}_1} s$. And since $\mathbf{G}_1$ is complete for $\mathcal{M}_1$, we conclude that $S \vdash_{\mathbf{G}_1} s$. □

Proposition 6.4.3 provides a technique to conservatively extend a given sequent calculus, as illustrated in the next example.

Example 6.4.4. The $\{\wedge, \vee, \supset, \neg\}$-fragment of the four-valued logic of bilattices from [3] (see Example 2.2.8) is characterized by the matrix $\mathcal{M}_4 = \langle \{t, f, \top, \bot\}, \{\top, \bot\}, \mathcal{O}_4 \rangle$, where $\mathcal{O}_4$ is given by:

$\mathcal{O}_4(\wedge)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\mathcal{O}_4(\vee)$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\mathcal{O}_4(\supset)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	t	t	t	t
$\top$	t	f	$\top$	$\bot$
$\bot$	t	t	t	t

	$\mathcal{O}_4(\neg)$
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

Let us consider the task of designing a calculus for this logic. What should be our starting point? It is routine to verify that the $\{\wedge, \supset\}$-fragment of $\mathcal{M}_4$ is a preserving **G**-rexpansion of the same fragment of $\mathcal{M}_{\mathbf{PAC}}$ (see Section 6.4.1), for $\mathbf{G}(t) = \{t\}$, $\mathbf{G}(f) = \{f, \bot\}$, and $\mathbf{G}(\top) = \{\top\}$. This means that we can start with the $\{\wedge, \supset\}$-fragment of $\mathbf{G_{PAC}}$ (or of **LK**), and focus on finding rules for $\vee$ and $\neg$. But perhaps there is an even better starting point? Define an expansion function F for $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$ by $F(f) = \{f, \bot\}$ and $F(t) = \{t, \top\}$. It is easy to see that $\mathcal{M}_4$ is an extension to $\{\wedge, \vee, \supset, \neg\}$ of a simple refinement of $(\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}})_F = \langle \{t, \top, f, \bot\}, \{t, \top\}, \mathcal{O} \rangle$, where

$\mathcal{O}(\wedge)$	t	f	$\top$	$\bot$
t	$\{t, \top\}$	$\{f, \bot\}$	$\{t, \top\}$	$\{f, \bot\}$
f	$\{f, \bot\}$	$\{f, \bot\}$	$\{f, \bot\}$	$\{f, \bot\}$
$\top$	$\{t, \top\}$	$\{f, \bot\}$	$\{t, \top\}$	$\{f, \bot\}$
$\bot$	$\{f, \bot\}$	$\{f, \bot\}$	$\{f, \bot\}$	$\{f, \bot\}$

$\mathcal{O}(\vee)$	t	f	$\top$	$\bot$
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{f, \bot\}$	$\{t, \top\}$	$\{f, \bot\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\bot$	$\{t, \top\}$	$\{f, \bot\}$	$\{t, \top\}$	$\{f, \bot\}$

$\mathcal{O}(\supset)$	t	f	$\top$	$\bot$
t	$\{t, \top\}$	$\{f, \bot\}$	$\{t, \top\}$	$\{f, \bot\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f, \bot\}$	$\{t, \top\}$	$\{f, \bot\}$
$\bot$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

We thus obtain that $\vdash_{\mathcal{M}_4}$ is conservative over $\mathbf{CL}^{\{\wedge,\vee,\supset\}}$, and that when constructing a calculus for it, one may focus on rules for negation. There is no need for new rules that involve disjunction (without negation). And indeed, [3] introduced the calculus $\mathbf{G}_4$ (see Example 2.2.8) for this logic, which is obtained from the positive fragment of **LK** by the addition of the new rules for $\mathbf{G_{PAC}}$, except for $(\Rightarrow \neg)$. Thus, $\mathbf{G}_4$ is conservative over the positive fragment of **LK**. Note that only soundness of the rules from $\mathbf{G}_4 \setminus \mathbf{LK}$ for $\mathcal{M}_4$ is needed in order to use Proposition 6.4.3 and obtain conservativity.

A similar process to the one described in Example 6.4.4 can be done for the construction of the other calculi presented in this chapter, by starting with the positive fragment of **LK**, and then adding additional rules, according to the particular rexpansion that was employed. Then, only soundness of the new rules for that rexpansion is needed for conservativity. For example, since the extension of **CLuN** with (**c**) from section 6.4.1 is conservative over positive classical logic, we may start with the positive fragment of **LK** and focus our search on rules that include negation. And indeed, the additional rule $(\neg\neg \Rightarrow)$ includes negation, and also, it is sound for $\mathcal{M}_c$.

As another example, notice that the single connective that causes $\mathcal{M}_{BKcal}$ to be only a preserving rexpansion of $\mathcal{M}_{BKca}$ (and not a strongly preserving one) is $\wedge$ (see Section 6.4.2). Thus, by Corollary 6.2.9, the logic that is induced by $\mathcal{M}_{BKcal}$ is conservative over the $\wedge$-free fragment of $\mathcal{M}_{BKca}$. Therefore, a calculus for the former logic only requires new rules that manipulate $\wedge$. And indeed, $\mathbf{G}_{BKcal}$ is obtained from $\mathbf{G}_{BKca}$ by the addition of the rule $(\neg\wedge \Rightarrow)_{(\mathbf{l})}$, that involves $\wedge$. By Proposition 6.4.3, the calculus $\mathbf{G}_{BKcal}$ is conservative over the $\wedge$-free fragment of $\mathbf{G}_{BKca}$. Note again that while $\mathbf{G}_{BKcal}$ is sound and complete for $\mathcal{M}_{BKcal}$, only soundness of $(\neg\wedge \Rightarrow)_{(\mathbf{l})}$ is required to establish conservativity.

6.5 Negations for Gödel Logic

The goal of this section is to develop reasonable logics in the language $\mathcal{CL}_\perp$ that simultaneously have two properties that were discussed in Section 6.3: paraconsistency and fuzziness, in a way that would then enable the construction of corresponding analytic Gentzen-type systems. The main problem we face in achieving this goal is that standard fuzzy logics (like the two described in Example 6.1.3) are defined via matrices with a single designated value. However, it is well known [4] that a logic which is induced by such a matrix cannot be paraconsistent. Therefore, none of the standard fuzzy logics is paraconsistent. In order to develop logics that are both paraconsistent and fuzzy, it is necessary to replace the standard method of defining a fuzzy consequence relation by a weaker one. An additional step that can be made is to take $\neg$ as a primitive connective, and use new semantic interpretations for it. (In the standard fuzzy logics $\neg\psi$ is *defined* as $\psi \supset \perp$.)

The first attempt to achieve the goal of this section according to the above lines was made by [52]. Its main idea was to follow a recent approach [36] to defining fuzzy consequence relations, that instead of preserving absolute truth (i.e. the truth value 1), preserve *degrees of truth*. Given a matrix $\mathcal{M}$ (whose truth values are $[0,1]$) which induces the ordinary (i.e. truth-preserving) fuzzy logic $\mathbf{L}$, this means that a formula ψ follows from a set of formulas $\mathcal{T}$ if there is a finite subset $\{\varphi_1, \dots, \varphi_n\} \subseteq \mathcal{T}$ such that the truth value which is assigned to ψ by some $\mathcal{M}$-valuation v is always greater than or equal to the minimal truth value that v assigns to $\varphi_1, \dots, \varphi_n$. For the standard matrices used in fuzzy logics, the latter condition is equivalent to demanding the formula $(\varphi_1 \wedge \dots \wedge \varphi_n) \supset \psi$ to be valid in the corresponding truth-preserving logic **L** (see, e.g., [36]). This fact implies that $\mathbf{L}^{\leq}$, the degree-preserving logic induced by $\mathcal{M}$, has the same set of valid formulas as the truth-preserving logic **L** which is induced by $\mathcal{M}$. This makes $\mathbf{L}^{\leq}$ a natural variant of **L**.

A good example of the method of [52] is provided by Łukasiewicz logic $Ł_\infty$ (Example 6.1.3). The interpretation of $\neg$ there (where $\neg$ is taken as a defined connective, as explained above) is: $\mathcal{O}(\neg)(a) = 1 - a$. As said above, $Ł_\infty$ itself cannot be paraconsistent. However, its degree-preserving variant $Ł_\infty{}^{\leq}$ *is* paraconsistent, as can be seen by any valuation v such that $v(p) = v(\neg p) = 1/2$ and $v(q) = \frac{1}{4}$. $Ł_\infty{}^{\leq}$ is also subclassical (as it is contained in $Ł_\infty$, which is subclassical), and thus it is even strictly (though not boldly) paraconsistent (see Definition 6.3.2). Moreover, it validates some basic classical equivalences connected with negation, like De Morgan laws and the double negation laws. Unfortunately, $Ł_\infty^{\leq}$ has some serious drawbacks as well. The main one is that Modus Ponens (M.P.) for $\supset$ is not valid in it: ψ does not necessarily follow from φ and $\varphi \supset \psi$. (This is exemplified by any valuation v in which $v(\varphi) = 0.5$, $v(\psi) = 0.4$ and $v(\varphi \supset \psi) = 0.9$.) Thus $\supset$ cannot be regarded in $Ł_\infty^{\leq}$ as an implication connective of any sort.

Is there a standard fuzzy logic **L** such that M.P. for $\supset$ is valid in $\mathbf{L}^{\leq}$? Of the three basic fuzzy logics (Łukasiewicz logic, Gödel logic and product logic), only in Gödel logic $\supset$ has this property (see, e.g., [60]). Hence it seems better to try to use $\mathbf{G}^{\leq}$ instead of $Ł_\infty{}^{\leq}$. However, in its original language (of $\{\wedge, \vee, \supset, \bot\}$) $\mathbf{G}^{\leq}$ is identical to **G**. In particular, $\mathbf{G}^{\leq}$ is not paraconsistent with respect to the official negation of **G**. To obtain a paraconsistent variant of **G**, one should employ also the second idea mentioned above (and used in [52]): to augment the language with a new negation connective. A particularly appealing choice is to augment $\mathbf{G}^{\leq}$ with the involutive Łukasiewicz negation. Denote the resulting logic by $\mathbf{G}_{\neg}^{\leq}$. As a paraconsistent fuzzy logic, $\mathbf{G}_{\neg}^{\leq}$ has all the nice properties of $Ł_\infty^{\leq}$ that were mentioned above. On the other hand it does not have its main shortcoming, because $\supset$ is in it a true implication connective: $\varphi \supset \psi$ follows in $\mathbf{G}_{\neg}^{\leq}$ from $\mathcal{T}$ iff ψ follows in it from $\mathcal{T} \cup \{\varphi\}$. What is more: $\mathbf{G}_{\neg}^{\leq}$ is a *conservative* extension to a richer language of the basic fuzzy logic **G**. However, even $\mathbf{G}_{\neg}^{\leq}$ still has some serious drawbacks. Thus like $Ł_\infty{}^{\leq}$ it is not boldly paraconsistent.[10] Even more significant is the fact that (again like $Ł_\infty{}^{\leq}$) $\varphi \vee \neg\varphi$ is not valid in it. As explained in Section 6.4.1, this is very important, as since we are seeking here paraconsistency and thus giving up the law of contradiction, we should keep intact at least the other basic principle of classical negation: the law of excluded middle.

