From the Subformula Property to Cut-Admissibility in Propositional Sequent Calculi

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Abstract

While the subformula property is usually a trivial consequence of cut-admissibility in sequent calculi, it is unclear in which cases the subformula property *implies* cut-admissibility. In this paper, we identify two wide families of propositional sequent calculi for which this is the case: the (generalized) subformula property is equivalent to cut-admissibility. For this purpose, we employ a semantic criterion for cut-admissibility, which allows us to uniformly handle a wide variety of calculi. Our results shed light on the relation between these two fundamental properties of sequent calculi, and can be useful to simplify cut-admissibility proofs in various calculi for non-classical logics, where the subformula property (equivalently, the property known as "analytic cut-admissibility") is easier to show than cut-admissibility.[1]

Keywords— Sequent Calculus, Subformula Property, Cut-elimination, Analyticity

1 Introduction

In his seminal paper [19], Gentzen introduced the first sequent calculi, systems LK and LJ, as technical devices for studying classical and intuitionistic logics (respectively). His "Main Theorem" (*Hauptsatz*) was the *cut-elimination* theorem, from which he, and later others, derived several important consequences. In particular, since the cut rule is the only rule of these systems whose premises may include a formula that is not a subformula of (some formula in) the conclusion, the admissibility of cut immediately implies *"the subformula property"* for these systems. That is: when deriving a sequent s it suffices to only consider the syntactic material contained in s, i.e., to compose the derivation only from subformulas of s. Other formulas may sometimes shorten the derivation, but are never necessary.

Since then the framework of sequent calculi (and its various extensions) is widely applied in proof-theory and automated deduction, especially when one is interested in the computational aspects of a logic (e.g., [18]). Various important non-classical logics—e.g., modal logics [40, 33], many-valued and fuzzy logics [9, 27], and paraconsistent logics [10]—admit a proof-theoretic presentation using some sequent calculus for which cut-admissibility holds. In all these cases, the subformula property (or some generalization of it) trivially follows.

In this paper, we are interested in the converse direction:

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[1]The main results of this paper were first announced in [26]. However, that extended abstract did not include all the proofs, and contained a number of mistakes, which are corrected here.

Is cut-admissibility a corollary of the subformula property?

If a calculus enjoys the subformula property, then one can easily deduce that *some* cuts are never needed (a.k.a. *non-analytic* cuts): these are the cuts that (read bottom-up) introduce some formula that is not included in the final sequent to be proved. Indeed, the subformula property ensures that such detours are always redundant. Nevertheless, the subformula property gives us no guarantees about the admissibility of, so-called, *analytic cuts*, where the cut formula is a subformula of the final sequent.

Clearly, one cannot expect a general affirmative answer to the question above:

1. There are several logics for which a cut-free sequent calculus appears beyond reach, but nevertheless logicians settled with a sequent calculus that enjoys the subformula property. In particular, this is the case for the modal logics $S5$ (of universal Kripke frames) and B (of symmetric Kripke frames), bi-intuitionistic logic [31, 32], as well as several paraconsistent logics [8] (see Example 5.6 below). Notably, these calculi are substantially more complicated than Gentzen's original systems. They all involve logical rules that restrict, in one way or another, the formulas that may appear as context while applying the rule, and combine such restrictions in non-trivial ways. For example, the calculus for bi-intuitionistic logic has logical rules that can be applied only without context formulas on the right-hand side, as well as rules that can be applied only when the left-hand side has no context formulas.

2. Even a simple syntactic manipulation on LK yields an alternative calculus for classical logic that still enjoys the subformula property, but requires cuts in various derivations. One may turn the logical rules of LK to *axiomatic* rules. For example, the right-introduction rule of negation will then have the form:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \qquad \rightsquigarrow \qquad \frac{}{\Gamma \Rightarrow \varphi, \neg\varphi, \Delta}$$

   The resulting calculus is equivalent to LK (e.g., $\Gamma \Rightarrow \neg\varphi, \Delta$ can be derived from $\Gamma \Rightarrow \varphi, \neg\varphi, \Delta$ and $\Gamma, \varphi \Rightarrow \Delta$ using a cut); it enjoys the subformula property; but it does not admit cut-admissibility (see Example 4.1 below).

The main contribution of this paper is an affirmative answer to the question above for two wide families of propositional multiple-conclusion sequent calculi. The first is a family of *pure* calculi (in terms of [4]) whose derivation rules do not impose any restrictions on context formulas, of which LK is the prototype example. The second is a family of sequent calculi, which we call *intuitionistic calculi*, in which premises of the form $\Gamma \Rightarrow \Delta$ with $\Gamma \neq \emptyset$ in right introduction rules forbid context formulas on the right-hand side. The well-known multiple-conclusion calculus for intuitionistic logic (see [38]) is the prototype example for an intuitionistic calculus. In both families, we further require the rules to have a certain "directed" structure (precisely defined below), thus avoiding the mentioned counterexamples above.

Besides its theoretical interest, we believe that our result can be useful in future investigation and development of sequent calculi. While the subformula property is traditionally proved as a consequence of cut-admissibility, there are other approaches to show that a certain calculus enjoys the subformula property. One prominent way is the *semantic* approach (see, e.g., [29]), often employed when cut-admissibility is beyond reach (see, e.g., [37]), or when general families of calculi are studied (see, e.g., [12, 23]). These are essentially completeness proofs: for a calculus and a sound semantics for it, one constructs a countermodel for a sequent s provided that s has no derivation that consists only of subformulas of s. In fact, such semantic proofs tend to be simpler than syntactic cut-elimination proofs. They are also much easier to generalize and uniformly apply to families of calculi. In particular, the paper [23] provides a sufficient criterion for the subformula property in a general family of calculi, which reduces it to showing that certain *partial* models can be always extended to full ones. Using this characterization, our recent paper [24] provides a method for proving the subformula property for a wide family of calculi for sub-classical logics. Using the result of the current paper, we obtain the admissibility of cut in all these calculi.

Our proof in this paper follows a similar semantic approach. It is based on two different semantics for a given calculus: (i) a semantics for the case that derivations are confined to consist only of subformulas of the end sequent; and (ii) a semantics for the cut-free fragment of the calculus. The second paves the way to a sufficient semantic criterion for cut-admissibility. Then, the crux of our proof is to show that this criterion is met when the calculus enjoys the subformula property. As a result, we obtain that the cut rule is admissible given that the calculus enjoys the subformula property. Note that we focus only on the *admissibility* of the cut rule, rather than offering some effective procedure to *eliminate* cuts.

In order for our result to cover a wide variety of logics, we consider here a generalization of the subformula property. This is necessary to accommodate, e.g., sequent calculi for many-valued logics and paraconsistent logics that do not admit the strict subformula property, but do admit a simple generalization of it. For example, in sequent calculi for the family of C-systems from [10], if a sequent s is derivable, then there exists a derivation of s that uses only subformulas of s and their negations. We therefore assume a general notion of *subformula*, based on a general ordering of propositional formulas.

The rest of this paper is organized as follows. First, in Section 2, we define the family of pure sequent calculi and cut-admissibility. Section 3 defines a generalized notion of the subformula property. In Section 4, we prove our result concerning pure calculi, by considering semantic characterizations of derivability in them, based on three-valued valuation functions. Finally, Section 5 introduces the family of *intuitionistic sequent calculi*, and proves a similar result for this family, again going through semantic characterizations of derivability, which is based on three-valued intuitionistic-like Kripke models.

Related Work Avron and Lev [12] introduced and studied the family of *canonical calculi*, a restricted sub-family of pure calculi, and proved the equivalence of the subformula property and cut-admissibility in them. The proof of this equivalence was semantic, based on the framework of *Nmatrices* [11], a simple generalization of logical matrices, which was shown to precisely account for canonical calculi.

Our work goes far beyond canonical calculi, and so Nmatrices do not suffice. Rather, the framework that we use here is an extension of *bivaluation semantics*. Bivaluation semantics, introduced by Béziau [14] (see also [15]), is a simple semantic framework for pure calculi that is based on two-valued valuation functions. In [14], it was shown how to associate a set of bivaluations to arbitrary pure calculi, in a way that will ensure soundness and completeness.

Later, Lahav and Avron [23] extended this framework in two respects: first, it was generalized from plain valuation functions to Kripke models, which uniformly accommodate a much wider family of sequent calculi, that includes, among others, pure and intuitionistic calculi; second, and more importantly for the purpose of the current paper, they provided semantic counterparts for important syntactic properties, including cut-admissibility. A general soundness and completeness theorem was proved, which associates a class of two-valued Kripke models for each calculus of a general family of sequent calculi, and allows the formulation of sufficient semantic criteria for cut-admissibility and the subformula property. The former amounts to the ability to refine three-valued valuations into two-valued ones, while the latter amounts to the ability to extend partial two-valued valuations into full valuations. Then, in a previous work [25], we showed that for pure calculi, the criterion for the subformula property is also necessary. For the present paper, however, the mere ability to extend partial two-valued valuations is not enough, and a constructive extension method is introduced. Moreover, the fact that this criterion is sufficient also for the Kripke-style counterpart is new. The semantic framework that we employ here closely follows the one of [23], adapted and simplified for the particular families of calculi studied in this paper.

Finally, in previous work [24], we studied general conditions for the subformula property in pure calculi, while cut-admissibility was not considered at all.

2 Pure Sequent Calculi

In this section, we define the family of pure sequent calculi [4], as well as the notion of cut-admissibility. Several examples of well-known calculi that belong to this family are provided as well.

2.1 Preliminaries

Let $At = \{p_1, p_2, \ldots\}$ denote a fixed infinite set of propositional variables. A *propositional language* $\mathcal{L}$ is given by a set $\diamondsuit_{\mathcal{L}}$ of connectives. *$\mathcal{L}$-formulas* are defined as usual, where *atomic $\mathcal{L}$-formulas* are the elements of At. We usually identify a propositional language with its set of formulas (e.g., when writing expressions like $\varphi \in \mathcal{L}$). For a set $\mathcal{F} \subseteq \mathcal{L}$, by *$\mathcal{F}$-formula* we mean a formula φ satisfying $\varphi \in \mathcal{F}$.

An *$\mathcal{L}$-substitution* is a function $\sigma : At \to \mathcal{L}$, naturally extended to all $\mathcal{L}$-formulas and to sets of $\mathcal{L}$-formulas (by putting $\sigma(\diamond(\varphi_1, \ldots, \varphi_n)) = \diamond(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ for every n-ary connective $\diamond \in \diamondsuit_{\mathcal{L}}$, and $\sigma(\Gamma) = \{\sigma(\varphi) \mid \varphi \in \Gamma\}$).

An *$\mathcal{L}$-sequent* is a pair of finite sets Γ and Δ of $\mathcal{L}$-formulas, denoted $\Gamma \Rightarrow \Delta$. We employ the standard sequent notations, e.g., when writing expressions like $\Gamma, \varphi \Rightarrow \Delta$ or $\Rightarrow \varphi$. The union of two sequents $(\Gamma_1 \Rightarrow \Delta_1) \cup (\Gamma_2 \Rightarrow \Delta_2)$ is the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. We denote by $frm[\Gamma \Rightarrow \Delta]$ the set $\Gamma \cup \Delta$, and extend this notation to sets of sequents, by putting $frm[S] = \bigcup\{frm[s] \mid s \in S\}$. $\mathcal{L}$-substitutions are extended to $\mathcal{L}$-sequents and sets of $\mathcal{L}$-sequents (by putting $\sigma(\Gamma \Rightarrow \Delta) = \sigma(\Gamma) \Rightarrow \sigma(\Delta)$ and $\sigma(S) = \{\sigma(s) \mid s \in S\}$).

In what follows, $\mathcal{L}$ denotes an arbitrary propositional language. When $\mathcal{L}$ can be inferred from the context, we omit the prefix "$\mathcal{L}$-" from the notions above (as well as from the ones introduced below).

2.2 Pure Sequent Calculi

Following [12], we find it technically convenient to use the object propositional language for specifying derivation rules. (One could use meta-variables and rule schemes instead.)

Definition 2.1. A *pure $\mathcal{L}$-rule* is a pair $\langle S, s\rangle$, denoted $S \,/\, s$, where S is a finite set of $\mathcal{L}$-sequents and s is an $\mathcal{L}$-sequent. The elements of S are called the *premises* of the rule and s is called the *conclusion* of the rule. We sometimes omit set braces around the premises, and separate them by semi-colons (e.g., when writing expressions like $\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow p_1 \wedge p_2$).

An *$\mathcal{L}$-application* of a pure $\mathcal{L}$-rule $s_1, \ldots, s_n \,/\, s$ is a pair of the form $\langle\{\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \ldots \cup c_n\rangle$ where σ is an $\mathcal{L}$-substitution, and $c_1, \ldots, c_n$ are $\mathcal{L}$-sequents (called the *context sequents* of the application). The sequents $\sigma(s_i) \cup c_i$ are called the *premises* of the application, and the sequent $\sigma(s) \cup c_1 \cup \ldots \cup c_n$ is called the *conclusion* of the application.

Example 2.2. The pure rules for introducing implication in classical logic are:

$$p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2 \qquad\qquad \Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow$$

Their applications take the form (respectively):

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta} \qquad\qquad \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2}$$

Examples for derivation rules that *cannot* be formulated as pure rules include the following rule schemes, which are employed in intuitionistic and modal logic:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad\qquad \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

Pure calculi, in turn, are simply finite sets of pure rules.

Definition 2.3. A *pure $\mathcal{L}$-calculus* is a finite set of pure $\mathcal{L}$-rules. A *derivation* of a sequent s from a set S of sequents (a.k.a. "assumptions" or "non-logical axioms") in a pure $\mathcal{L}$-calculus $\mathbf{G}$ is a finite sequence of sequents, where each sequent in the sequence is either one of the following:
(i) an element of S;
(ii) the conclusion of an application of a rule of $\mathbf{G}$, all premises of which are preceding elements of the sequence;
(iii) the conclusion of one of the following standard structural rules,[2] again where all premises are preceding elements of the sequence:

$$\text{(ID)}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \text{(CUT)}\ \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \qquad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad \text{(WEAK)}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}$$

In (CUT), φ is called the *cut formula*.

We write $S \vdash_{\mathbf{G}} s$ if there exists a derivation of a sequent s from a set S of sequents in $\mathbf{G}$.

In what follows, unless stated otherwise, we may refer to pure rules and pure calculi simply as *rules* and *calculi*.

The most well-studied property of sequent calculi is the admissibility of the cut rule. When cut is admissible the calculus is generally considered well-behaved, and reasoning about the calculus becomes much easier. Moreover, proof-search algorithms have no need to "guess" the cut formulas. Next, we precisely define cut-admissibility.

Definition 2.4. A derivation of s from S in a calculus $\mathbf{G}$ is called *cut-limited* if the cut formula is in $frm[S]$ in every application of (CUT). We write $S \vdash^{\mathrm{cf}}_{\mathbf{G}} s$ if such a derivation exists. A calculus $\mathbf{G}$ enjoys *cut-admissibility* if $\vdash_{\mathbf{G}} = \vdash^{\mathrm{cf}}_{\mathbf{G}}$.

What we call here cut-admissibility is actually known as *strong* cut-admissibility, in which cuts are allowed, but they are confined to cases where the cut formula appears in the set of assumptions [5]. Usual cut-admissibility, which we call here *weak cut-admissibility*, only requires that $\vdash_{\mathbf{G}} s$ iff $\vdash^{\mathrm{cf}}_{\mathbf{G}} s$ for every sequent s. For pure calculi, however, the two notions turn out to be equivalent (see [5]). (This is not the case for intuitionistic calculi, studied in Section 5.)

Next, we present several examples of pure calculi (they all enjoy cut-admissibility).

[2] By defining sequents to be pairs of *sets* we implicitly include other standard structural rules, such as exchange and contraction.

Example 2.5 (Classical Logic). The propositional language $\mathcal{CL}$ consists of three binary connectives $\wedge$, $\vee$, $\supset$, and one unary connective $\neg$. The propositional fragment of Gentzen's fundamental sequent calculus for classical logic [19] can be directly presented as a pure $\mathcal{CL}$-calculus, denoted $\mathbf{LK}$, that consists of the following $\mathcal{CL}$-rules:

$$\begin{array}{rr}
\Rightarrow p_1 \,/\, \neg p_1 \Rightarrow & p_1 \Rightarrow \;/\; \Rightarrow \neg p_1 \\
p_1, p_2 \Rightarrow \;/\, p_1 \wedge p_2 \Rightarrow & \Rightarrow p_1 \,;\; \Rightarrow p_2 \,/\; \Rightarrow p_1 \wedge p_2 \\
p_1 \Rightarrow \;;\, p_2 \Rightarrow \;/\, p_1 \vee p_2 \Rightarrow & \Rightarrow p_1, p_2 \,/\; \Rightarrow p_1 \vee p_2 \\
\Rightarrow p_1 \,;\, p_2 \Rightarrow \;/\, p_1 \supset p_2 \Rightarrow & p_1 \Rightarrow p_2 \,/\; \Rightarrow p_1 \supset p_2
\end{array}$$

Example 2.6 (Paraconsistent Logics). In [10], sequent calculi for various paraconsistent logics are presented. For example, a pure calculus for da Costa's historical paraconsistent logic C_1, which we call $\mathbf{G}_{C_1}$, consists of the rules of $\mathbf{LK}$ except for the left introduction rule of negation that is replaced by the following pure $\mathcal{CL}$-rules:

$$\begin{array}{rr}
p_1 \Rightarrow \;/\, \neg\neg p_1 \Rightarrow & \\
\Rightarrow p_1 \,;\; \Rightarrow \neg p_1 \,/\, \neg(p_1 \wedge \neg p_1) \Rightarrow & \neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\, \neg(p_1 \wedge p_2) \Rightarrow \\
\neg p_1 \Rightarrow \;;\, p_2, \neg p_2 \Rightarrow \;/\, \neg(p_1 \vee p_2) \Rightarrow & p_1, \neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\, \neg(p_1 \vee p_2) \Rightarrow \\
p_1 \Rightarrow \;;\, p_2, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow & p_1, \neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow
\end{array}$$

Similarly, a pure calculus $\mathbf{G}_{P_1}$ for the *atomic* paraconsistent logic P_1 was given in [3]. It is obtained by replacing the left introduction rule of negation in $\mathbf{LK}$ with the following alternative rules:

$$\begin{array}{rr}
\Rightarrow p_1 \,;\; \Rightarrow p_2 \,/\, \neg(p_1 \wedge p_2) \Rightarrow & \Rightarrow p_1, p_2 \,/\, \neg(p_1 \vee p_2) \Rightarrow \\
p_1 \Rightarrow p_2 \,/\, \neg(p_1 \supset p_2) \Rightarrow & \Rightarrow \neg p_1 \,/\, \neg\neg p_1 \Rightarrow
\end{array}$$

Example 2.7 (Many-valued Logics). In [6], pure sequent calculi for well-known many-valued logics are presented. For example, a calculus for Łukasiewicz three-valued logic, which we call $\mathbf{G}_3$, has the following rules for implication:

$$\begin{array}{rr}
\neg p_1 \Rightarrow \;;\, p_2 \Rightarrow \;;\; \Rightarrow p_1, \neg p_2 \,/\, p_1 \supset p_2 \Rightarrow & p_1 \Rightarrow p_2 \,;\, \neg p_2 \Rightarrow \neg p_1 \,/\; \Rightarrow p_1 \supset p_2 \\
p_1, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow & \Rightarrow p_1 \,;\; \Rightarrow \neg p_2 \,/\; \Rightarrow \neg(p_1 \supset p_2)
\end{array}$$

A pure calculus for the $\mathcal{CL}$-fragment of the logic of bilattices [2] (whose implication-free fragment coincides with the logic of first-degree entailments [1]), which we call $\mathbf{G}_4$, is obtained in a similar manner, by augmenting the negation-free fragment of $\mathbf{LK}$ with the following rules:

$$\begin{array}{rr}
p_1, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow & \Rightarrow p_1 \,;\; \Rightarrow \neg p_2 \,/\; \Rightarrow \neg(p_1 \supset p_2) \\
\neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\, \neg(p_1 \wedge p_2) \Rightarrow & \Rightarrow \neg p_1, \neg p_2 \,/\; \Rightarrow \neg(p_1 \wedge p_2) \\
\neg p_1, \neg p_2 \Rightarrow \;/\, \neg(p_1 \vee p_2) \Rightarrow & \Rightarrow \neg p_1 \,;\; \Rightarrow \neg p_2 \,/\; \Rightarrow \neg(p_1 \vee p_2) \\
p_1 \Rightarrow \;/\, \neg\neg p_1 \Rightarrow & \Rightarrow p_1 \,/\; \Rightarrow \neg\neg p_1
\end{array}$$

Example 2.8 (Logic for efficient access control). Primal infon logic [16] was designed to efficiently reason about access control policies. The quotations-free fragment of its sequent calculus [13] can be presented as a pure calculus, which we denote by $\mathbf{P}$. It is obtained from the negation-free fragment of $\mathbf{LK}$ by adding the axiomatic rules $\emptyset \,/\; \Rightarrow \top$ and $\emptyset \,/\, \bot \Rightarrow$, dismissing the left introduction rule of disjunction, and replacing the right introduction rule of implication with the following weaker rule:

$$\Rightarrow p_2 \,/\; \Rightarrow p_1 \supset p_2$$

While all the calculi in the examples above admit cut-admissibility, some of them do not enjoy the subformula property, but do enjoy a simple generalization of it, which we introduce in the next section.