[10] In both logics $q \vee \neg q$ is not valid, but it follows from $\{p, \neg p\}$, as the minimum value assigned to $\{p, \neg p\}$ is at most $1/2$, while the value assigned to $q \vee \neg q$ is at least $1/2$.

In this section we use rexpansions of the Gödel matrix $\mathcal{M}_{\mathbf{G}}$ for constructing even better paraconsistent fuzzy logics which are based on **G**. Based on the ideas described in previous sections, the use of rexpansions paves the way not only for constructing such logics, but also for obtaining proof systems for them (see Section 6.5.4 below). Before describing our method, here is the list of properties that we would like a paraconsistent fuzzy logic **L** to have:

(i) **L** should be boldly paraconsistent;

(ii) **L** should be subclassical (and so, by (i), strictly paraconsistent);

(iii) **L** should be conservative over **G**;

(iv) $\supset$, $\wedge$, and $\vee$ should respectively be an implication, a conjunction, and a disjunction for **L**. This means that for every $\mathcal{T}$, φ, ψ, and ϕ we should have:

(iv).A $\mathcal{T} \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ iff $\mathcal{T} \vdash_{\mathbf{L}} \varphi \supset \psi$;

(iv).B $\mathcal{T} \vdash_{\mathbf{L}} \varphi \wedge \psi$ iff $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T} \vdash_{\mathbf{L}} \psi$;

(iv).C $\mathcal{T} \cup \{\varphi \vee \psi\} \vdash_{\mathbf{L}} \phi$ iff $\mathcal{T} \cup \{\varphi\} \vdash_{\mathbf{L}} \phi$ and $\mathcal{T} \cup \{\psi\} \vdash_{\mathbf{L}} \phi$;

(v) **L** should validate $\varphi \vee \neg\varphi$;

(vi) **L** should validate the basic classical equivalences concerning $\neg$, $\vee$, and $\wedge$: $\varphi \equiv \neg\neg\varphi$, $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$, and $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$;

(vii) **L** should validate the following connections between negation and implication:

(vii).A $\varphi \supset (\neg\psi \supset \neg(\varphi \supset \psi))$

(vii).B $\neg(\varphi \supset \psi) \supset \neg\psi$

(vii).C $(\varphi \supset \psi) \supset (\neg(\varphi \supset \psi) \supset \varphi)$

A word of explanation is needed for the last item in this list. Ideally, we would have liked to add to item (vi) of the list above also the classical equivalence that connects $\neg$ and $\supset$: $\neg(\varphi \supset \psi) \equiv (\varphi \wedge \neg\psi)$. This, in turn, is equivalent to the validity of (vii).A, (vii).B and

(vii).C' $\neg(\varphi \supset \psi) \supset \varphi$

Unfortunately, we cannot include (vii).C' in our list, since together with items (iv) and (v), it immediately entails the validity of $\varphi \vee (\varphi \supset \psi)$, contradicting item (iii) of our list. So instead of (vii).C' we include a weaker version, which is valid in $\mathbf{G}^{\leqq}_{\neg}$, as well as in all the standard fuzzy logics (in which $\neg\psi$ is taken as $\psi \supset \bot$).

6.5.1 The Nmatrix $\mathcal{M}_{\mathbf{G}_\neg}^{t}$ and Its Refinements

The method of rexpansions allows us to present a better approach to the construction of paraconsistent conservative extensions of Gödel logic, which stays within the framework of truth-preservation. This is achieved by relaxing the principle of truth-functionality, and the preservation of absolute truth. The former is done by basing our construction on Nmatrices, and the latter by replacing "completely true" with "true enough", that is, taking a larger set of designated truth values. Formally:

Definition 6.5.1. Let $0 < t \leq 1$. $\mathcal{M}_{\mathbf{G}}{}^{t}$ is the Nmatrix for $\{\wedge, \vee, \supset, \perp\}$ obtained from $\mathcal{M}_{\mathbf{G}}$ by:

1. Taking $[t, 1]$ as the designated values.

2. Changing $\mathcal{O}(\supset)$ to $\mathcal{O}(\supset)(a, b) = \begin{cases} [t, 1] & a \leq b \text{ or } b \geq t \\ \{b\} & a > b \text{ and } b < t \end{cases}$.

$\mathcal{M}_{\mathbf{G}_\neg}^{t}$ is the extension of $\mathcal{M}_{\mathbf{G}}{}^{t}$ to $\{\wedge, \vee, \supset, \perp, \neg\}$, in which $\mathcal{O}(\neg)(a) = 1 - a$.

The next theorem shows that $\mathcal{M}_{\mathbf{G}_\neg}^{t}$ provides a satisfactory basis for constructing paraconsistent fuzzy logics.

Theorem 6.5.2. Let $0 < t \leq 1$ and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a simple refinement of $\mathcal{M}_{\mathbf{G}_\neg}^{t}$. Then:

1. $\vdash_{\mathcal{M}}$ satisfies (iii), (iv).B, (iv).C, and (vi) (that is, it is conservative over $\mathbf{G}$, $\wedge$ is a conjunction, $\vee$ is a disjunction, and De Morgan and double negation laws are valid).[11]

2. If $1 \in \mathcal{O}(\supset)(x, y)$ whenever either $x = 0$ or $y = 1$ then $\vdash_{\mathcal{M}}$ satisfies (ii).

3. If $t > 1/2$ then $\vdash_{\mathcal{M}}$ satisfies neither of (i), (iv), and (v).

4. If $t \leq 1/2$ then $\vdash_{\mathcal{M}}$ satisfies (i) and (v).

Proof. Suppose $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ and $\mathcal{M}_{\mathbf{G}_\neg}^{t} = \langle \mathcal{V}^t, \mathcal{D}^t, \mathcal{O}^t \rangle$.

1. It is straight-forward to verify (iv).B, (iv).C and (vi). As for (iii), one verifies that $\mathcal{M}_{\mathbf{G}}{}^{t}$ is a simple F-expansion of $\mathcal{M}_{\mathbf{G}}$, for $F = \lambda x \in [0, 1] . \begin{cases} [t, 1] & x = 1 \\ \{t \cdot x\} & x < 1 \end{cases}$. By Corollary 6.2.9, every simple refinement of $\mathcal{M}_{\mathbf{G}_\neg}^{t}$ induces a logic that is conservative over $\mathbf{G}$.

[11] While the left-to-right direction of (iv).A may not hold, its right-to-left direction (namely M.P.) does hold.

2. Suppose that $\mathcal{T} \nvdash_{\mathbf{CL}} \varphi$. Then there exists a classical valuation v such that $v(\psi) = 1$ for every $\psi \in \mathcal{T}$ and $v(\varphi) = 0$. v is also an $\mathcal{M}$-valuation, and thus $\mathcal{T} \nvdash_{\mathcal{M}} \varphi$.

3. If $v \vDash^{\mathcal{M}} p$ and $v \vDash^{\mathcal{M}} \neg p$, we must have $v(p), 1 - v(p) \geq t$, which is impossible for $t > 1/2$. Therefore, $p, \neg p \vdash_{\mathcal{M}} q$ and (i) fails. Moreover, $\nvdash_{\mathcal{M}} (p \wedge \neg p) \supset q$ (by assigning $1/2$ to p and 0 to q), and thus also (iv) fails. Finally, $v \nvDash^{\mathcal{M}} p \vee \neg p$ for $v(p) = v(\neg p) = v(p \vee \neg p) = 1/2$.

4. We start with (i): First we show that $\neg$ is a weak negation in $\vdash_{\mathcal{M}}$. Since $0 < t \leq 1/2$ there exists $a < t$ such that $1 - a \geq t$. Any $\mathcal{M}$-valuation v in which $v(p) = 1 - a$ satisfies p but not $\neg p$, and any $\mathcal{M}$-valuation v in which $v(p) = a$ satisfies $\neg p$ but not p. Thus $p \nvdash_{\mathcal{M}} \neg p$ and $\neg p \nvdash_{\mathcal{M}} p$. Second, in any $\mathcal{M}$-valuation v in which $v(p) = v(\neg p) = 1/2$ and $v(q) = 0$, we have $v \vDash^{\mathcal{M}} \{p, \neg p\}$ and $v \nvDash^{\mathcal{M}} q$. Therefore $p, \neg p \nvdash_{\mathcal{M}} q$. Next, we show that $\vdash_{\mathcal{M}}$ is boldly paraconsistent. Suppose $\nvdash_{\mathcal{M}} \varphi$ and $p \notin At(\varphi)$. Then there exists an $\mathcal{M}$-valuation v such that $v(\varphi) < t$. Define a function v' as follows: $v'(\psi) = v(\psi)$ for every subformula ψ of φ (including φ itself), and $v'(p) = v'(\neg p) = 1/2$. Now extend v' to an $\mathcal{M}$-valuation, and obtain that $p, \neg p \nvdash_{\mathcal{M}} \varphi$. As for (v), for every $\mathcal{M}$-valuation v, if $v(\varphi) < t$ then $v(\neg\varphi) > t$ and vice verse, and hence $\vdash_{\mathcal{M}} \neg\varphi \vee \varphi$. □

The proof of Theorem 6.5.2 actually provides another interesting result regarding the Gödel matrix: the same logic would be obtained if the designated values were taken to be any interval of the form $[t, 1]$ for any $0 < t < 1$.

Proposition 6.5.3. Let $0 < t < 1$ and $\mathcal{M}^t = \langle \mathcal{V}^t, \mathcal{D}^t, \mathcal{O}^t \rangle$, where $\mathcal{V}^t = [0, 1]$, $\mathcal{D}^t = [t, 1]$, and $\mathcal{O}^t = \mathcal{O}_{\mathbf{G}}$. Then $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathcal{M}^t}$.

Proof. $\mathcal{M}^t$ is a simple refinement of ${\mathcal{M}_{\mathbf{G}}}^t$, which is a simple rexpansion of $\mathcal{M}_{\mathbf{G}}$. By Corollary 6.2.8, we have $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathcal{M}^t}$. □

Other negations can be considered for **G**, and rexpansions can be used in order to prove that the result is conservative over **G**. This is obtained in a similar manner to Proposition 6.4.3.

Lemma 6.5.4. Let A be a set of axioms in $\mathcal{CL}_\bot$. If A is valid in ${\mathcal{M}_{\mathbf{G}}}^t_\neg$ then $\mathbf{G}^A$, the axiomatic extension of **G** with A, is conservative over **G**.