3 Analyticity: A Generalized Subformula Property

Roughly speaking, analyticity of a propositional calculus provides a computable bound on the formulas that may appear in derivations of a sequent s from a set S of sequents. The special case of the *subformula property* is obtained when the set of subformulas of formulas in $S \cup \{s\}$ constitutes such a bound. Many useful calculi, however, do not admit this strict property, while still allowing some other effective bound. Here, we generalize the subformula property, by assuming a given ordering of $\mathcal{L}$-formulas, denoted $\prec$, which has to satisfy certain properties, as defined next.

Notation 3.1. Given a binary relation R on $\mathcal{L}$, we denote by $R[\varphi]$ the set $\{\psi \in \mathcal{L} \mid \langle \psi, \varphi \rangle \in R\}$. This notation is naturally extended to sets ($R[\Gamma] = \bigcup_{\varphi \in \Gamma} R[\varphi]$), sequents ($R[\Gamma \Rightarrow \Delta] = R[\Gamma] \cup R[\Delta]$), and sets of sequents ($R[S] = \bigcup_{s \in S} R[s]$).

Definition 3.2. An order relation (i.e., an irreflexive and transitive binary relation) $\prec$ on $\mathcal{L}$ is called:

- *prefinite* if $\prec[\varphi]$ is finite for every $\varphi \in \mathcal{L}$.
- *structural* if $\varphi \prec \psi$ implies $\sigma(\varphi) \prec \sigma(\psi)$ for every substitution σ.

In what follows, $\prec$ denotes an arbitrary prefinite and structural order relation over $\mathcal{L}$. The prefiniteness of $\prec$ will be used in order to enumerate the formulas of the language according to this order. Structurality is essential in order to lift arguments about pure rules to their applications.

Example 3.3. The usual subformula relation over $\mathcal{CL}$, which we denote by $\prec_0$, is a prefinite structural order relation. Another useful prefinite structural order relation on $\mathcal{CL}$, denoted $\prec_1$, is given by $\varphi \prec_1 \psi$ iff $\varphi \prec_0 \psi$ or ($\varphi \neq \psi$ and $\varphi = \neg\psi'$ for some $\psi' \prec_0 \psi$).

The above definition allows us to present a generalization of the subformula property, which we call $\prec$-analyticity.

Definition 3.4. We call a derivation of a sequent s from a set S of sequents in a calculus $\mathbf{G}$ *$\prec$-analytic* if it consists solely of $\preceq[S \cup \{s\}]$-formulas ($\preceq$ denotes the reflexive closure of $\prec$), and write $S \vdash_{\mathbf{G}}^{\preceq} s$ if there exists a $\prec$-analytic derivation of s from S in $\mathbf{G}$. A calculus $\mathbf{G}$ is called *$\prec$-analytic* if $\vdash_{\mathbf{G}} = \vdash_{\mathbf{G}}^{\preceq}$.

This generalization of the subformula property does not necessarily inherit its most important consequence, which is decidability. To ensure decidability, it is also required that the function $\lambda\varphi \in \mathcal{L}.\prec[\varphi]$ is computable. Clearly, if S is finite and $\prec$ admits this property, it is decidable whether $S \vdash_{\mathbf{G}}^{\preceq} s$. When $\mathbf{G}$ is $\prec$-analytic, the same holds for $\vdash_{\mathbf{G}}$.

Remark 3.5. The relation $\vdash_{\mathbf{G}}^{\preceq}$ is non standard, as it may lack a natural transitivity property. For example, consider a calculus $\mathbf{G}$ consisting of the rules $\Rightarrow p_1 \,/\, \Rightarrow p_1 \circ p_2$ and $\Rightarrow p_1 \circ p_2 \,/\, \Rightarrow p_2$, and define $\prec$ so that $\varphi_1 \prec \varphi_1 \circ \varphi_2$ and $\varphi_2 \prec \varphi_1 \circ \varphi_2$ for every φ_1, φ_2. While we have both $\Rightarrow p_1 \vdash_{\mathbf{G}}^{\preceq} \Rightarrow p_1 \circ p_2$ and $\Rightarrow p_1 \circ p_2 \vdash_{\mathbf{G}}^{\preceq} \Rightarrow p_2$, we do *not* have $\Rightarrow p_1 \vdash_{\mathbf{G}}^{\preceq} \Rightarrow p_2$.

Considering the examples above, $\mathbf{LK}$, $\mathbf{G}_{P_1}$ and $\mathbf{P}$ are $\prec_0$-analytic; while $\mathbf{G}_{C_1}$, $\mathbf{G}_3$ and $\mathbf{G}_4$ are not $\prec_0$-analytic, but are $\prec_1$-analytic. These facts can be derived from cut-admissibility, and also by the method presented in [24]. The infinite family of calculi for weak double negations from [20], presented in the next example, goes beyond $\prec_0$ and $\prec_1$.

Example 3.6. In [20], Kamide provides a way of constructing sequent calculi for paraconsistent logics that admit the double negation principle, as well as its weaker forms (e.g., $\neg\neg\neg\psi \leftrightarrow \neg\psi$). For this purpose, the paper investigates a hierarchy of weak double negations, by presenting an infinite set $\{L2^{n+2} \mid n \in \mathbb{N}\}$ of pure calculi, all of which admit cut-admissibility. For example, $L4$ is identical to the calculus $\mathbf{G}_4$ from Example 2.7, and is $\prec_1$-analytic. Furthermore, for every n, let $\prec_n$ be the transitive closure of the relation $\lhd_n$, defined by: $\varphi \lhd_n \psi$ iff either $\psi = \neg\varphi$, or $\psi = \varphi_1 \sharp \varphi_2$ and $\varphi = \neg^m \varphi_i$ for some φ_1, φ_2, $\sharp \in \{\wedge, \vee, \supset\}$, $0 \leq m \leq n$, and $i \in \{1,2\}$ (where $\neg^n$ is defined by: $\neg^0\varphi = \varphi$ and $\neg^{n+1}\varphi = \neg(\neg^n\varphi)$). Each $L2^{n+2}$ is $\prec_{n+1}$-analytic. Clearly, the previous definitions of $\prec_0$ and $\prec_1$ coincide with the new ones.

For many calculi, including those presented above, all rules except (CUT) are *$\prec$-ordered*: in every application of the rule, every formula φ that appears in the premises satisfies $\varphi \preceq \psi$ for some formula ψ that appears in the conclusion. For such calculi, cut-admissibility immediately entails $\prec$-analyticity, as every cut-limited derivation is $\prec$-analytic. Whether or not the converse holds is the subject of the next section.

4 From Analyticity to Cut-admissibility

In this section we address the question of the equivalence of $\prec$-analyticity and cut-admissibility in pure calculi. First, note that $\prec$-analyticity may not imply cut-admissibility:

Example 4.1. Consider the calculus $\mathbf{LK}_{AX}$ that consists of the following axiomatic rules:

$$
\begin{array}{lll}
\emptyset \,/\, p_1, p_2 \Rightarrow p_1 \wedge p_2 & \emptyset \,/\, p_1 \wedge p_2 \Rightarrow p_1 & \emptyset \,/\, p_1 \wedge p_2 \Rightarrow p_2 \\
\emptyset \,/\, p_1 \vee p_2 \Rightarrow p_1, p_2 & \emptyset \,/\, p_1 \Rightarrow p_1 \vee p_2 & \emptyset \,/\, p_2 \Rightarrow p_1 \vee p_2 \\
\emptyset \,/\, p_2 \Rightarrow p_1 \supset p_2 & \emptyset \,/\, \Rightarrow p_1, p_1 \supset p_2 & \emptyset \,/\, p_1, p_1 \supset p_2 \Rightarrow p_2 \\
\emptyset \,/\, \Rightarrow p_1, \neg p_1 & \emptyset \,/\, p_1, \neg p_1 \Rightarrow &
\end{array}
$$

Its rules, besides the two rules of implication, were already identified by Gentzen as equivalent to their original counterparts (see Remark 2.2 in [19]). It can be easily shown that $\mathbf{LK}_{AX}$ is $\prec_0$-analytic (using the fact that $\mathbf{LK}$ is $\prec_0$-analytic). However, it does not admit cut-admissibility (for instance, the sequent $p_1 \wedge p_2 \Rightarrow p_1 \vee p_2$ has no cut-free derivation).

Next, we identify a family of calculi in which analyticity does imply cut-admissibility.

Definition 4.2. A rule S/s is called *$\prec$-ordered* if *frm*$[S] \subseteq \prec [s]$. It is called *$\prec$-directed* if it is $\prec$-ordered, and s has the form $\Rightarrow \varphi$ or $\varphi \Rightarrow$ for some formula φ. A calculus $\mathbf{G}$ is called *$\prec$-ordered* (*$\prec$-directed*) if all its rules are $\prec$-ordered ($\prec$-directed).

The calculi $\mathbf{LK}$, $\mathbf{G}_{P_1}$ and $\mathbf{P}$ are $\prec_0$-directed, $\mathbf{G}_{C_1}$, $\mathbf{G}_3$ and $\mathbf{G}_4$ are $\prec_1$-directed, and for every n, $L2^{n+2}$ is $\prec_{n+1}$-directed. In contrast, $\mathbf{LK}_{AX}$ is not $\prec$-directed for any $\prec$, as its conclusions include several formulas.

Our first main result is that $\prec$-analyticity guarantees cut-admissibility in the family of $\prec$-directed pure calculi.

Theorem 4.3. *Every $\prec$-analytic $\prec$-directed pure calculus enjoys cut-admissibility.*

We prove Theorem 4.3 in Section 4.1. Note that for all the calculi mentioned above (except $\mathbf{LK}_{AX}$), this theorem allows one to obtain cut-admissibility as a consequence of them being $\prec$-directed and $\prec$-analytic for some (prefinite and structural) order $\prec$.

It is worth mentioning that the equivalence of $\prec_0$-analyticity and cut-admissibility was proved in [12] for a subfamily of pure calculi, called *canonical calculi*. These are $\prec_0$-directed pure calculi in which the premises only contain atomic variables and the formula in the conclusion includes a single connective. A simple syntactic criterion called *coherence* was defined in [12] and was proved to be equivalent both to $\prec_0$-analyticity and cut-admissibility.