Proof. Clearly, $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}^A}$. Now let $\mathcal{T} \subseteq \mathcal{CL}_\bot \setminus \{\neg\}$ and $\varphi \in \mathcal{CL}_\bot \setminus \{\neg\}$. If $\mathcal{T} \vdash_{\mathbf{G}^A} \varphi$, then $\mathcal{T} \vdash_{{\mathcal{M}_{\mathbf{G}}}^t_\neg} \varphi$. And since $\vdash_{{\mathcal{M}_{\mathbf{G}}}^t_\neg}$ is conservative over **G**, we must have $\mathcal{T} \vdash_{\mathbf{G}} \varphi$. □

Like before, finding a new semantics for the augmented logic is not required, as only *soundness* is needed for the proof.

Example 6.5.5. Let A be a set consisting of the axioms from property (vi) above. Then G^A is an axiomatic extension of $\mathbf{G}$ with a negation that satisfies the usual double negation and De Morgan laws, and is conservative over $\mathbf{G}$.

6.5.2 Two Particular Refinements of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$

Theorem 6.5.2 shows that simple refinements of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$ enjoy many desirable properties one would expect from a paraconsistent fuzzy logic. However, they may lack some of the properties mentioned above. In particular, the formulas in (vii) are not valid in the logic that is induced by $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$ itself (for example, if $v(\varphi) = 0.7$, $v(\psi) = 0.8$, and $v(\varphi \supset \psi) = 0.7$, then v does not satisfy (vii).B). Moreover, (iv) does not hold in the simple refinement $\mathcal{M}^{0.7-0.8}$ of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$, obtained by setting $\mathcal{O}(\supset)(a,b) = 0.7$ whenever $b \geq 1/2$ or $a = b$, and $\mathcal{O}(\supset)(a,b) = 0.8$ whenever $b < 1/2$ and $a < b$. Indeed, $v(\neg(\varphi \supset \varphi)) = 0.3 < 1/2$ for every $\mathcal{M}^{0.7-0.8}$-valuation v, which means that $\neg(\varphi \supset \varphi) \vdash_{\mathcal{M}^{0.7-0.8}} \psi$. However, the $\mathcal{M}^{0.7-0.8}$-valuation u in which $u(\varphi) = u(\psi) = 0$ and $u(\varphi \supset \varphi) = 0.7$, shows that $\nvdash_{\mathcal{M}^{0.7-0.8}} \neg(\varphi \supset \varphi) \supset \psi$. Property (ii) also does not hold in $\vdash_{\mathcal{M}^{0.7-0.8}}$, as q follows from $\neg(p \supset p) \supset \neg(\bot \supset \neg(p \supset p))$ in it, but not in classical logic.

We present two particularly interesting simple refinements of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$. The first is obtained by refining the interpretation of $\supset$ back to its original interpretation in $\mathcal{M}_{\mathbf{G}}$. The second is a reconstruction of a well-known semi-relevant logic [2, 49], in which all properties (i)–(vii) hold.

Closest to The Original: $det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})$

If we refine the interpretation of $\supset$ in $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$ to its original interpretation in $\mathcal{M}_{\mathbf{G}}$, we obtain a *matrix* for a paraconsistent fuzzy logic (denoted $det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})$), whose $\{\wedge, \vee, \supset, \bot\}$-fragment differs from $\mathcal{M}_{\mathbf{G}}$ solely in the choice of designated values. This seems as close as one can get to adding a paraconsistent involutive negation to $\mathcal{M}_{\mathbf{G}}$. Moreover, $\vdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})}$ strictly extends $\vdash_{\mathcal{M}_{\mathbf{G}_\neg}^{1/2}}$, and satisfies all properties listed above, except for (vii).A. The only property whose verification is not routine is (iv).A, which we now prove.

Proposition 6.5.6 (Deduction Theorem for $\vdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})}$)**.** $\mathcal{T} \vdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_1 \supset \varphi_2$ iff $\mathcal{T}, \varphi_1 \vdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_2$.

Proof. The fact that $\mathcal{T} \vdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_1 \supset \varphi_2$ implies $\mathcal{T}, \varphi_1 \vdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_2$ is easily verified using the interpretation of $\supset$. For the converse, suppose $\mathcal{T} \nvdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_1 \supset \varphi_2$. We prove that $\mathcal{T}, \varphi_1 \nvdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_2$. By our assumption, there exists a $det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})$-valuation v such that $v(\psi) \geq 1/2$ for every $\psi \in \mathcal{T}$, and $v(\varphi_1 \supset \varphi_2) < 1/2$. Let $r_1 = v(\varphi_1)$ and $r_2 = v(\varphi_2)$. Then: $v(\varphi_1 \supset \varphi_2) = r_2 < 1/2$, $1 - r_2 > 1/2$, and $r_1 > r_2$. If $r_1 \geq 1/2$ then $v \models^{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_1$ and $v \not\models^{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_2$, and thus $\mathcal{T}, \varphi_1 \nvdash_{det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})} \varphi_2$. Hence we assume in addition that $r_1 < 1/2$. We construct an appropriate countermodel by "fixing" v so that it satisfies $\mathcal{T}$ and φ_1, but still does not satisfy φ_2. This is done by replacing r_1 by $1/2$, and then making other necessary adjustments to keep the resulting valuation a $det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})$-valuation. Let

$$f = \lambda x \in [0,1] . \begin{cases} \frac{1}{2r_1}x & x < r_1 \\ 1/2 & r_1 \leq x \leq 1 - r_1 \\ \frac{x-1+2r_1}{2r_1} & x > 1 - r_1 \end{cases}$$

and let $v'(\varphi) = f(v(\varphi))$ for every φ. f is clearly an increasing function from $[0,1]$ to $[0,1]$. Now, $v'(\psi) \geq 1/2$ for every $\psi \in \mathcal{T}$, as $v(\psi) \geq 1/2 > r_1$ for every such ψ. Also, $v'(\varphi_1) = 1/2$, and $v'(\varphi_2) < 1/2$, as $v(\varphi_2) = r_2 < r_1$. It is left to prove that v' is a $det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})$-valuation. Suppose $det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2}) = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$.

1. $v'(\varphi \wedge \psi) = f(v(\varphi \wedge \psi)) = f(\min\{v(\varphi), v(\psi)\}) = \min\{v'(\varphi), v'(\psi)\}$, as f is increasing.

2. Disjunction is shown similarly.

3. If $v'(\varphi) \leq v'(\psi)$ then $v(\varphi) \leq v(\psi)$, and then $v'(\varphi \supset \psi) = f(v(\varphi \supset \psi)) = f(1) = 1$. $v'(\varphi \supset \psi) = f(v(\varphi \supset \psi)) = f(v(\psi)) = v'(\psi)$.

4. We show that $v'(\neg\varphi) = 1 - v'(\varphi)$. If $v(\varphi) < r_1$, then $v(\neg\varphi) > 1 - r_1$. In such a case, $v'(\varphi) = \frac{1}{2r_1}v(\varphi)$ and $v'(\neg\varphi) = \frac{v(\neg\varphi)-1+2r_1}{2r_1} = \frac{2r_1 - v(\varphi)}{2r_1}$. In particular, $v'(\varphi) + v'(\neg\varphi) = \frac{v(\varphi)+2r_1-v(\varphi)}{2r_1} = 1$. If $r_1 \leq v(\varphi) \leq 1 - r_1$, then $v'(\varphi) = v'(\neg\varphi) = 1/2$. And if $v(\varphi) > 1 - r_1$, then this case is symmetric to the first case.

5. $v'(\bot) = f(v(\bot)) = f(0) = 0$. □

A Semi-relevant Refinement: $\mathcal{M}_{\mathbf{RM}^\supset}$

The matrix for the logic $\mathbf{RM}^\supset$ [8], that we denote by $\mathcal{M}_{\mathbf{RM}^\supset}$ is a simple refinement of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$ in which implication is interpreted by: $\mathcal{O}(\supset)(a,b) = \begin{cases} \{1-a\} & a \leq b \leq 1-a \\ \{b\} & otherwise \end{cases}$.

$\mathbf{RM}^\supset$ is shown in [8] to be equivalent to the famous Dunn-Meyer semi-relevant logic **RM** [2]. $\mathbf{RM}^\supset$ satisfies all of the properties listed above. (All properties but (iv).A and (vii) follow from Theorem 6.5.2. (iv).A and (vii) were proved in [8].) In particular, it strictly extends $\vdash_{\mathcal{M}_{\mathbf{G}_\neg}^{1/2}}$.[12] Moreover, we show that $\mathbf{RM}^{\supset}$ is unique with respect to the properties above:

Proposition 6.5.7. $\mathbf{RM}^{\supset}$ is the only finitary[13] logic that satisfies all properties (i)–(vii) above.

Proof. Let $\mathbf{L}$ be such a logic. Denote by $\mathcal{H}$ the Hilbert-type calculus for $\mathbf{G}$ from [48], and by $\mathcal{H}_{\mathbf{RM}^{\supset}}$ the Hilbert-type calculus obtained from $\mathcal{H}$ by the addition of the axiom schemes of (v), (vi) and (vii). Note that the only rule of inference in $\mathcal{H}$ and $\mathcal{H}_{\mathbf{RM}^{\supset}}$ is M.P. It was shown in [8] that the set of theorems of $\mathcal{H}_{\mathbf{RM}^{\supset}}$ is the same as the set of formulas that are valid in $\mathbf{RM}^{\supset}$. Since $\mathbf{RM}^{\supset}$ is finitary[14] and admits the deduction theorem, it follows that $\mathcal{H}_{\mathbf{RM}^{\supset}}$ is sound and complete for $\mathbf{RM}^{\supset}$, that is, φ is derivable from $\mathcal{T}$ in $\mathcal{H}_{\mathbf{RM}^{\supset}}$ iff $\mathcal{T} \vdash_{\mathbf{RM}^{\supset}} \varphi$. Now, to satisfy (iii), all axiom schemes from $\mathcal{H}$ must be valid in $\mathbf{L}$, as otherwise, it would not be conservative over $\mathbf{G}$. To satisfy (v), (vi) and (vii), the axioms they include must be valid in $\mathbf{L}$ also. For (iv), M.P. must be valid in $\mathbf{L}$. Thus $\mathcal{H}_{\mathbf{RM}^{\supset}}$ is sound for $\mathbf{L}$, and in particular, $\mathbf{RM}^{\supset}$ is contained in $\mathbf{L}$. Now, if $\mathbf{L}$ strictly contains $\mathbf{RM}^{\supset}$, then since M.P. is valid in $\mathbf{RM}^{\supset}$ and $\mathbf{L}$ is both finitary and admits the deduction theorem (by (iv)), there exists a formula φ that is valid in $\mathbf{L}$ but not in $\mathbf{RM}^{\supset}$. It is then a corollary of [8] that $\mathbf{L}$ has a finite characteristic matrix, and in particular, so does its $\neg$-free fragment. Since $\mathbf{G}$ cannot be finitely characterized by a matrix [48], $\mathbf{L}$ is not a conservative extension of $\mathbf{G}$, and thus (iii) fails. Therefore, we must have that $\mathbf{L}$ and $\mathbf{RM}^{\supset}$ are identical. □

Table 6.1 summarizes the various logics and properties discussed in this section, and in particular, specifies the properties that hold in each logic. In the table, "t" means that the property holds, and "f" means that it does not. The column in the middle (titled "Simple Refinements of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$") includes some cells with the symbol "$\top$". For the corresponding properties, the meaning is that some simple refinements of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$ satisfy them, and some do not. In contrast, property (iii) is not relevant for $\text{Ł}_\infty^{\leq}$, as it does not include any new connective. Thus it is marked with "$\perp$".