Before turning to the proof of Theorem 4.3, we outline two possible applications of it in cut-admissibility proofs:

Simpler Semantic Proofs of Cut-admissibility

Theorem 4.3 reduces the burden in proving cut-admissibility to establishing only *analytic cut-admissibility*. An application of (CUT) in a derivation of s from S is called a *$\prec$-analytic cut* if the cut formula is in $\prec[S \cup \{s\}]$. In turn, *$\prec$-analytic cut-admissibility* concerns only the admissibility of non-$\prec$-analytic cuts. Proving this property is often easier than showing full cut-admissibility. For example, it is straightforward to prove that **LK** is complete for the classical truth tables, when applications of (CUT) are restricted to be $\prec_0$-analytic. Indeed, assuming s is not derivable from S in **LK** using $\prec_0$-analytic cuts, one extends s to a maximal underivable sequent s^* that consists solely of $\prec_0[S \cup \{s\}]$-formulas. Then, a countermodel v can be defined simply by setting $v(\varphi) = \mathbf{T}$ for every φ on the left-hand side of s^*, and $v(\psi) = \mathbf{F}$ for every ψ on its right-hand side. Using $\prec_0$-analytic cuts, it immediately follows that $frm[s^*] = \prec_0[S \cup \{s\}]$, which makes it easy to prove that v respects the classical truth tables, and can therefore be extended to a full classical countermodel. By Theorem 4.3, we may conclude that **LK** enjoys cut-admissibility. Providing a semantic proof of cut-admissibility without going through $\prec_0$-analytic cuts is possible, but more complicated.

Sufficient Criterion for Cut-admissibility

The results of this paper are useful in combination with our recent paper [24], where we provided a general method for proving $\prec_n$-analyticity (see Example 3.6 for the definition of $\prec_n$) in a wide family of pure calculi. Concretely, we showed that the $\prec_n$-analyticity of a $\prec_n$-directed calculus $\mathbf{G}$ is guaranteed if the following property holds:[3]

> $(*)$ For every two rules of $\mathbf{G}$ of the forms $S_1 \,/ \Rightarrow \varphi_1$ and $S_2 \,/\, \varphi_2 \Rightarrow$, and substitutions σ_1, σ_2 such that $\sigma_1(\varphi_1) = \sigma_2(\varphi_2)$, the empty sequent is derivable from $\sigma(S_1) \cup \sigma(S_2)$ using only (CUT).

A direct consequence of Theorem 4.3 and the results of [24] is a simple, syntactic and decidable criterion for cut-admissibility:

Corollary 4.4. *Every pure $\prec_n$-directed calculus for which $(*)$ holds enjoys cut-admissibility.*

Cut-admissibility for all calculi considered above (except for $\mathbf{LK}_{AX}$) can be obtained by applying Corollary 4.4.

4.1 Proof of Theorem 4.3

Cut-admissibility is traditionally proved syntactically, by some form of induction on derivations. In this case, what is actually shown is *cut-elimination*: a method to eliminate cuts from derivations. However, going back at least to [35], semantic methods have also shown to be useful to prove cut-admissibility. We follow the semantic approach, and base our proof on several "semantic maneuvers". We start by presenting the bivaluation semantic framework from [14] that accounts for derivability in pure sequent

[3]This generalization of the coherence condition from [12] is also similar to the absorption of cut from [30].

calculi. Then, we generalize this framework to obtain a characterization of $\prec$-analytic derivations, as well as of cut-limited derivations. The latter characterization allows us to define a semantic sufficient condition for cut-admissibility, which we show to hold in $\prec$-analytic $\prec$-directed calculi.

The soundness and completeness theorems of this section can be obtained using the general framework of [23]. To assist the reader, we outline in the appendix direct proofs of these theorems, while employing our current notations and definitions.

4.1.1 Semantics of Pure Sequent Calculi

We start by presenting the bivaluations semantics from [14] that captures derivability in pure calculi, and is based on a two-valued interpretation of pure rules.

Definition 4.5. An *$\mathcal{L}$-bivaluation* is a function v from $\mathcal{L}$ to $\{-1, 1\}$. We say that v *satisfies a sequent* $\Gamma \Rightarrow \Delta$, denoted $v \models \Gamma \Rightarrow \Delta$, if either $v(\varphi) = -1$ for some $\varphi \in \Gamma$ or $v(\psi) = 1$ for some $\psi \in \Delta$. We say that v *satisfies a set S of sequents*, denoted $v \models S$, if $v \models s$ for every $s \in S$.

Definition 4.6. A bivaluation v *respects* a rule $S \,/\, s$ if $v \models \sigma(s)$ whenever $v \models \sigma(S)$ for every substitution σ. v is called $\mathbf{G}$-*legal* for a calculus $\mathbf{G}$ if it respects all rules of $\mathbf{G}$.

Depending on $\mathbf{G}$, this semantics may not be truth-functional, that is, the value of a compound formula is *not* uniquely determined by the values of its immediate subformulas. For this reason valuation functions are defined over the entire language rather than only over atomic formulas.

Example 4.7 (Semantics of Classical Logic). It is easy to see that a $\mathcal{CL}$-bivaluation v is $\mathbf{LK}$-legal iff it respects the classical truth tables. For example, the first line of the truth table for conjunction is obtained as follows: Suppose that $v(\varphi) = v(\psi) = 1$. Then $v \models \{\Rightarrow \varphi, \ \Rightarrow \psi\}$, and since v respects the right introduction rule of conjunction, we have $v \models \ \Rightarrow \varphi \wedge \psi$, and so $v(\varphi \wedge \psi) = 1$.

The semantic reading of rules as constraints on bivaluations provides an equivalent semantic view of derivations:

Theorem 4.8. *$S \vdash_{\mathbf{G}} s$ iff $v \models S$ implies $v \models s$ for every $\mathbf{G}$-legal bivaluation v.*

Example 4.9. $\mathbf{G}_3$-legal bivaluations provide an alternative semantics to Łukasiewicz three-valued logic (Example 2.7). For example, the rule

$$\Rightarrow p_1 \,;\ \Rightarrow \neg p_2 \,/\, \Rightarrow \neg(p_1 \supset p_2)$$

translates into the following semantic condition:

$$\text{If } v(\varphi) = 1 \text{ and } v(\neg\psi) = 1 \text{ then } v(\neg(\varphi \supset \psi)) = 1.$$

This semantics is two-valued, but not truth-functional. Another two-valued semantics for this logic was presented in [36], and was then used to construct a different calculus for it in [14].

4.1.2 Semantics of Analytic Derivations

Theorem 4.8 establishes a strong connection between derivations in pure calculi and bivaluations (two-valued assignments). Analytic derivations in such calculi induce a similar semantics, which is based on *three-valued* truth assignments. Roughly speaking, in the case that $\prec$-analytic derivations are taken into account, only $\preceq[S \cup \{s\}]$-formulas are allowed to occur in derivations, and thus we only need these formulas to be assigned with a truth-value. To implement this idea, we include a third truth-value marking formulas that are not assigned with any "real" truth-value. Thus, the semantics for analytic derivations is based on three truth-values: "false", "true", and "indeterminate", represented as -1, 1, and 0, respectively.

Definition 4.10. An *$\mathcal{L}$-trivaluation* is a function v from $\mathcal{L}$ to $\{-1, 0, 1\}$. We say that v *satisfies a sequent* $\Gamma \Rightarrow \Delta$, denoted $v \models \Gamma \Rightarrow \Delta$, if either $v(\varphi) < 1$ for some $\varphi \in \Gamma$ or $v(\psi) > -1$ for some $\psi \in \Delta$. v *satisfies a set S of sequents*, denoted $v \models S$, if $v \models s$ for every $s \in S$. $\mathbf{G}$-legal trivaluations are defined like in the case of bivaluations using the new definition of $\models$.

Example 4.11. Using trivaluations, the semantic condition that is induced by the rule of Example 4.9 is:

$$\text{If } v(\varphi) > -1 \text{ and } v(\neg\psi) > -1 \text{ then } v(\neg(\varphi \supset \psi)) > -1.$$

When semantically describing the existence of a $\prec$-analytic derivation of a sequent s from a set S of sequents in a calculus $\mathbf{G}$, some formulas must be assigned with either 1 or -1. These are the formulas that are allowed to appear in such a derivation, that is, the formulas in $\preceq[S \cup \{s\}]$. For this requirement, we introduce the following definition:

Definition 4.12. The *support* of a trivaluation v, denoted $\mathrm{supp}(v)$, is the set $\{\varphi \in \mathcal{L} \mid v(\varphi) \neq 0\}$. v is called:

- *$\mathcal{F}$-determined* (for $\mathcal{F} \subseteq \mathcal{L}$) if $\mathcal{F} \subseteq \mathrm{supp}(v)$; and
- *fully determined* if it is $\mathcal{L}$-determined.

Note that every trivaluation is $\emptyset$-determined, and that bivaluations are fully determined trivaluations (and they will be called this way from now on).

Trivaluations correspond to analytic derivations in the following way:

Theorem 4.13. *If $S \vdash^{\preceq}_{\mathbf{G}} s$ then $v \models S$ implies $v \models s$ for every $\preceq[S \cup \{s\}]$-determined $\mathbf{G}$-legal trivaluation v. Moreover, if $\mathbf{G}$ is $\prec$-ordered, then the converse also holds.*

Note that only *soundness* of analytic derivations with respect to $\preceq[S \cup \{s\}]$-determined trivaluations is needed in the proof of Theorem 4.3.

4.1.3 Semantics in the Absence of Cut

The semantics of cut-limited derivations is similar to that of analytic derivations. The only difference is that cut-limited derivations of s from S are tied to $\mathit{frm}[S]$-determined trivaluations, rather than to $\preceq[S \cup \{s\}]$-determined ones. Intuitively, if φ cannot serve as a cut formula, we may need a trivaluation v that satisfies $\Rightarrow \varphi$ and $\varphi \Rightarrow$, which is possible iff $v(\varphi) = 0$.

For cut-limited derivations, we have the following:

Theorem 4.14. *$S \vdash^{\mathrm{cf}}_{\mathbf{G}} s$ iff $v \models S$ implies $v \models s$ for every $\mathit{frm}[S]$-determined $\mathbf{G}$-legal trivaluation v.*

Note that only *completeness* of cut-limited derivations with respect to $\mathit{frm}[S]$-determined trivaluations is used in the proof of Theorem 4.3.

Remark 4.15. The three-valued semantics for the cut-free fragment of $\mathbf{LK}$ that is obtained from Theorem 4.14 induces the same set of trivaluations as the Nmatrix semantics from [22].

Theorem 4.14 gives rise to a sufficient semantic criterion for cut-admissibility, which is based on the following notion of *determination*:

Definition 4.16. We say that a trivaluation v' is a *determination* of a trivaluation v (alternatively, we say that v' *determines* v) if $v(\varphi) = v'(\varphi)$ for every $\varphi \in \mathrm{supp}(v)$. v' is called an *$\mathcal{F}$-determination of* v if, in addition, it is $\mathcal{F}$-determined. If v' is fully determined, we call it a *full determination* of v.