[12] The axiomatic extension of $\mathbf{RM}^{\supset}$ with (vii).C', that we did not include in our list of requirements, is also considered in [8], and is proven to be equivalent to the 3-valued logic **PAC** (see Section 6.4.1), that is also known as $\mathbf{RM}_3$.

[13] A logic $\mathbf{L}$ is called finitary if the compactness theorem holds for it, that is: $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ iff $\Gamma \vdash_{\mathbf{L}} \varphi$ for some finite $\Gamma \subseteq \mathcal{T}$.

[14] This follows from the equivalence between $\mathbf{RM}^{\supset}$ and $\mathbf{RM}$ shown in [8], together with the fact that $\mathbf{RM}$ itself is finitary. The latter follows from the (strong) soundness and completeness theorem that was proven for $\mathbf{RM}$ in [18].

Property	$Ł_\infty{}^{\leq}$	$\mathbf{G}^{\leq}$	Simple Refinements of $\mathcal{M}_{\mathbf{G}_\neg}^{1/2}$	$det(\mathcal{M}_{\mathbf{G}_\neg}^{1/2})$	$\mathbf{RM}^{\supset}$
(i)	f	f	t	t	t
(ii)	t	t	$\top$	t	t
(iii)	$\perp$	t	t	t	t
(iv)	f	t	$\top$	t	t
(v)	f	f	t	t	t
(vi)	t	t	t	t	t
(vii)	t	f	$\top$	f	t

Figure 6.1: Summary of properties for paraconsistent fuzzy logics

6.5.3 What is the Cardinality of $\left\{\vdash_{\mathcal{M}_{\mathbf{G}_\neg}^t} \mid 0 < t \leq 1\right\}$?

In this section we investigate the relation between the different logics that are induced by the Nmatrices $\mathcal{M}_{\mathbf{G}_\neg}^t$ (see Definition 6.5.1) themselves. These logics are *minimal* in the family of logics that are studied in Theorem 6.5.2, as different refinements of them may induce different extensions. The main result of this section can be summarized as follows:

1. All the Nmatrices $\mathcal{M}_{\mathbf{G}_\neg}^t$ for $0 < t \leq 1/2$ induce the same logic;
2. There are exactly two logics that are induced for $t > 1/2$.
3. All together, the answer to the above question is: three.[15]

The rest of this section is devoted to the proof of this result. We start by introducing the notion of $\langle t_1, t_2\rangle$*-expanding functions* in Definition 6.5.8, and prove that they characterize all strongly preserving rexpansions between elements of $\left\{\mathcal{M}_{\mathbf{G}_\neg}^t \mid 0 < t \leq 1\right\}$ in Lemma 6.5.9. This fact is then used in Lemma 6.5.10, where the logics that are induced by these Nmatrices are identified, thus obtaining the aforementioned result in Corollary 6.5.11.

Definition 6.5.8. Let $0 < t_1 < t_2 \leq 1$. An expansion function $F : [0,1] \to P([0,1])$ is called $\langle t_1, t_2\rangle$*-expanding* if:

(1) $\bigcup_{x\in[0,1]} F(x) = [0,1]$ and $\bigcup_{x\in[t_2,1]} F(x) = [t_1, 1]$.

[15]We stress that the logics that are considered here are those that are induced by the Nmatrices $\mathcal{M}_{\mathbf{G}_\neg}^t$ themselves, not their refinements.

(2) F is increasing: if $x < y$ then $x' < y'$ for every $x' \in F(x)$ and $y' \in F(y)$.

(3) $F(1-x) = \{1-y \mid y \in F(x)\}$ for every $x \in [0,1]$ (that is, $y \in F(x)$ iff $1-y \in F(1-x)$).

(4) $F(x)$ is a singleton whenever $x < t_2$ (and so because of (3), also when $x > 1-t_2$).

Lemma 6.5.9. Let $F : [0,1] \to P([0,1])$ and $0 < t_1 < t_2 \leq 1$. Then the following statements are equivalent:

1. F is $\langle t_1, t_2\rangle$-expanding.
2. $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$.
3. $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is a strongly preserving F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$.

Proof. Suppose $\mathcal{M}_{\mathbf{G}_\neg}^{t_1} = \langle [0,1], [t_1,1], \mathcal{O}_1\rangle$, $\mathcal{M}_{\mathbf{G}_\neg}^{t_2} = \langle [0,1], [t_2,1], \mathcal{O}_2\rangle$ and $(\mathcal{M}_{\mathbf{G}_\neg}^{t_2})_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F\rangle$.

$(\mathbf{1} \Rightarrow \mathbf{3})$: Let us calculate $(\mathcal{M}_{\mathbf{G}_\neg}^{t_2})_F$: $\mathcal{V}_F = [0,1]$ and $\mathcal{D}_F = \mathcal{D}_1 = [t_1,1]$, because of property (1). As for $\mathcal{O}_F$: Using property (2), we have that $\mathcal{O}_F(\wedge)(x,y) = F(\min\left\{\widetilde{F}[x], \widetilde{F}[y]\right\})$ and $\mathcal{O}_F(\vee)(x,y) = F(\max\left\{\widetilde{F}[x], \widetilde{F}[y]\right\})$. By property (3), we have $\mathcal{O}_F(\neg)(x) = F(1-\widetilde{F}[x])$. Combining properties (2) and (4) gives us $\mathcal{O}_F(\supset)(x,y) = \begin{cases} [t_1,1] & x \leq y \text{ or } y \geq t_1 \\ \{y\} & x > y \text{ and } y < t_1 \end{cases}$.
Finally, $\mathcal{O}_F(\bot) = F(0)$.

We show that $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is a (simple) refinement of $(\mathcal{M}_{\mathbf{G}_\neg}^{t_2})_F$:

1. $\mathcal{O}_1(\wedge)(x,y) \subseteq \mathcal{O}_F(\wedge)(x,y)$: Assume w.l.g. that $x \leq y$. Since F is increasing, $\widetilde{F}[x] \leq \widetilde{F}[y]$, which means that $\mathcal{O}_1(\wedge)(x,y) = \{x\} \subseteq F(\widetilde{F}[x]) = F(\min\left\{\widetilde{F}[x], \widetilde{F}[y]\right\}) = \mathcal{O}_F(\wedge)(x,y)$.
2. $\mathcal{O}_1(\vee)(x,y) \subseteq \mathcal{O}_F(\vee)(x,y)$: this is shown similarly.
3. $\mathcal{O}_1(\neg)(x) \subseteq \mathcal{O}_F(\neg)(x)$: using property (3), we have that $\mathcal{O}_1(\neg)(x) = \{1-x\} \subseteq \left\{1-y \mid y \in F(\widetilde{F}[x])\right\} = F(1-\widetilde{F}[x]) = \mathcal{O}_F(\neg)(x)$.
4. $\mathcal{O}_1(\supset)(x,y) \subseteq \mathcal{O}_F(\supset)(x,y)$: If $x \leq y$ or $y \in [t_1,1]$ then $\mathcal{O}_1(\supset)(x,y) = [t_1,1] = \mathcal{O}_F(\supset)(x,y)$. Otherwise, $\mathcal{O}_1(\supset)(x,y) = \{y\}$, which conforms with the calculation of $O_F(\supset)$ above.

5. $\mathcal{O}_1(\bot) \subseteq \mathcal{O}_F(\bot)$: We show that $0 \in F(0)$ $(= \mathcal{O}_F(\bot))$. Since $Im(F) = [0,1]$, $0 \in F(x)$ for some x. Assume for contradiction that $x > 0$. Since F is an expansion function, there exists some $y \in F(0)$. By property (2), $0 > y$, which is a contradiction.

Next, we prove that $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is a strongly preserving F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$. Clearly, it is preserving (as it is simple). The interpretations of all the connectives in $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ are deterministic, with the exception of $\supset$. Therefore, the only thing that needs to be verified is that $F(z) \cap \mathcal{O}_1(\supset)(x,y) \neq \emptyset$ whenever $z \in \mathcal{O}_2(\supset)(\widetilde{F}[x], \widetilde{F}[y])$ and either $x \leq y$ or $y \in [t_1, 1]$. Let $z \in \mathcal{O}_2(\supset)(\widetilde{F}[x], \widetilde{F}[y])$. Since $x \leq y$ or $y \in [t_1, 1]$, we have $\widetilde{F}[x] \leq \widetilde{F}[y]$ or $\widetilde{F}[y] \in [t_2, 1]$. Therefore, $z \in [t_2, 1]$, and so $F(z) \subseteq [t_1, 1]$. Since in this case, $\mathcal{O}_1(\supset)(x,y) = [t_1, 1]$, we have $F(z) \cap \mathcal{O}_1(\supset)(x,y) = F(z) \neq \emptyset$.

$(\mathbf{3} \Rightarrow \mathbf{2})$: Clearly, every strongly preserving F-rexpansion is an F-rexpansion.
$(\mathbf{2} \Rightarrow \mathbf{1})$: Suppose $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$. We prove that F is $\langle t_1, t_2 \rangle$-expanding, by verifying the four properties:

1. The correctness of property (1) is trivial.

2. If F is not increasing, then there exist $x, x', y, y' \in [0,1]$ such that $x < y$, $x' \in F(x)$, $y' \in F(y)$ and $x' \geq y'$. Since F is an expansion function, $x' \neq y'$, thus $x' > y'$. Now, since $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$, $y' \in \mathcal{O}_1(\wedge)(x', y') \subseteq \mathcal{O}_F(\wedge)(x', y') = F(x)$. This is impossible, as $x \neq y$, $y' \in F(y)$, and F is a rexpansion function.

3. Let $x \in [0,1]$. We prove that $F(1-x) = \{1 - y \mid y \in F(x)\}$. For every $z \in F(1-x)$, since $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$, $1 - z \in \mathcal{O}_1(\neg)(z) \subseteq \bigcup_{z' \in \mathcal{O}_2(\neg)(1-x)} F(z') = F(x)$, and therefore $z \in \{1 - y \mid y \in F(x)\}$. And for every $z \in \{1 - y \mid y \in F(x)\}$, $1 - z \in F(x)$, and therefore $z \in \mathcal{O}_1(\neg)(1-z) \subseteq \bigcup_{z' \in \mathcal{O}_2(\neg)(x)} F(z') = F(1-x)$.