It immediately follows from our definitions that:

Proposition 4.17. *Suppose that v' determines v. Then for every sequent s, if $v' \models s$ then $v \models s$. The converse holds as well when v is $frm[s]$-determined.*

The sufficient semantic criterion for cut-admissibility is given in the following corollary:

Corollary 4.18. *If every* $\mathbf{G}$*-legal trivaluation has a* $\mathbf{G}$*-legal full determination, then* $\mathbf{G}$ *enjoys cut-admissibility.*

Proof. Suppose that $S \nvdash^{\text{cf}}_{\mathbf{G}} s$. By Theorem 4.14, there exists some $frm[S]$-determined $\mathbf{G}$-legal trivaluation v such that $v \models S$ and $v \not\models s$. Let v' be a $\mathbf{G}$-legal full determination of v. By Proposition 4.17, $v' \models S$ and $v' \not\models s$, and by Theorem 4.8, it follows that $S \nvdash_{\mathbf{G}} s$. □

4.1.4 From Analyticity to Cut-admissibility

We now prove Theorem 4.3, by utilizing Corollary 4.18. Thus, given a $\prec$-analytic $\prec$-directed pure calculus, we show that every $\mathbf{G}$-legal trivaluation has a $\mathbf{G}$-legal full determination. This is done in two steps: Lemma 4.19 below shows that it is possible to add a single formula to the support of a $\mathbf{G}$-legal trivaluation. The extended trivaluation is proved to still be $\mathbf{G}$-legal. Then, in Lemma 4.20, we iteratively apply Lemma 4.19 using an enumeration of all formulas that respects $\prec$, thus obtaining a full determination.

Lemma 4.19. *Let* $\mathbf{G}$ *be a $\prec$-analytic $\prec$-directed calculus, v a* $\mathbf{G}$*-legal trivaluation and ψ a formula such that $\prec[\psi] \subseteq \mathrm{supp}(v)$. Then v has a* $\mathbf{G}$*-legal* $\mathrm{supp}(v) \cup \{\psi\}$*-determination.*

Proof. If $\psi \in \mathrm{supp}(v)$, then this is trivial, as v determines itself. We therefore assume that $\psi \notin \mathrm{supp}(v)$. Let $\Gamma_v = \{\varphi \in \prec[\psi] \mid v(\varphi) = 1\}$ and $\Delta_v = \{\varphi \in \prec[\psi] \mid v(\varphi) = -1\}$. We first show that $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \Delta_v$: v is $\preceq[\Gamma_v \Rightarrow \Delta_v]$-determined, as $\preceq[\Gamma_v \Rightarrow \Delta_v] \subseteq \prec[\psi] \subseteq \mathrm{supp}(v)$. Also, $v \not\models \Gamma_v \Rightarrow \Delta_v$. By Theorem 4.13, $\nvdash^{\prec}_{\mathbf{G}} \Gamma_v \Rightarrow \Delta_v$, and since $\mathbf{G}$ is $\prec$-analytic, $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \Delta_v$. Let v' be the trivaluation obtained from v by setting

$$v'(\psi) = \begin{cases} 1 & \nvdash_{\mathbf{G}} \Gamma_v, \psi \Rightarrow \Delta_v \\ -1 & otherwise \end{cases}$$

Clearly, v' is a $\mathrm{supp}(v) \cup \{\psi\}$-determination of v. We prove that it is $\mathbf{G}$-legal. Let $s_1, \ldots, s_n \,/\, s$ be a rule of $\mathbf{G}$ and σ a substitution. Suppose that $v' \not\models \sigma(s)$. We prove that $v' \not\models \sigma(s_i)$ for some $1 \leq i \leq n$. By our assumption, $frm[\sigma(s)] \subseteq \mathrm{supp}(v') = \mathrm{supp}(v) \cup \{\psi\}$. If $frm[\sigma(s)] \subseteq \mathrm{supp}(v)$, then this follows from the fact that v is $\mathbf{G}$-legal and v' determines v. Otherwise, $frm[\sigma(s)] \not\subseteq \mathrm{supp}(v)$. Since $\mathbf{G}$ is $\prec$-directed, we have $frm[\sigma(s)] = \{\psi\}$. Also, for every $1 \leq i \leq n$, we have $frm[s_i] \subseteq \prec[s]$, and since $\prec$ is structural, $frm[\sigma(s_i)] \subseteq \sigma(\prec[s]) \subseteq \prec[\sigma(s)] = \prec[\psi]$.

To show that there exists $1 \leq i \leq n$ such that $v' \not\models \sigma(s_i)$, we first prove that $\nvdash_{\mathbf{G}} \sigma(s) \cup (\Gamma_v \Rightarrow \Delta_v)$: by the above, either $\sigma(s) = (\psi \Rightarrow)$ or $\sigma(s) = (\Rightarrow \psi)$. In the first case, $v'(\psi) = 1$, which means that $\nvdash_{\mathbf{G}} \Gamma_v, \psi \Rightarrow \Delta_v$. In the second, $v'(\psi) = -1$, which means that $\vdash_{\mathbf{G}} \Gamma_v, \psi \Rightarrow \Delta_v$. Using (CUT), since $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \Delta_v$, we have $\nvdash_{\mathbf{G}} \Gamma_v \Rightarrow \psi, \Delta_v$. Either way, we have $\nvdash_{\mathbf{G}} \sigma(s) \cup (\Gamma_v \Rightarrow \Delta_v)$.

Since $\nvdash_{\mathbf{G}} \sigma(s) \cup (\Gamma_v \Rightarrow \Delta_v)$, we must have some $1 \leq i \leq n$ such that $\nvdash_{\mathbf{G}} \sigma(s_i) \cup (\Gamma_v \Rightarrow \Delta_v)$. Suppose that $s_i = \Gamma_i \Rightarrow \Delta_i$. Then using the fact that $frm[\sigma(s_i)] \subseteq \prec[\psi] = \Gamma_v \cup \Delta_v$, we have $\sigma(\Gamma_i) \subseteq \Gamma_v$ and $\sigma(\Delta_i) \subseteq \Delta_v$. Thus $v \not\models \sigma(s_i)$, and therefore $v' \not\models \sigma(s_i)$. □

Lemma 4.20. *Let* $\mathbf{G}$ *be a $\prec$-analytic $\prec$-directed calculus and v a* $\mathbf{G}$*-legal trivaluation. Then there exists a* $\mathbf{G}$*-legal full determination of v.*

Proof. For every trivaluation u and formula ψ, let u_ψ denote an arbitrary $\mathbf{G}$-legal $\mathrm{supp}(u) \cup \{\psi\}$-determination of u, if such exists (otherwise, u_ψ is undefined). Note that whenever u is $\mathbf{G}$-legal and $\prec[\psi] \subseteq \mathrm{supp}(u)$, Lemma 4.19 ensures that u_ψ is defined.

Let $\psi_1, \psi_2, \dots$ be an enumeration of the formulas of $\mathcal{L}$, such that $i < j$ whenever $\psi_i \prec \psi_j$. Such enumeration exists by the prefiniteness of $\prec$. For every i, denote the set $\{\psi_1, \dots, \psi_i\}$ by $\Phi_{\leq i}$ (in particular, $\Phi_{\leq 0} = \emptyset$).

Define a sequence $v^0, v^1, \dots$ of trivaluations as follows: $v^0 = v$, and $v^i = v^{i-1}_{\psi_i}$ for every $i > 0$. We prove by induction on i that v^i is defined, and is a $\mathbf{G}$-legal $\Phi_{\leq i}$-determination of v and of v^{i-1} (for $i \geq 1$).

For $i = 0$, this trivially holds. Let $i > 0$. By the induction hypothesis, v^{i-1} is $\mathbf{G}$-legal and is a $\Phi_{\leq i-1}$-determination of v. By the enumeration, $\prec[\psi_i] \subseteq \Phi_{\leq i-1} \subseteq \mathrm{supp}(v^{i-1})$. Thus, v^i is defined, and is a $\mathbf{G}$-legal $\mathrm{supp}(v^{i-1}) \cup \{\psi_i\}$-determination of v^{i-1}, and thus also of v. Now, $\Phi_{\leq i} = \Phi_{\leq i-1} \cup \{\psi_i\} \subseteq \mathrm{supp}(v^{i-1}) \cup \{\psi_i\} \subseteq \mathrm{supp}(v^i)$, and therefore v^i is actually a $\Phi_{\leq i}$-determination of v^{i-1} and of v.

We now define v', a $\mathbf{G}$-legal full determination of v. For every $\varphi \in \mathcal{L}$, let i_φ be the index of φ in the enumeration (that is, $\varphi = \psi_{i_\varphi}$). Define $v' = \lambda\varphi \in \mathcal{L}.v^{i_\varphi}(\varphi)$.

For every $\varphi \in \mathcal{L}$, $v'(\varphi) = v^{i_\varphi}(\varphi) \neq 0$, as v^{i_φ} is $\Phi_{\leq i_\varphi}$-determined. Also, for every $\varphi \in \mathrm{supp}(v)$, $v'(\varphi) = v^{i_\varphi}(\varphi) = v(\varphi)$, as v^{i_φ} determines v. Finally, let $S \,/\, s$ be a rule of $\mathbf{G}$ and σ be a substitution. Let $k = \min\{i \mid \mathit{frm}[\sigma(S \cup \{s\})] \subseteq \Phi_{\leq i}\}$. Then $v'(\varphi) = v^k(\varphi)$ for every $\varphi \in \mathit{frm}[\sigma(S \cup \{s\})]$. Therefore, if $v' \models \sigma(S)$, then $v^k \models \sigma(S)$ as well. Since v^k is $\mathbf{G}$-legal, $v^k \models \sigma(s)$, and therefore $v' \models \sigma(s)$. $\square$

Putting all pieces together we have proved Theorem 4.3: Assuming that G is $\prec$-analytic and $\prec$-directed, by Lemma 4.20, every $\mathbf{G}$-legal $\mathcal{L}$-trivaluation has a $\mathbf{G}$-legal full determination, and cut-admissibility follows by Corollary 4.18.

5 Intuitionistic Calculi

For various important non-classical logics, such as intuitionistic logic, there is no known cut-free pure calculus. Gentzen's original calculus for this logic, LJ, is not pure, as it does not meet the requirement of allowing arbitrary context formulas in applications of rules: it manipulates *single-conclusion sequents*, in which the right-hand side includes at most one formula. There exists, however, an equivalent multiple-conclusion cut-free sequent calculus, which we call $\mathbf{LJ}'$ (see [38]). This calculus restricts only the right introduction rules of implication and negation to single-conclusion sequents. In other words, $\mathbf{LJ}'$ is obtained from $\mathbf{LK}$ by restricting applications of $p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$ and $p_1 \Rightarrow \,/\, \Rightarrow \neg p_1$ to have the forms:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$$

Put differently, $\mathbf{LJ}'$ is obtained from $\mathbf{LK}$ by *forbidding right context formulas in all premises with a non-empty left side of right introduction rules.*

Another well-known calculus that follows this pattern, which we call $\mathbf{G}'_4$, is obtained by extending the negation-free fragment of $\mathbf{LJ}'$ with the rules for negation of $\mathbf{G}_4$ (see Example 2.7). $\mathbf{G}'_4$, investigated in [7, 39], is sound and complete for Nelson's paraconsistent constructive logic $N4$ [28].