4. If $F(x)$ is not a singleton for some $x < t_2$, then let $y_1, y_2 \in F(x)$ such that $y_1 < y_2$. In particular, $y_1, y_2 < t_1$. Therefore, since $\mathcal{M}_{\mathbf{G}_\neg}^{t_1}$ is an F-rexpansion of $\mathcal{M}_{\mathbf{G}_\neg}^{t_2}$, $y_1 \in \mathcal{O}_1(\supset)(y_2, y_1) \subseteq \bigcup_{z \in \mathcal{O}_2(\supset)(x,x)} F(z) = [t_1, 1]$, which is a contradiction. □

Now we apply Lemma 6.5.9 and Theorem 6.2.7 to the matrices $\mathcal{M}_{\mathbf{G}_\neg}^{t}$ for various values of t.

Lemma 6.5.10.

1. $\vdash_{\mathcal{M}_{\mathbf{G}_\neg}^{t}} = \vdash_{\mathcal{M}_{\mathbf{G}_\neg}^{1/2}}$ for every $0 < t < 1/2$.

2. $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t_1}_{\neg}} = \vdash_{\mathcal{M}_{\mathbf{G}}{}^{t_2}_{\neg}}$ for every $1/2 < t_1 < t_2 < 1$.

3. $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \subsetneq \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}}$ for every $1/2 < t < 1$.

4. $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{1/2}_{\neg}} \not\subseteq \vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}}$ and $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \not\subseteq \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1/2}_{\neg}}$ for every $1/2 < t \leq 1$.

Proof.

1. We construct an expansion function F that maps $[0, 1/2)$ to $[0, t)$ and $(1/2, 1]$ to $(1-t, 1]$. The remaining value $1/2$ is duplicated to the remaining segment $[t, 1-t]$. Namely:

$$F = \lambda x \in [0,1] . \begin{cases} \{2tx\} & x < 1/2 \\ [t, 1-t] & x = 1/2 \\ \{2tx + 1 - 2t\} & x > 1/2 \end{cases}$$

   $F : [0,1] \to P([0,1])$ since $t < 1/2$. By Lemma 6.5.9 and Theorem 6.2.7, it suffices to prove that F is $\langle t, 1/2 \rangle$-expanding. F is clearly an increasing expansion function with $\bigcup Im(F) = [0,1]$ and $\bigcup_{x \in [1/2, 1]} F(x) = [t, 1]$. To see that property (3) is satisfied, we distinguish three cases: If $x < 1/2$, $1 - x > 1/2$, and then $F(1-x) = \{1 - 2tx\} = \{1 - y \mid y \in F(x)\}$. If $x = 1/2$, $1 - x = 1/2$ and then $F(1-x) = [t, 1-t]$. Note that for every y, $y \in [t, 1-t]$ iff $1 - y \in [t, 1-t]$. Hence in this case, $F(1-x) = \{1 - y \mid y \in [t, 1-t]\} = \{1 - y \mid y \in F(x)\}$. If $x > 1/2$, $1 - x < 1/2$, and then $F(1-x) = \{2t(1-x)\} = \{1 - y \mid y \in \{2tx + 1 - 2t\}\} = \{1 - y \mid y \in F(x)\}$. Finally, property (4) clearly holds, as $F(x)$ is a singleton whenever $x < 1/2$.

2. We construct a rexpansion function that maps $[t_2, 1]$ to $[t_1, 1]$, $[0, 1-t_2)$ to $[0, 1-t_1)$, and $[1-t_2, t_2)$ to $[1-t_1, t_1)$. Consider the following function:

$$f = \lambda x \in [0,1] . \begin{cases} \frac{1-t_1}{1-t_2} \cdot x & x < 1 - t_2 \\ \frac{2t_1 - 1}{2t_2 - 1} \cdot x + \frac{t_2 - t_1}{2t_2 - 1} & 1 - t_2 \leq x < t_2 \\ \frac{1-t_1}{1-t_2} \cdot x + \frac{t_1 - t_2}{1 - t_2} & x \geq t_2 \end{cases}$$

   Let $F = \lambda x \in [0,1] . \{f(x)\}$. By Lemma 6.5.9 and Theorem 6.2.7, it suffices to show that F is $\langle t_1, t_2 \rangle$-expanding. F is clearly an increasing expansion function, $\bigcup Im(F) = [0,1]$, and $F(x)$ is always a singleton. In addition, $\bigcup_{x \in [t_2, 1]} F(x) = f([t_2, 1]) = [t_1, 1]$. Finally, $F(1-x) = \{1 - y \mid y \in F(x)\}$, as $f(1-x) = 1 - f(x)$ for every $x \in [0,1]$.

3. To show that $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \subseteq \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}}$, we prove that for every $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation v there exists

a $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$-valuation v' such that for every formula φ, $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \varphi$ iff $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \varphi$.[16] Let v be a $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation. We construct v' by mapping the values that are strictly below 1 to being strictly below t, in a way that conforms with the interpretation of $\neg$. By making this mapping an increasing one, we conform with the interpretation of the other connectives. This is defined as follows: Let f be defined by:

$$f = \lambda x \in [0,1] \,.\begin{cases} 0 & x = 0 \\ (2t-1)x + 1 - t & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$

$f : [0,1] \to [0,1]$ is strongly increasing and $f(1-x) = 1 - f(x)$ for every $x \in [0,1]$.

Define $v'(\psi) = f(v(\psi))$ for every ψ. First, we prove that v' is a $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$-valuation: For $\wedge$, we have $v'(\varphi \wedge \psi) = f(v(\varphi \wedge \psi)) = f(\min\{v(\varphi), v(\psi)\}) = \min\{v'(\varphi), v'(\psi)\}$, as f is increasing. $\vee$ is shown similarly. In addition, $v'(\neg\psi) = f(v(\neg\psi)) = f(1 - v(\psi)) = 1 - f(v(\psi)) = 1 - v'(\psi)$ and $v'(\bot) = f(v(\bot)) = f(0) = 0$. Next, we show that the implication constraints are satisfied: If $v'(\varphi) \leq v'(\psi)$, then since f is increasing, $v(\varphi) \leq v(\psi)$. Since v is a $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation, $v(\varphi \supset \psi) = 1$, and hence $v'(\varphi \supset \psi) = 1 > t$. If $v'(\psi) \geq t$, then by the definition of f, $v'(\psi) = 1$, which means that $v(\psi) = 1$, and again, $v'(\varphi \supset \psi) = 1 > t$. Finally, if $v'(\varphi) > v'(\psi)$ and $v'(\psi) < t$ then we have $v(\varphi) > v(\psi)$ and $v(\psi) < 1$. Since v is a $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation, $v(\varphi \supset \psi) = v(\psi) < 1$, and hence $v'(\varphi \supset \psi) = f(v(\psi)) = v'(\psi)$. Second, we prove that $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \psi$ iff $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \psi$, for every formula ψ. If $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \psi$ then $v(\psi) = 1$. In this case, $v'(\psi) = 1$ as well, and in particular, $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \psi$. In addition, if $v' \vDash^{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \psi$, then $v'(\psi) \geq t$, that is, $f(v(\psi)) \geq t$. By f's definition, we must have $v'(\psi) = f(v(\psi)) = 1$, which means that $v(\psi) = 1$. Therefore, $v \vDash^{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \psi$.

To show that $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \neq \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}}$, note that $p \supset q \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}} \neg q \supset \neg p$, as for every $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$-valuation v such that $v(p \supset q) = 1$, we must have that $v(p) \leq v(q)$. In particular, $v(\neg q) \leq v(\neg p)$, and thus $v(\neg q \supset \neg p) = 1$. However, $p \supset q \nvdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \neg q \supset \neg p$, as can be seen by the following $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$-valuation: $v(p) = 1, v(q) = t, v(\neg p) = 0, v(\neg q) = 1 - t, v(p \supset q) = t, v(\neg q \supset \neg p) = 0$.

4. By Theorem 6.5.2, we have that $p, \neg p \vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} q$ but $p, \neg p \nvdash_{\mathcal{M}_{\mathbf{G}}{}^{1/2}_{\neg}} q$, and $\vdash_{\mathcal{M}_{\mathbf{G}}{}^{1/2}_{\neg}} \neg\varphi \vee \varphi$ but $\nvdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \neg\varphi \vee \varphi$. □

[16]The proof of this item does not use rexpansions: by Proposition 6.2.4, $\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}$ is not a rexpansion of $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$, as there is no function $f : [0,1] \to [0,1]$ satisfying $x \in \{1\}$ iff $f(x) \in [t, 1]$.

Corollary 6.5.11. $\{\vdash_{\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}} \mid 0 < t \leq 1\} = \left\{\vdash_{\mathcal{M}_{\mathbf{G}}{}^{1/2}_{\neg}}, \vdash_{\mathcal{M}_{\mathbf{G}}{}^{\frac{3}{4}}_{\neg}}, \vdash_{\mathcal{M}_{\mathbf{G}}{}^{1}_{\neg}}\right\}$, and its cardinality is 3.

6.5.4 On The Construction of Corresponding Proof Systems

We conclude by describing the current and potential development of analytic proof systems for the logics investigated in this section. We shall not repeat all general arguments laid down in previous sections regarding the usage of rexpansions for the construction of proof systems, but focus instead on an example that illustrates these principles in the context of paraconsistent fuzzy logics.

Since all of the logics that were investigated here are conservative over **G**, our starting point should be a proof system for it. There are several Gentzen-type systems for this logic (see, e.g., [7, 10, 19, 43, 50, 51, 94]), and each of them can be used as a starting point. As an example, we employ here the system from [10], which is the most important of this list, and was already used for extending Gödel logic with quantifiers and modalities (see, e.g. [31]). A (additive) variant of the system from [10], which we call $\mathbf{G_{LC}}$ is presented in Figure 6.2. $\mathbf{G_{LC}}$ goes beyond ordinary sequent calculi, and employs a more complex data structure: *hypersequents*. In our variant of $\mathbf{G_{LC}}$, hypersequents are taken to be finite sets of sequents. For a hypersequent H and a sequent s, we write $H \mid s$ to denote the hypersequent $H \cup \{s\}$. The hypersequents in $\mathbf{G_{LC}}$ are *single-conclusion*, in the sense that for every hypersequent in a derivation of $\mathbf{G_{LC}}$, all sequents comprising it are single-conclusion. The structural rules of $\mathbf{G_{LC}}$ are obtained from obvious variants of the usual structural rules of ordinary sequent calculi, by the addition of the communication rule (com) and external weakening (ew), that operate on the hypersequent level. All the logical rules of $\mathbf{G_{LC}}$ except for ($\Rightarrow \vee$) are obtained from those of **LK** (with the same names) by:

(1) Restricting them to single-conclusion sequents

(2) Adding a context hypersequent that is copied from the premises to the conclusion.

Equivalently, they are obtained from the logical rules of the propositional fragment of Gentzen's calculus LJ for intuitionistic logic by (2) alone.