Next, we define a general family of calculi, which we call *intuitionistic calculi*, of which $\mathbf{LJ}'$ and $\mathbf{G}'_4$ are particular examples. Like $\mathbf{LJ}'$ and $\mathbf{G}'_4$, intuitionistic calculi are sets of pure rules, but some of the applications of these rules are restricted. As in the previous section, we show that cut-admissibility is a consequence of $\prec$-analyticity in calculi of this family. Our proof, presented in Section 5.3, has a similar general structure as the proof for pure calculi, but is more challenging, because simple valuation functions do not suffice to characterize the calculi of this family. Instead, we introduce in Section 5.2 a more complex semantic interpretation, which is based on Kripke models.

5.1 Intuitionistic Derivations

In this section we precisely define the family of intuitionistic calculi, of which $\mathbf{LJ}'$ and $\mathbf{G}_4'$ are particular examples. Note that both $\mathbf{LJ}'$ and $\mathbf{G}_4'$ employ pure rules: $\mathbf{LJ}'$ is based on the rules of $\mathbf{LK}$, and $\mathbf{G}_4'$ is based on the rules of $\mathbf{G}_4$. The difference lies in the allowed applications of the rules: both calculi forbid right context formulas in premises of the form $\Gamma \Rightarrow \Delta$ with $\Gamma \neq \emptyset$ of applications of rules that introduce some formula on the right-hand side. This is formalized as follows:

Definition 5.1. A pure rule is called *positive* if its conclusion has the form $\Gamma \Rightarrow \Delta$ for some $\Delta \neq \emptyset$. A derivation in a pure calculus $\mathbf{G}$ is called *intuitionistic* if in every application $\langle \{\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \ldots \cup c_n \rangle$ of a positive rule $s_1, \ldots, s_n \,/\, s$, for every $1 \leq i \leq n$ we have that if s_i has the form $\Gamma_i \Rightarrow \Delta_i$ with $\Gamma_i \neq \emptyset$, then c_i has the form $\Gamma_i' \Rightarrow$.

Derivability, cut-admissibility and $\prec$-analyticity are adapted to intuitionistic derivations in the obvious way:

Definition 5.2. For a pure calculus $\mathbf{G}$, we write $S \vdash_{\mathbf{G_{Int}}} s$ if there is an intuitionistic derivation of a sequent s from a set S of sequents in $\mathbf{G}$. We write $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} s$ if there is such a derivation that is also cut-limited, and $S \vdash^{\prec}_{\mathbf{G_{Int}}} s$ if there is such a derivation that is $\prec$-analytic (see Definitions 2.4 and 3.4). We say that $\mathbf{G}$ enjoys *Int-cut-admissibility* if $\vdash_{\mathbf{G_{Int}}} = \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}}$ and is *Int-$\prec$-analytic* if $\vdash_{\mathbf{G_{Int}}} = \vdash^{\prec}_{\mathbf{G_{Int}}}$.

The difference between pure and intuitionistic calculi is not in the *rules*, but rather in *applications* that are allowed to appear in derivations. Thus, any pure calculus has an intuitionistic counterpart, obtained by considering only intuitionistic derivations.

Example 5.3. Derivations in $\mathbf{LJ}'$ are exactly the intuitionistic derivations of $\mathbf{LK}$. Indeed, a formula φ follows from a finite set Γ of formulas in intuitionistic logic iff $\vdash_{\mathbf{LK_{Int}}} \Gamma \Rightarrow \varphi$. In contrast, φ follows from Γ in classical logic iff $\vdash_{\mathbf{LK}} \Gamma \Rightarrow \varphi$. Similarly, derivations in $\mathbf{G}_4'$ coincide with intuitionistic derivations of $\mathbf{G}_4$.

Our main theorem concerning intuitionistic calculi is presented next.

Theorem 5.4. *Every Int-$\prec$-analytic $\prec$-directed pure calculus enjoys Int-cut-admissibility.*

Theorem 5.4, which we prove below, allows one to derive the fact that cut is admissible in $\mathbf{LJ}'$ from the fact that $\mathbf{LJ}'$ enjoys the subformula property. More precisely, Int-cut-admissibility of $\mathbf{LK}$ follows from its Int-$\prec_0$-analyticity. Such entailment also holds for the pure calculi presented in the examples above, as well as for the calculi of the next example.

Example 5.5 (Constructive Negations). In [7], Avron provides sequent calculi for logics that replace classical negation with several non-classical negations. One of the families investigated there consists of calculi that are obtained from the negation-free fragment of $\mathbf{LJ}'$ by augmenting it with pure rules for negation. All calculi of this family, except those described in Example 5.6 below, allow only intuitionistic derivations, and are $\prec_1$-directed and Int-$\prec_1$-analytic. From these facts, Theorem 5.4 allows us to conclude that cut is admissible in them. These calculi include a calculus for Nelson's constructive logic N_3 [28], as well as the calculus $\mathbf{G}_4'$ presented above for its paraconsistent variant N_4.

Intuitionistic derivations disallow right context formulas in premises of positive rules, in which the left-hand side is not empty. A natural question that arises regarding Theorem 5.4 is: Does it still hold if we allow right context formulas for certain premises of a right introduction rule with a non-empty left-hand side, and forbid them in others? The answer is negative as the next example demonstrates.

Example 5.6 (Beyond Intuitionistic Derivations). Following Example 5.5, we note that [7, 8] investigate also several calculi that include *both* the single-conclusion right introduction rule of implication and the multiple-conclusion right introduction rule of negation. The former conforms with the restriction to intuitionistic derivations, as right context formulas are forbidden. The latter allows for non-intuitionistic derivations, as it allows right context formulas in a premise that has a non-empty left side. Such calculi are therefore left out from the scope of Theorems 4.3 and 5.4. Indeed, as shown in [8], all of them are $\prec_1$-analytic, but none of them enjoys cut-admissibility. A particular example for such a calculus is the calculus from [34] for da-Costa's paraconsistent logic C_w [17], obtained from the positive fragment of $\mathbf{LJ}'$ by adding the pure rules $p_1 \Rightarrow \;/\; \neg\neg p_1 \Rightarrow$ and $p_1 \Rightarrow \;/\; \Rightarrow \neg p_1$.

The rest of this section is devoted to the proof of Theorem 5.4. Similarly to the case of pure calculi, we go through a semantic interpretation of intuitionistic derivations, which is defined next.

5.2 Semantics of Intuitionistic Calculi

The syntactic restriction on context sequents that are enforced in intuitionistic derivations requires elevating bivaluation semantics (Section 4.1) to Kripke-style semantics.

Definition 5.7. An *$\mathcal{L}$-trimodel* is a triple $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$, where W is a non-empty set (whose elements are called "worlds"), R is a transitive and reflexive relation over W, and $\mathcal{V}$ is a function that assigns a trivaluation $\mathcal{V}_w$ to every $w \in W$, such that $\mathcal{V}$ is *persistent*, that is, for every $\varphi \in \mathcal{L}$ and $w \in W$, if $\mathcal{V}_w(\varphi) = 1$ then $\mathcal{V}_u(\varphi) = 1$ for every u such that wRu.

Next we introduce several useful notations and definitions concerning trimodels.

Notation 5.8. Let $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ be a trimodel and $w \in W$. We denote the set $\{u \in W \mid wRu\}$ by $R\,(w)$, and the set $\{\mathcal{V}_u \mid u \in R\,(w)\}$ by $\mathcal{V}_{R(w)}$.

Definition 5.9. Let Σ be a set of trivaluations. Σ *satisfies* a sequent s, denoted $\Sigma \models s$, if $v \models s$ for every $v \in \Sigma$ (see Definition 4.10). Σ *satisfies* a set S of sequents, denoted $\Sigma \models S$, if $\Sigma \models s$ for every $s \in S$. For a trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$, we write $\mathcal{M} \models s$ ($\mathcal{M} \models S$) if $\{\mathcal{V}_w \mid w \in W\} \models s$ ($\{\mathcal{V}_w \mid w \in W\} \models S$).

We turn to the semantic reading of pure rules in trimodels. Given a trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ and a world w in it, the interpretation of a pure rule in w may involve not only w itself, but also the elements of $R\,(w)$:

Definition 5.10. A trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ *respects* a rule $r = S \,/\, s$ if one of the following holds for every $w \in W$ and substitution σ:

- r is positive, and $\mathcal{V}_w \models \sigma(s)$ whenever both of the followings hold:
    1. $\mathcal{V}_w \models \sigma(\Rightarrow \Delta)$ for every $\Rightarrow \Delta \in S$.
    2. $\mathcal{V}_{R(w)} \models \sigma(s')$ for every $s' \in S$ with a non-empty left side.
- r is not positive, and $\mathcal{V}_w \models \sigma(s)$ whenever $\mathcal{V}_w \models \sigma(S)$.

Example 5.11. Let $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ be a trimodel. $\mathcal{M}$ respects the right introduction rule of conjunction in $\mathbf{LK}$ iff for every $w \in W$ we have that $\mathcal{V}_w(\varphi \wedge \psi) > -1$ whenever $\mathcal{V}_w(\varphi) > -1$ and $\mathcal{V}_w(\psi) > -1$. $\mathcal{M}$ respects the left introduction rule of implication in $\mathbf{LK}$ iff for every $w \in W$ we have that $\mathcal{V}_w(\varphi \supset \psi) < 1$ whenever $\mathcal{V}_w(\varphi) > -1$ and $\mathcal{V}_w(\psi) < 1$. $\mathcal{M}$ respects the right introduction rule of implication in $\mathbf{LK}$ iff for every $w \in W$ we have that $\mathcal{V}_w(\varphi \supset \psi) > -1$ whenever either $\mathcal{V}_u(\varphi) < 1$ or $\mathcal{V}_u(\psi) > -1$ for every $u \in R\,(w)$.

Definition 5.12. For a pure calculus $\mathbf{G}$, a trimodel $\mathcal{M}$ is called $\mathbf{G}$-legal if it respects the rules of $\mathbf{G}$.

The semantic reading of derivation rules in Definitions 5.10 and 5.12 conforms with the essence of Kripke's semantics for intuitionistic logic, according to which "constructive truth" is more demanding than "classical truth". Thus, when introducing a formula on the right-hand side, some premises should be satisfied not only in the particular world in question, but also in all its accessible worlds.