We are thus again in a situation where a semantics and a corresponding Gentzen-type system exist for some basic logic (in this case, Gödel logic), and we would like to extend this logic in several ways, thus obtaining a family of logics (in this case, logics induced by simple refinements of $\mathcal{M}_{\mathbf{G}}{}^{1/2}_{\neg}$), as well as corresponding proof systems.

Structural Rules:

$$(\text{ID}) \quad \frac{}{H \mid \Gamma, \varphi \Rightarrow \varphi} \qquad\qquad (\text{CUT}) \quad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi}$$

$$(\text{IW}) \quad \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma, \Gamma' \Rightarrow \varphi} \qquad\qquad (\text{EW}) \quad \frac{H}{H \mid \Gamma \Rightarrow \varphi}$$

$$(\text{COM}) \quad \frac{H \mid \Gamma, \Gamma' \Rightarrow \varphi \quad H \mid \Gamma, \Gamma' \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \mid \Gamma' \Rightarrow \psi}$$

Logical Rules:

$$(\wedge \Rightarrow) \quad \frac{H \mid \Gamma, \varphi, \psi \Rightarrow \phi}{H \mid \Gamma, \varphi \wedge \psi \Rightarrow \phi} \qquad\qquad (\Rightarrow \wedge) \quad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \wedge \psi}$$

$$(\vee \Rightarrow) \quad \frac{H \mid \Gamma, \varphi \Rightarrow \phi \quad H \mid \Gamma, \psi \Rightarrow \phi}{H \mid \Gamma, \varphi \vee \psi \Rightarrow \phi} \qquad\qquad (\Rightarrow \vee) \quad \frac{H \mid \Gamma \Rightarrow \varphi \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \vee \psi}$$

$$(\supset \Rightarrow) \quad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \psi \Rightarrow \phi}{H \mid \Gamma, \varphi \supset \psi \Rightarrow \phi} \qquad\qquad (\supset \Rightarrow) \quad \frac{H \mid \Gamma, \varphi \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \supset \psi}$$

$$(\bot \Rightarrow) \quad \frac{}{H \mid \Gamma, \bot \Rightarrow \phi}$$

Figure 6.2: The hypersequent calculus $\mathbf{G_{LC}}$

The first step, which is performing the right rexpansion for the base logic, was already done, and the various properties of the various refinements were identified in Theorem 6.5.2. What is left is to augment $\mathbf{G_{LC}}$ with appropriate rules for each logic.

All the arguments laid down in Section 6.4, and in particular Proposition 6.4.3, can be extended to other families of proof systems besides pure calculi, and in particular to hypersequent calculi. Accordingly, since every simple refinement of $\mathcal{M}_{\mathbf{G}_{\neg}^{1/2}}$ is conservative over $\mathbf{G}$, only rules that involve $\neg$ should be considered when constructing calculi for refinements of $\mathcal{M}_{\mathbf{G}_{\neg}^{1/2}}$.

As an example, we consider $\mathbf{RM}^{\supset}$, which is the most promising refinement (taking into consideration properties (i)–(vii) above). A calculus for this logic was recently constructed, which we call $\mathbf{G_{RM^{\supset}}}$.[17] As expected by the above arguments, $\mathbf{G_{RM^{\supset}}}$ is obtained from $\mathbf{G_{LC}}$ by the addition of rules that involve negation. These additional rules are described in Figure 6.3. There is a strong connection between $\mathbf{G_{RM^{\supset}}}$ and $\mathbf{G}_4$ (see Example 2.2.8), which is similar to the connection between $\mathbf{G_{LC}}$ and $\mathbf{LK}$: the rules $(\neg\neg \Rightarrow)$, $(\Rightarrow \neg\neg)$, $(\neg\wedge \Rightarrow)$, $(\neg\vee \Rightarrow)$ and $(\Rightarrow \neg\vee)$ of $\mathbf{G_{RM^{\supset}}}$ are obtained from the rules with the same names of $\mathbf{G}_4$ by making the changes (1) and (2) described above.

[17] $\mathbf{G_{RM^{\supset}}}$ was given by Arnon Avron in an unpublished manuscript. It is presented here with his permission.

$$(\Rightarrow \neg)\ \frac{}{H \mid \Gamma \Rightarrow \varphi \mid \Gamma \Rightarrow \neg\varphi}$$

$$(\neg\neg \Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow \psi}{H \mid \Gamma, \neg\neg\varphi \Rightarrow \psi} \qquad (\Rightarrow \neg\neg)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow \neg\neg\varphi}$$

$$(\neg\wedge \Rightarrow)\ \frac{H \mid \Gamma, \neg\varphi \Rightarrow \phi \quad H \mid \Gamma, \neg\psi \Rightarrow \phi}{H \mid \Gamma, \neg(\varphi \wedge \psi) \Rightarrow \phi} \qquad (\Rightarrow \neg\wedge)\ \frac{H \mid \Gamma \Rightarrow \neg\varphi \mid \Gamma \Rightarrow \neg\psi}{H \mid \Gamma \Rightarrow \neg(\varphi \wedge \psi)}$$

$$(\neg\vee \Rightarrow)\ \frac{H \mid \Gamma, \neg\varphi, \neg\psi \Rightarrow \phi}{H \mid \Gamma, \neg(\varphi \vee \psi) \Rightarrow \phi} \qquad (\Rightarrow \neg\vee)\ \frac{H \mid \Gamma \Rightarrow \neg\varphi \quad H \mid \Gamma \Rightarrow \neg\psi}{H \mid \Gamma \Rightarrow \neg(\varphi \vee \psi)}$$

$$(\neg \supset \Rightarrow)_1\ \frac{H \mid \Gamma, \varphi \Rightarrow \psi \quad H \mid \Gamma, \varphi \Rightarrow \phi}{H \mid \Gamma, \neg(\varphi \supset \psi) \Rightarrow \phi} \qquad (\Rightarrow \neg \supset)_1\ \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \neg\psi}{H \mid \Gamma \Rightarrow \neg(\varphi \supset \psi)}$$

$$(\neg \supset \Rightarrow)_2\ \frac{H \mid \Gamma, \neg\psi \Rightarrow \phi}{H \mid \Gamma, \neg(\varphi \supset \psi) \Rightarrow \phi} \qquad (\Rightarrow \neg \supset)_2\ \frac{H \mid \Gamma, \varphi \supset \psi \Rightarrow \neg(\varphi \supset \psi)}{H \mid \Gamma \Rightarrow \neg(\varphi \supset \psi)}$$

Figure 6.3: Additional rules for $\neg$ in $\mathbf{G_{RM^{\supset}}}$

When searching for proof systems for other refinements of $\mathcal{M}_{\mathbf{G}_{\neg}^{1/2}}$, we have an even better starting point than $\mathbf{G_{LC}}$. Indeed, let $\mathcal{M}$ be such a refinement. Then $\mathcal{M}$ differs from $\mathcal{M}_{\mathbf{RM^{\supset}}}$ solely in the interpretation of $\supset$. By Proposition 6.1.8, $\vdash_{\mathcal{M}}$ is conservative over the $\supset$-free fragment of $\mathbf{RM^{\supset}}$. Thus, when designing a hypersequential calculus for $\mathcal{M}$, it suffices to focus on finding alternatives to the implication rules of $\mathbf{G_{RM^{\supset}}}$.

Chapter 7

Summary and Further Work

In this thesis we studied and utilized analyticity in general families of propositional Gentzen-type proof systems. Our research included pure calculi and their extensions with modal operators, as well as intuitionistic calculi. For each of them, we defined a parametrized notion of a subformula that led to a general definition of analyticity, that, while preserving the main advantages of the usual subformula property, applies for a much wider collection of calculi. Decidable sufficient criteria for analyticity in pure calculi were given, and analyticity was used in order to develop and implement a uniform SAT-based decision procedure for the family of analytic pure calculi. The connection between cut-admissibility and analyticity was also studied, and it was shown that the two properties are equivalent for a wide sub-family. This result was also proved for intuitionistic calculi. We then extended pure calculi with well-known impure rules for modal operators, and proved that this extension preserves the analyticity of the base pure calculus. An extension of the decision procedure for pure calculi was given, that accommodates the addition of modal operators of a certain type. While all of the aforementioned results are syntactic in nature, their proofs were based on general semantic frameworks. Thus, general soundness and completeness theorems were used to identify and prove semantic counterparts of the desired syntactic properties. In particular, we investigated a useful semantic framework, namely Nmatrices, and introduced the rexpansion operation on them. This operation was shown to be useful in the construction of pure analytic sequent calculi for families of non-classical logics. The constructed calculi, being pure and analytic, are of course subject to the reduction to SAT, as well as to the (analyticity-preserving) extension with modal operators. We have also shown that this operation is interesting in its own right, and used it to introduce a general method to obtain conservative extensions of a given logic. In particular, we constructed paraconsistent conservative extensions of Gödel fuzzy logic using rexpansions. For them, hypersequent calculi were considered, rather than ordinary sequent calculi.

We believe that the usefulness of our generic approach was demonstrated in this thesis, and that future Gentzen-type systems for new non-classical logics will be able to be proven analytic, and also to be given decision procedures using our results. The quest for generic theorems and tools that capture families of logics at once is still at its early stages of development, and there are several open problems and directions for further research. The general direction is to further extend, refine and generalize the results of this thesis, thus making them more robust across families of non-classical logics. In particular, this includes the following:

Modal and Intuitionistic Calculi For pure calculi, the picture is quite clear. Our investigation provides a comprehensive study of analyticity and cut-admissibility, and also a uniform SAT-based decision procedure. These results should be imported in their full generality also to the extension of pure calculi with modal operators, and to intuitionistic calculi. For example, while the SAT-based decision procedure for pure calculi was extended here to *Next*-operators, the inclusion of other modal operators, as well as intuitionistic calculi are both left for future work. As another example, the results of Section 2.6 and chapter 5 were proven for a notion of analyticity that is parametrized by a binary relation $\prec$. To what extent this could be done for other parts of the thesis is another area for future research. In particular, a corresponding "$\prec$-local" relation should be found, that will carry the results of Section 4.4. The connection between cut-admissibility and analyticity in calculi for modal logics should also be investigated, using a similar approach to the one taken here for intuitionistic calculi. We note, however, that such an approach is expected to have certain limitations, as some analytic calculi for modal logics (e.g., $S5$ and B [85, 96, 101]) do not admit cut-admissibility. The connection between pure calculi and their intuitionistic variants should also be studied, with the following questions in mind: Does $\prec$-analyticity imply Int-$\prec$-analyticity? Does cut-admissibility imply Int-cut-admissibility? Do either of the converses hold?

First-order Languages Our general investigation focused on the propositional level of non-classical logics. We believe, however, that the ideas presented here could be elevated to the first-order level as well. In particular, the investigation of analyticity and cut-admissibility in a first-order setting could be accomplished by studying partial first-order structures, as well as three-valued first-order structures. Note that for the case of classical logic, the latter was done already by Schütte in [90]. In turn, a uniform decision procedure for first-order non-classical logics would rely on automated first-order theorem provers, rather than on SAT-solvers.