Finally, the notion of determined trivaluations is generalized to trimodels in the most natural way:

Definition 5.13. For a set $\mathcal{F} \subseteq \mathcal{L}$, a trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ is called *$\mathcal{F}$-determined* if $\mathcal{V}_w$ is $\mathcal{F}$-determined for every $w \in W$. $\mathcal{M}$ is called *fully determined* if $\mathcal{V}_w$ is fully determined for every $w \in W$.

Example 5.14 (Semantics of Intuitionistic Logic). Recall that $\mathbf{LJ}'$ is obtained by considering only intuitionistic derivations in $\mathbf{LK}$. Accordingly, Definition 5.12 associates $\mathbf{LJ}'$ with $\mathbf{LK}$-legal *trimodels*. Following Example 5.11, it is easy to see that a fully determined $\mathcal{CL}$-trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ is $\mathbf{LK}$-legal iff it respects the usual truth conditions of the intuitionistic connectives. For example, the truth condition for implication is obtained as follows: Let $w \in W$, and suppose that for every $u \in W$ such that wRu, we have that if $\mathcal{V}_u(\varphi) = 1$ then also $\mathcal{V}_u(\psi) = 1$. Then $\mathcal{V}_{R(w)} \models \varphi \Rightarrow \psi$. Since $\mathcal{M}$ is $\mathbf{LK}$-legal, it respects the positive rule $p_1 \Rightarrow p_2 \,/ \Rightarrow p_1 \supset p_2$, and so $\mathcal{V}_w \models \ \Rightarrow \varphi \supset \psi$, which gives us $\mathcal{V}_w(\varphi \supset \psi) = 1$. For the other direction of the semantics of $\supset$, suppose $\mathcal{V}_u(\varphi) = 1$ and $\mathcal{V}_u(\psi) = -1$ for some $u \in W$ such that wRu. Since $\mathcal{M}$ is $\mathbf{LK}$-legal, it respects the non-positive rule $\Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow$, which means that $\mathcal{V}_u \models \varphi \supset \psi \Rightarrow$. This gives us $\mathcal{V}_u(\varphi \supset \psi) = -1$, and by the persistence condition, since $\mathcal{M}$ is fully determined, it follows that $\mathcal{V}_w(\varphi \supset \psi) = -1$.

After setting all required adjustments in definitions, Theorems 4.8, 4.13 and 4.14 are now given an intuitionistic variant. Like in the case of pure calculi, the following theorem can be obtained using the general framework of [23]. We outline in the appendix a direct proof of this theorem.

Theorem 5.15.

1. $S \vdash_{\mathbf{G_{Int}}} s$ *iff* $\mathcal{M} \models S$ *implies* $\mathcal{M} \models s$ *for every fully determined* $\mathbf{G}$*-legal trimodel* $\mathcal{M}$.
2. *If* $S \vdash^{\prec}_{\mathbf{G_{Int}}} s$ *then* $\mathcal{M} \models S$ *implies* $\mathcal{M} \models s$ *for every* $\preceq[S \cup \{s\}]$*-determined* $\mathbf{G}$*-legal trimodel* $\mathcal{M}$*. Moreover, if* $\mathbf{G}$ *is* $\prec$*-ordered, then the converse holds as well.*
3. $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} s$ *iff* $\mathcal{M} \models S$ *implies* $\mathcal{M} \models s$ *for every* $frm[S]$*-determined* $\mathbf{G}$*-legal trimodel* $\mathcal{M}$.

Theorem 5.15 gives rise to a sufficient semantic criterion for Int-cut-admissibility that is a variant of Corollary 4.18:

Definition 5.16. We say that a trimodel $\mathcal{M}' = \langle W', R', \mathcal{V}'\rangle$ is a *determination* of a trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$ (or that $\mathcal{M}'$ *determines* $\mathcal{M}$) if $W = W'$, $R = R'$, and $\mathcal{V}'_w$ is a determination of $\mathcal{V}_w$ for every $w \in W$. $\mathcal{M}'$ is called an *$\mathcal{F}$-determination* of $\mathcal{M}$ if, in addition, it is $\mathcal{F}$-determined. If $\mathcal{M}'$ is fully determined, we call it a *full determination* of $\mathcal{M}$.

Corollary 5.17. *If every* $\mathbf{G}$*-legal trimodel has a* $\mathbf{G}$*-legal full determination, then* $\mathbf{G}$ *enjoys Int-cut-admissibility.*

Proof. Suppose that $S \nvdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} s$. By Item 3 in Theorem 5.15, there exists some $frm[S]$-determined $\mathbf{G}$-legal trimodel $\mathcal{M}$ such that $\mathcal{M} \not\models s$ while $\mathcal{M} \models S$. $\mathcal{M}$ has a $\mathbf{G}$-legal full determination $\mathcal{M}'$. By Proposition 4.17, $\mathcal{M}' \not\models s$ and $\mathcal{M}' \models S$. By Item 1 in Theorem 5.15, we have that $S \nvdash_{\mathbf{G_{Int}}} s$. $\square$

5.3 Proof of Theorem 5.4

The semantic view of intuitionistic calculi via Kripke-style semantics enables us to provide appropriate variants of Lemmas 4.19 and 4.20 above, and then, together with Corollary 5.17, to derive Theorem 5.4. We start by showing that a single formula can be added to the support of a given trimodel.

Definition 5.18. The support of a trimodel $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$, denoted $\mathrm{supp}(\mathcal{M})$, is the set $\bigcap\{\mathrm{supp}(\mathcal{V}_w) \mid w \in W\}$.

Lemma 5.19. *Let* $\mathbf{G}$ *be an Int-*$\prec$*-analytic* $\prec$*-directed calculus,* $\mathcal{M}$ *a* $\mathbf{G}$*-legal trimodel and* ψ *a formula such that* $\prec[\psi] \subseteq \mathrm{supp}(\mathcal{M})$*. Then* $\mathcal{M}$ *has a* $\mathbf{G}$*-legal* $\mathrm{supp}(\mathcal{M}) \cup \{\psi\}$*-determination.*

Proof. Let $\mathcal{M} = \langle W, R, \mathcal{V}\rangle$. For every $w \in W$, define the following sets:

- $\Gamma_w = \{\varphi \in \prec[\psi] \mid \mathcal{V}_w(\varphi) = 1\}$;
- $\Delta_w = \{\varphi \in \prec[\psi] \mid \mathcal{V}_w(\varphi) = -1\}$; and
- $S_w = \{s \mid \mathit{frm}[s] \subseteq \prec[\psi], \mathcal{V}_{R(w)} \models s\}$.

We start by proving that for every $w \in W$, $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w \Rightarrow \Delta_w$. Let $w \in W$. Define the following trimodel $\mathcal{M}^* = \langle W^*, R^*, \mathcal{V}^*\rangle$: $W^* = R(w)$, $R^* = R \cap (W^* \times W^*)$, and $\mathcal{V}^*_u = \mathcal{V}_u$ for every $u \in W^*$. $\mathcal{M}^*$ is a $\mathbf{G}$-legal trimodel, as every requirement that is induced by the rules of $\mathbf{G}$ or by the definition of a trimodel only concerns accessible worlds. In addition, $\mathcal{M}^* \models S_w$ and $\mathcal{V}^*_w \not\models \Gamma_w \Rightarrow \Delta_w$ (and thus $\mathcal{M}^* \not\models \Gamma_w \Rightarrow \Delta_w$). $\mathcal{M}^*$ is also $\preceq[S_w \cup \{\Gamma_w \Rightarrow \Delta_w\}]$-determined, as $\preceq[S_w \cup \{\Gamma_w \Rightarrow \Delta_w\}] \subseteq \prec[\psi] \subseteq \mathrm{supp}(\mathcal{M}^*)$. By Item 2 in Theorem 5.15, $S_w \nvdash^{\prec}_{\mathbf{G_{Int}}} \Gamma_w \Rightarrow \Delta_w$, and since $\mathbf{G}$ is Int-$\prec$-analytic, $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w \Rightarrow \Delta_w$.

Define a $\mathbf{G}$-legal $\mathrm{supp}(\mathcal{M}) \cup \{\psi\}$-determination $\mathcal{M}' = \langle W, R, \mathcal{V}'\rangle$ of $\mathcal{M}$ as follows: for every $w \in W$ and $\varphi \in \mathcal{L}$, $\mathcal{V}'_w(\varphi)$ is defined by:

- if $\varphi \neq \psi$ or $\varphi \in \mathrm{supp}(\mathcal{V}_w)$, then $\mathcal{V}'_w(\varphi) = \mathcal{V}_w(\varphi)$;
- if $\varphi = \psi$, $\varphi \notin \mathrm{supp}(\mathcal{V}_w)$, and either $\mathcal{V}_u(\varphi) = -1$ or $S_u \vdash_{\mathbf{G_{Int}}} \Gamma_u, \varphi \Rightarrow \Delta_u$ for some $u \in R(w)$, then $\mathcal{V}'_w(\varphi) = -1$;
- otherwise, $\mathcal{V}'_w(\varphi) = 1$.

Let us show that $\mathcal{M}'$ is a trimodel. Suppose for contradiction that $\mathcal{V}'_w(\varphi) = 1$ and $\mathcal{V}'_u(\varphi) \neq 1$ for some formula φ and $u \in R(w)$. If $\varphi \neq \psi$ or $\varphi \in \mathrm{supp}(\mathcal{V}_w)$, then this contradicts the fact that $\mathcal{M}$ is a trimodel. If $\varphi \in \mathrm{supp}(\mathcal{V}_u)$, then $\mathcal{V}_u(\varphi) = -1$, and then by definition $\mathcal{V}'_w(\varphi) = -1$, which cannot be the case. Otherwise, $\varphi = \psi$, $\varphi \notin \mathrm{supp}(\mathcal{V}_w)$, and $\varphi \notin \mathrm{supp}(\mathcal{V}_u)$. Since $\psi \in \mathrm{supp}(\mathcal{M}')$ and $\mathcal{V}'_u(\varphi) \neq 1$, we have $\mathcal{V}'_u(\varphi) = -1$, and thus by the definition of $\mathcal{V}'$ we must have either $\mathcal{V}_z(\varphi) = -1$ for some $z \in R(u) \subseteq R(w)$, or $S_z \vdash_{\mathbf{G_{Int}}} \Gamma_z, \varphi \Rightarrow \Delta_z$ for some $z \in R(u) \subseteq R(w)$. Either way, we obtain a contradiction to the fact that $\mathcal{V}'_w(\varphi) = 1$.