Further Development of Gen2sat Our tool Gen2sat, that implements the suggested decision procedure, could be further developed in several ways. Having logic researchers in mind, the ability to generate actual derivations of provable sequents could be of great help. As an artifact of its semantic approach, however, Gen2sat currently cannot provide such derivations. This can be overcome by integrating Gen2sat with other existing propositional theorem provers so that for unprovable sequents, the theorem prover will not have to search for a proof, while for provable sequents, the search space can be potentially reduced by exploiting Gen2sat's capability of supplying a sufficient subset of rules. For applications, it is interesting to study more useful logics that can be reduced to efficient fragments of SAT (e.g., 2SAT). Finally, the sufficient criteria for analyticity that were proposed in Section 2.5 could be implemented and incorporated into the tool, thus asking the user for an analyticity assumption only in case the tool itself did not succeed in finding one.

An Extended Study of Rexpansions Theorem 6.2.7 provides a sufficient condition for two Nmatrices to induce the same consequence relation. However, in Section 6.4.1 we have seen that $\mathcal{M}^3_{\mathbf{CLuN}}$ induces the same logic as $\mathcal{M}^2_{\mathbf{CLuN}}$ without being a strongly preserving rexpansion of it. An interesting direction for further research is to find a necessary and sufficient criterion for two Nmatrices to induce the same logic, of which being a strongly preserving rexpansion is a particular instance. In addition, rexpansions can be used to construct paraconsistent conservative extensions of fuzzy logics other than $\mathbf{G}$, using a similar methodology to the one of Section 6.5. Finally, new sequent and hypersequent calculi for non-classical logics, and in particular, for more interesting refinements of $\mathcal{M}_{\mathbf{G}^{1/2}_{\neg}}$ should be constructed, taking the guidelines of Sections 6.4 and 6.5.4 into account.

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פרסומים

להלן רשימת הפרסומים עליהם מבוססת עבודה זו:

1. Ori Lahav and Yoni Zohar. SAT-based decision procedure for analytic pure sequent calculi. IJCAR 2014.

2. Ori Lahav and Yoni Zohar. On the construction of analytic sequent calculi for sub-classical logics. WoLLIC 2014.

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4. Arnon Avron and Yoni Zohar. Non-deterministic matrices in action: Expansions, refinements, and rexpansions. ISMVL 2017.

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טהורים והרחבתם מאפשר לנו גם להכליל את האלגוריתם הגנרי שניתן עבור תחשיבים טהורים, גם להרחבתם על ידי אופרטורים מודאליים מסוג מסויים.

הכלי הסמנטי המרכזי שמשמש לחקירה של תחשיבים מסוג זה הוא מסגרות קריפקה דו־ערכיות חלקיות. בכל עולם הסמנטיקה נקבעת לפי התחשיב הטהור המקורי, בעוד שהכללים המודאליים קובעים את הסמנטיקה הגלובלית של האופרטורים המודאליים. כך, ההפרדה הסינטקטית בין החלק הטהור והחלק המודאלי משתקפת גם בסמנטיקה. העובדה שאנליטיות של תחשיב טהור גוררת את האנליטיות של התחשיב המורחב נובעת מהעובדה שהיכולת להרחיב השמות חלקיות מבטיחה את היכולת להרחיב מסגרות קריפקה חלקיות.

תחשיבים אינטואיציוניסטיים

תחשיב הסקוונטים LJ של גנצן עבור הלוגיקה האינטואיציוניסטית אינו טהור, משום שהוא מאפשר אך ורק סקוונטים חד־מסקנתיים. אולם, תחשיב מאוחר יותר ששקול לו, LJ', מצמצם את הדרישה הזו אך ורק לכללי ההכנסה מימין של שלילה וגרירה. אנו מגדירים משפחה של תחשיבים שמכלילה את LJ', וכוללת תחשיבים רבים עבור לוגיקות קונסטרוקטיביות ידועות אחרות. אנו מכנים תחשיבים אלה בשם תחשיבים אינטואיציוניסטיים.

עבור תחשיבים אלה, בדומה לתחשיבים טהורים, אנו מראים שסילוק חתך ואנליטיות הן תכונות שקולות עבור תת משפחה רחבה.

הכלי הסמנטי המרכזי בו אנו משתמשים עבור משפחה זו הוא מסגרות קריפקה תלת־ערכיות, בהן כל עולם כולל השמה תלת־ערכית. שלא כמו מסגרות קריפקה דו־ערכיות, השפה בה אנו משתמשים בפרק זה לא כוללת אופרטורים מודאליים, והסמנטיקה הגלובלית משמשת תת קבוצה של הקשרים של השפה המקורית.

מוגבלת. אפיון זה מקשר בין תכונת האנליטיות ובין היכולת להרחיב השמות חלקיות להשמות מלאות. כמו כן, הוא הרכיב המרכזי שמאפשר את הרדוקציה לבעיית הספיקות הקלאסית. השמות תלת־ערכיות מלאות מתאימות לתחשיבים טהורים ללא כלל החתך. הקשר בין התחום החלקי של ההשמות הדו־ערכיות ובין השימוש בערך השלישי בהשמות התלת־ערכיות מאפשר לנו לחקור את הקשר בין סילוק חתך ואנליטיות. כמקרה פרטי ושימושי, אנו חוקרים לעומק מטריצות לא דטרמיניסטיות, ובפרט מציגים פעולות שימושיות שניתן לבצע עליהן. חקירה זו מובילה לשיטה כללית לבנייה של תחשיבי סקוונטים טהורים ואנליטיים, דרך מטריצות לא דטרמיניסטיות. כפי שאנו מראים, תחשיבים רבים עבור לוגיקות לא קלאסיות נבנו לפי שיטה זו מבלי לציין זאת במפורש. בנוסף, פעולות אלה על מטריצות לא דטרמיניסטיות מאפשרות לנו להציג שיטה כללית לבניית הרחבות משמרות של לוגיקה נתונה. השימוש החשוב ביותר של שיטה זו מוביל לבנייה מסוג חדש של לוגיקות פרה־קונסיסטנטיות ועמומות, שכולל יתרונות רבים על פני פתרונות קיימים לבעיה זו.

תחשיבים טהורים עם אופרטורים מודאליים

כללי הסקוונטים הידועים עבור אופרטורים מודאליים אינם טהורים, שכן הם כוללים מספר מגבלות על נוסחאות ההקשר. תחשיבים כאלה, לפיכך, אינם כלולים בחקירה שתוארה לעיל. אף על פי כן, אנו מראים כיצד ניתן להכליל תוצאות על תחשיבים טהורים עבור הרחבתם עם כללים לא טהורים לאופרטורים מודאליים. דבר זה נעשה על ידי הפרדה בין החלק הטהור של התחשיב ובין החלק שכולל את האופרטורים המודאליים. אנו מוכיחים שלמעט מקרים מנוונים, הרחבה של תחשיב טהור עם כללים לאופרטורים מודאליים משמרת אנליטיות. כלומר, אם תחשיב טהור הוא אנליטי, הוא יישאר כך לאחר הרחבתו עם כללים עבור אופרטורים מודאליים. תוצאה זו, יחד עם תוצאות הפרק על תחשיבים טהורים, נותנת קריטריונים פשוטים לזיהוי אנליטיות גם בתחשיבים מהסוג הזה, כמו גם שיטה לבניית תחשיבים כאלה. קשר זה בין תחשיבים

תרומתנו המרכזית היא חקירה שיטתית של תכונת האנליטיות והשלכותיה בשלוש משפחות כלליות של תחשיבי סקוונטים:

תחשיבים טהורים

אלה תחשיבי סקוונטים בעלי מבני צורני דומה לחלק הפסוקי של LK. הם כוללים את כל הכללים המבניים הנפוצים (זהות, חתך, החלשה, כיווץ והחלפה), וכללי ההיסק הקשורים לקשרים הלוגים לא מגבילים כלל את נוסחאות ההקשר. משפחה זו שימושית מאוד עבור לוגיקות תלת־ערכיות וארבע־ערכיות, וכן ללוגיקות פרה־קונסיסטנטיות רבות.

עבור משפחה זו, אנו מספקים תנאים פשוטים וכריעים שמבטיחים אנליטיות. תנאים אלה יכולים להיבדק בקלות על ידי חוקרים או בצורה אוטומטית, באופן פשוט בהרבה מהוכחות סילוק חתך. תנאים אלה גם מספקים שיטות לבניה של תחשיבים טהורים ואנליטיים חדשים. תכונת האנליטיות של תחשיבים ידועים רבים, כמו גם של תחשיבים חדשים שאנו מציגים בעבודה זו, מוכחת באמצעות תנאים כללים אלה.

בנוסף, אנו מראים שעבור תת משפחה רחבה, אנליטיות וסילוק חתך למעשה שקולים. תוצאה זו מאפשרת להשתמש בחלק מן הקריטריונים שאנו מספקים עבור אנליטיות גם להוכחה פשוטה של סילוק חתך. מעבר לכך, לתוצאה זו יש ערך תיאורטי חשוב, שכן היא מזהה בין שתי תכונות יסודיות של תחשיבי סקוונטים, שבמבט ראשון נראות שונות זו מזו.

בצד האלגוריתמי, אנו מספקים אלגוריתם גנרי להכרעת יכיחות של סקוונטים בכל תחשיב טהור ואנליטי, שמבוסס על רדוקציה לבעיית הספיקות הקלאסית, לה כלים רבים ויעילים. אלגוריתם זה מומש, ופרטי המימוש, כמו גם מדידה וניתוח של זמני ריצה מתוארים גם כן.

שני הכלים הסמנטים המרכזיים עליהם מסתמכות רוב תוצאותנו לגבי תחשיבים טהורים הם השמות דו־ערכיות חלקיות, וכן השמות תלת־ערכיות מלאות. השמות דו־ערכיות חלקיות מאפיינות הוכחות בהן קבוצת הנוסחאות שמותר להשתמש בהן

שימושיות להוכחת כריעות וקונסיסטנטיות בדיוק כמו תכונת התת־נוסחה המקורית, אף על פי שהן כוללות מגוון רחב יותר של תחשיבים. בפרט, תחשיבים רבים עבור לוגיקות רב־ערכיות ולוגיקות פרה־קונסיסטנטיות אינם נהנים מתכונת התת־נוסחה, אך כן נהנים מהתכונה המוכללת שזה עתה תוארה.

מאפיין חשוב של מרבית ההוכחות בעבודה זו הוא העובדה שהן מבוססות על כלים סמנטיים. הוכחות סמנטיות נוטות להיות קריאות יותר, חשופות לפחות טעויות, ומאפשרות הכללות רבות יותר מאשר הוכחות סינטקטיות. בנוסף, השיטה הסמנטית מספקת נקודת מבט נוספת על התחשיבים השונים, שיכולה להוביל לתובנות נוספות לגביהם.

המתכון הכללי להוכחת תוצאתנו הוא:

1. ניסוח היפותזה סינטקטית לגבי משפחה של מערכות סקוונטים

2. הוכחת משפט שלמות ונאותות כללי עבור משפחה זו

3. בהסתמך על שלמות ונאותות, מציאת היפותזה סמנטית שנכונתה תבטיח את נכונות ההיפותזה המקורית

4. הוכחת ההיפותזה הסמנטית בכלים סמנטיים

5. הסקת ההיפותזה המקורית

העיקרון המרכזי שמשותף לכל הכלים הסמנטיים בהם אנו משתמשים בעבודה זו הוא עיקרון האי־דטרמיניזם. לפי עיקרון זה, ערך האמת של נוסחה מורכבת אינו נקבע באופן יחיד על פי ערכי האמת של תתי הנוסחאות המיידיות שלה. תנאים אחרים, שלאו דווקא קשורים לתתי נוסחאות אלה, יכולים להשפיע על המודלים השונים. עיקרון האי־דטרמיניזם מאפשר מודולריות רבה, שמתאימה במיוחד לחקירה של משפחות של מערכות הוכחה. כך, כל כלל היסק מתאים לתנאי סמנטי מקומי, ואין צורך לשנות את כל המסגרת הסמנטית עבור שינוי קטן במערכת ההוכחה.

וכן לוגיקות פרה־קונסיסטנטיות. לרוב, תהליך מציאת התחשיב והשימוש בו נראה כך:

1. תחשיב סקוונטים נמצא עבור לוגיקה מסויימת
2. כלל החתך מוכח כגזיר בתחשיב
3. תכונת התת־נוסחה מתקבלת כמסקנה
4. לעתים, אלגוריתם לקביעת תקפות בלוגיקה נבנה על סמך התחשיב שנמצא

ישנם שני חסרונות מרכזיים בתהליך שזה עתה תואר. ראשית, כאשר מוכיחים את תכונת התת־נוסחה כמסקנה מסילוק חתך, הדבר לא מאפשר להוכיח את תכונת התת־נוסחה עבור תחשיבים שמקיימים אותה אך אינם נהנים מסילוק חתך (כגון מספר תחשיבים ללוגיקות מודאליות). יתרה מזאת, אפילו כאשר שתי התכונות מתקיימות בתחשיב מסויים, פעמים רבות קל יותר להוכיח את תכונת התת־נוסחה מאשר את משפט סילוק החתך עבור התחשיב. שנית, פיתוח של אלגוריתמים המבוססים על חיפוש הוכחה נוטה לכלול שיטות שמפותחות אד־הוק ולא ניתנות להכללה עבור תחשיבים ללוגיקות אחרות. כך, כל לוגיקה דורשת בנייה מחדש של אלגוריתם עבורה.

לאור זאת, עבודה זו מתרכזת בעיקר בתכונת התת־נוסחה, מבלי להסתמך על סילוק חתך, ומשתמשת בתכונה זו בכדי לפתח אלגוריתמים גנריים יותר.

עבודה זו

בכדי לכסות מגוון רחב ככל שניתן של לוגיקות לא קלאסיות, איננו מגבילים את עצמנו לתכונת התת־נוסחה בלבד, אלא מאפשרים גם מספר הכללות שימושיות שלה. כך, חקירתנו תכלול, למשל, תחשיבי סקוונטים בהם נדרשות לא רק תתי נוסחאות של הסקוונט המוכח, אלא גם שלילות שלהן. את תכונת התת־נוסחה המוכללת על צורותיה השונות נכנה בשם אנליטיות, ותחשיבים שמקיימים אותה ייקראו אנליטיים. הכללות אלה

תקציר

בשנת 1934 גנצן הציג את שתי מערכות הדדוקציה הטבעית הראשונות: NK ו־NJ עבור לוגיקה קלאסית ולוגיקה אינטואיציוניסטית. התחשיב עבור הלוגיקה האינטואיציוניסטית, NJ, מקיים את תכונת התת־נוסחה: אם אפשר להוכיח טענה מקבוצה של הנחות, אז יש הוכחה של אותה הטענה מאותה קבוצת הנחות שמשתמשת אך ורק בחומר הסינטקטי המצוי בטענה ובהנחות. אולם, תכונה חשובה זו לא מתקיימת עבור NK.

בכדי להשיג תחשיבים שמקיימים את תכונת התת־נוסחה גם במקרה הקלאסי וגם במקרה האינטואיציוניסטי, גנצן הציג שני תחשיבים אלטרנטיביים: LK ו־LJ. תחשיבים אלה היו לתחשיבי הסקוונטים הראשונים. סקוונטים הם ביטויים מהצורה $\Gamma \Rightarrow \Delta$, כך ש־$\Gamma$ ו־Δ הן רשימות סופיות של נוסחאות, ו־$\Rightarrow$ הוא סימן מיוחד ששמור להפרדה ביניהן (נעיר כי בעבודה זו אנחנו משתמשים בגירסה אחרת לפיה סקוונטים מורכבים מקבוצות סופיות של נוסחאות ולא מרשימות). העבודה הישירה עם סקוונטים במקום עם נוסחאות איפשרה לגנצן להוכיח ששני התחשיבים LK ו־LJ נהנים מתכונת התת־נוסחה. בכדי להוכיח זאת, היה צורך להוכיח שכלל החתך, שהוא הכלל היחיד שכולל בהנחות נוסחה שעלולה להיות כלל לא קשורה למסקנה, הינו גזיר בשתי המערכות. משפט זה, המכונה משפט סילוק החתך, דרש הוכחה סינטקטית המורכבת מאינדוקציה כפולה על הוכחות בשני התחשיבים. ההוכחה כללה חלוקה למקרים (ותתי מקרים) רבים. תכונת התת־נוסחה נובעת מיידית ממשפט זה, ומבטיחה שהפרגמנט הפסוקי של התחשיבים כריע, וכן שהתחשיבים הם קונסיסטנטיים (כלומר, הסקוונט הריק אינו יכיח בהם). הוכחת

מאז, תחשיבי סקוונטים (וכן הכללות שלהם, כגון תחשיבי היפר־סקוונטים) מהווים כלי מרכזי בתורת ההוכחה של לוגיקות לא קלאסיות. בנוסף ללוגיקה הקלאסית והלוגיקה האינטואיציוניסטית, נמצאו תחשיבי סקוונטים בעלי תכונת התת־נוסחה ללוגיקות לא קלאסיות רבות אחרות, כגון לוגיקות מודאליות, לוגיקות רב־ערכיות, לוגיקות עמומות,

כאן כמקרה פרטי של תוצאה זו. בנוסף, הערך של הקריטריונים המוצעים מודגם על ידי הוכחת האנליטיות של מספר תחשיבים חדשים ומועילים שאנו מציגים.

- אנו מראים שעבור מגוון רחב של תחשיבים, אנליטיות וסילוק חתך הן תכונות שקולות. תוצאה זו יכולה לפשט הוכחות של סילוק חתך, במידה ואנליטיות כבר הוכחה. באמצעות תוצאה זו, אנו מראים שחלק מהקריטריונים המספיקים לאנליטיות מבטיחים גם סילוק חתך.

- אנו משתמשים בתכונת האנליטיות בכדי לבנות אלגוריתם יוניפורמי עבור משפחה רחבה של תחשיבי סקוונטים. האלגוריתם מבוסס על רדוקציה יעילה לבעיית הספיקות הקלאסית, לה קיימים כלים רבים ויעילים. העבודה כוללת גם תיאור של מימוש האלגוריתם וניתוח זמני הריצה שלו.

- לבסוף, אנו חוקרים את המסגרת הסמנטית של מטריצות לא דטרמיניסטיות, ומשתמשים בה ככלי לבנייה של תחשיבי סקוונטים עבור לוגיקות שמוגדרות באמצעים אחרים. אנו מציגים את פעולת ה־$rexpansion$, ומראים שלה תפקיד מכריע (אם כי עד כה סמוי) בשימוש של מסגרת סמנטית זו בבנייתן של תחשיבי סקוונטים אנליטיים ללוגיקות לא קלאסיות.

תמצית

תחשיבי סקוונטים מהווים כלי מרכזי בתורת ההוכחה, שמתאים למגוון רחב של לוגיקות, לרבות לוגיקה קלאסית ולוגיקה אינטואיציוניסטית, וכן לוגיקות לא קלאסיות רבות אחרות. התכונה החשובה ביותר של תחשיבי סקוונטים שימושיים היא אנליטיות, אשר במקרים רבים מסתכמת בתכונת התת־נוסחה. כאשר תחשיב סקוונטים הוא אנליטי, מתקבלת מגבלה על מרחב החיפוש של ההוכחות בתחשיב. עבור תחשיבים פסוקיים, הדבר לרוב מוביל לחסם סופי על מרחב החיפוש, ואז כריעותו של התחשיב מתקבלת באופן מיידי.

הדרך הסטנדרטית להוכחת אנליטיות עוברת דרך הוכחת סילוק חתך, כלומר, הוכחת גזירותו של כלל החתך בתחשיב. אכן, כאשר ניתן לסלק את כל החתכים מהוכחות, וכל יתר כללי ההיסק כוללים בהנחותיהם אך ורק חומר סינטקטי מתוך המסקנות שלהם, מיד נובע שהסקוונט הסופי יכיח תוך שימוש בחומר הסינטקטי המצוי בו. מוטיבציה נוספת להוכחת סילוק חתך היא העובדה שתחשיבים ללא חתך מתאימים במיוחד לחיפוש הוכחה.

למרות השימושיות שבהוכחת סילוק חתך, הסתמכות רק על כך כאשר מוכיחים אנליטיות אינה כוללת תחשיבים רבים שהינם אנליטיים, אך לא נהנים מסילוק חתך. יתרה מזאת, אפילו כאשר ניתן להוכיח סילוק חתך, ייתכן שקל יותר להוכיח אנליטיות באופן ישיר.

הנושא המרכזי של עבודה זו הוא אנליטיות של תחשיבי סקוונטים. תרומתנו המרכזית היא חקירה כללית של של תכונה זו, שכוללת את הדברים הבאים:

- אנו מספקים קריטריונים פשוטים וכריעים שמבטיחים את קיומה של תכונת האנליטיות. קריטריונים אלה יכולים להחליף הוכחות מסובכות של אנליטיות שעוברות דרך סילוק חתך. למעשה, האנליטיות של תחשיבים ידועים רבים מוכחת

אוניברסיטת תל אביב
הפקולטה למדעים מדוייקים ע"ש ריימונד ובברלי סאקלר
בית הספר למדעי המחשב ע"ש בלבטניק

מערכות גנצן עבור לוגיקות לא קלאסיות

חיבור לשם קבלת תואר דוקטור לפילוסופיה

מאת

יהונתן זוהר

עבודה זו נעשתה בהנחייתו של:

פרופ' ארנון אברון

הוגש לסנאט של אוניברסיטת תל אביב
ינואר 2018