Clearly, $\mathcal{M}'$ is a $\mathrm{supp}(\mathcal{M}) \cup \{\psi\}$-determination of $\mathcal{M}$. It is therefore left to prove that $\mathcal{M}'$ is $\mathbf{G}$-legal. Let $s_1, \dots, s_n \,/\, s$ be a rule of $\mathbf{G}$, σ a substitution and $w \in W$. Suppose that $\mathcal{V}'_w \not\models \sigma(s)$. Then $\mathit{frm}[\sigma(s)] \subseteq \mathrm{supp}(\mathcal{V}_w) \cup \{\psi\}$. First, assume that $\mathit{frm}[\sigma(s)] \subseteq \mathrm{supp}(\mathcal{V}_w)$. In this case, $\mathcal{V}_w \not\models \sigma(s)$ as well. If $s_1, \dots, s_n \,/\, s$ is positive, then since $\mathcal{M}$ is $\mathbf{G}$-legal we have that either $\mathcal{V}_w \not\models \sigma(s_i)$ for some $1 \leq i \leq n$ such that s_i has an empty left side, or $\mathcal{V}_u \not\models \sigma(s_j)$ for some $1 \leq j \leq n$ and $u \in R(w)$ such that s_j has a non-empty left side. Since $\mathcal{M}'$ determines $\mathcal{M}$, the same holds for $\mathcal{V}'$. If $s_1, \dots, s_n \,/\, s$ is not positive, then similarly, we obtain that $\mathcal{V}'_w \not\models \sigma(s_i)$ for some $1 \leq i \leq n$.

We therefore assume that $\mathit{frm}[\sigma(s)] \not\subseteq \mathrm{supp}(\mathcal{V}_w)$, and thus $\psi \notin \mathrm{supp}(\mathcal{V}_w)$, and so by the fact that $\mathbf{G}$ is $\prec$-directed, also $\mathit{frm}[\sigma(s)] = \{\psi\}$, and by the structurality of $\prec$, $\mathit{frm}[\sigma(s_i)] \subseteq {\prec}[\psi]$ for every $1 \leq i \leq n$.

First, suppose that $s_1, \ldots, s_n / s$ is not positive. Then $\sigma(s) = \psi \Rightarrow$. We prove that $\mathcal{V}'_w \not\models \sigma(s_i)$ for some $1 \leq i \leq n$. By our assumption, $\mathcal{V}'_w(\psi) = 1$. This in particular means that $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w, \psi \Rightarrow \Delta_w$, and so there exists $1 \leq i \leq n$ such that $S_w \nvdash_{\mathbf{G_{Int}}} \Gamma_w, \sigma(\Gamma_i) \Rightarrow \sigma(\Delta_i), \Delta_w$, where $s_i = \Gamma_i \Rightarrow \Delta_i$. Since $\mathit{frm}[\sigma(s_i)] \subseteq {\prec}[\psi] = \mathit{frm}[\Gamma_w \Rightarrow \Delta_w]$, we must have $\sigma(\Gamma_i) \subseteq \Gamma_w$ and $\sigma(\Delta_i) \subseteq \Delta_w$, which means that $\mathcal{V}_w \not\models \sigma(s_i)$. $\mathcal{V}'_w$ determines $\mathcal{V}_w$, and so $\mathcal{V}'_w \not\models \sigma(s_i)$.

Next, suppose that $s_1, \ldots, s_n / s$ is positive. Then $\sigma(s) = \Rightarrow \psi$. For $\mathcal{M}'$ to respect $s_1, \ldots, s_n / s$, we need to prove that:

> $(*)$ There exists $1 \leq i \leq n$ such that either s_i has an empty left side and $\mathcal{V}'_w \not\models \sigma(s_i)$, or s_i has a non-empty left side and $\mathcal{V}'_z \not\models \sigma(s_i)$ for some $z \in R(w)$.

We prove the following claim, $(**)$, and then show that $(*)$ follows from $(**)$:

> $(**)$ There exists $1 \leq i \leq n$ such that $\mathcal{V}_z \not\models \sigma(s_i)$ for some $z \in R(w)$.

Proof of $(**)$: By our assumption, $\mathcal{V}'_w(\psi) = -1$. According to the definition of $\mathcal{V}'$, there exists some $u \in R(w)$ such that either $\mathcal{V}_u(\psi) = -1$ or $S_u \vdash_{\mathbf{G_{Int}}} \Gamma_u, \psi \Rightarrow \Delta_u$. We consider each case separately:

1. $\mathcal{V}_u(\psi) = -1$: Then $\mathcal{V}_u \not\models \Rightarrow \psi$. Since $\mathcal{M}$ is $\mathbf{G}$-legal, we must have some $1 \leq i \leq n$ such that either s_i has a non-empty left side and $\mathcal{V}_z \not\models \sigma(s_i)$ for some $z \in R(u) \subseteq R(w)$, or s_i has an empty left side and $\mathcal{V}_u \not\models \sigma(s_i)$.

2. $S_u \vdash_{\mathbf{G_{Int}}} \Gamma_u, \psi \Rightarrow \Delta_u$: Using (CUT) and the fact that $S_u \nvdash_{\mathbf{G_{Int}}} \Gamma_u \Rightarrow \Delta_u$, we must have that $S_u \nvdash_{\mathbf{G_{Int}}} \Rightarrow \psi$, and thus there exists some $1 \leq i \leq n$ such that $S_u \nvdash_{\mathbf{G_{Int}}} \sigma(s_i)$. Hence, $\sigma(s_i) \notin S_u$. Since $\mathit{frm}[\sigma(s_i)] \subseteq {\prec}[\psi]$, we have that $\mathcal{V}_z \not\models \sigma(s_i)$ for some $z \in R(u) \subseteq R(w)$.

Proof that $(**)$ implies $(*)$: Let i and z as in $(**)$. Since $\mathcal{M}'$ determines $\mathcal{M}$, we also have $\mathcal{V}'_z \not\models \sigma(s_i)$. If s_i happens to have a non-empty left side, then we are done, as the second disjunct of $(*)$ holds. Otherwise, s_i has the form $\Rightarrow \Delta_i$. We prove that $\mathcal{V}'_w \not\models \sigma(\Rightarrow \Delta_i)$. Let $\varphi \in \Delta_i$. We prove that $\mathcal{V}'_w(\sigma(\varphi)) = -1$. Since $\mathcal{V}'_z \not\models \sigma(\Rightarrow \Delta_i)$, we have that $\mathcal{V}'_z(\sigma(\varphi)) = -1$. $\mathcal{M}'$ is a trimodel and wRz, and so $\mathcal{V}'_w(\sigma(\varphi)) \neq 1$. Now, $\sigma(\varphi) \in \mathit{frm}[\sigma(s_i)] \subseteq {\prec}[\psi] \subseteq \mathrm{supp}(\mathcal{M}) \subseteq \mathrm{supp}(\mathcal{M}')$, and so $\mathcal{V}'_w(\sigma(\varphi)) \neq 0$. We therefore have $\mathcal{V}'_w(\sigma(\varphi)) = -1$. □

Next, a repeated application of Lemma 5.19 allows us to fully determine trimodels, in a similar manner to the determination of trivaluations:

Lemma 5.20. *Let $\mathbf{G}$ be an Int-$\prec$-analytic $\prec$-directed calculus and $\mathcal{M}$ a $\mathbf{G}$-legal trimodel. Then there exists a $\mathbf{G}$-legal full determination of $\mathcal{M}$.*

Proof. For every trimodel $\mathcal{M}$ and formula ψ, $\mathcal{M}_\psi$ denotes an arbitrary $\mathbf{G}$-legal $\mathrm{supp}(\mathcal{M}) \cup \{\psi\}$-determination of $\mathcal{M}$, if such exists. Otherwise, $\mathcal{M}_\psi$ is undefined. Whenever $\mathcal{M}$ is $\mathbf{G}$-legal and ${\prec}[\psi] \subseteq \mathrm{supp}(\mathcal{M})$, Lemma 5.19 provides us with such a trimodel, in which case $\mathcal{M}_\psi$ is defined. Let $\psi_1, \psi_2, \ldots$ be an enumeration of $\mathcal{L}$ satisfying $i < j$ whenever $\psi_i \prec \psi_j$. For every i, denote the set $\{\psi_1, \ldots, \psi_i\}$ by $\Phi_{\leq i}$ ($\Phi_{\leq 0} = \emptyset$).

Let $\mathcal{M} = \langle W, R, \mathcal{V} \rangle$ be a $\mathbf{G}$-legal trimodel. Define a sequence $\mathcal{M}^0, \mathcal{M}^1, \ldots$ of trimodels as follows: $\mathcal{M}^0 = \mathcal{M}$ and for every $i > 0$, $\mathcal{M}^i = \mathcal{M}^{i-1}_{\psi_i}$. We prove by induction on i that $\mathcal{M}^i$ is defined, and is a $\mathbf{G}$-legal $\Phi_{\leq i}$-determination of $\mathcal{M}$, and also of $\mathcal{M}^{i-1}$ (for $i \geq 1$).

For $i = 0$, this trivially holds. Now let $i > 0$. By the induction hypothesis, $\mathcal{M}^{i-1}$ is $\mathbf{G}$-legal and is a $\Phi_{\leq i-1}$-determination of $\mathcal{M}$. By the enumeration, $\prec[\psi_i] \subseteq \Phi_{\leq i-1} \subseteq \mathrm{supp}(\mathcal{M}^{i-1})$. Thus, $\mathcal{M}^i$ is defined, and is a $\mathbf{G}$-legal $\mathrm{supp}(\mathcal{M}^{i-1}) \cup \{\psi_i\}$-determination of $\mathcal{M}^{i-1}$, and therefore also of $\mathcal{M}$. Also, $\Phi_{\leq i} \subseteq \mathrm{supp}(\mathcal{M}^i)$, and thus $\mathcal{M}^{i-1}$ is actually a $\Phi_{\leq i}$-determination of $\mathcal{M}^{i-1}$ and of $\mathcal{M}$.

We now define $\mathcal{M}' = \langle W, R, \mathcal{V}' \rangle$, a $\mathbf{G}$-legal full determination of $\mathcal{M}$. For every $\varphi \in \mathcal{L}$, let i_φ be the index of φ in the enumeration. For every $w \in W$, define $\mathcal{V}'_w = \lambda\varphi \in \mathcal{L}.\mathcal{V}^{i_\varphi}_w(\varphi)$, where $\mathcal{V}^{i_\varphi}$ denotes the trivaluation assignment of $\mathcal{M}^{i_\varphi}$.

First, we show that $\mathcal{M}'$ is a trimodel. Suppose that $\mathcal{V}'_w(\varphi) = 1$ and let $u \in R(w)$. Then $\mathcal{V}^{i_\varphi}_w(\varphi) = 1$, and since $\mathcal{M}^{i_\varphi}$ is a trimodel, $\mathcal{V}^{i_\varphi}_u(\varphi) = 1$. Therefore, $\mathcal{V}'_u(\varphi) = 1$. Next, we show that $\mathcal{M}'$ fully determines $\mathcal{M}$. For every $\varphi \in \mathcal{L}$ and $w \in W$, $\mathcal{V}'_w(\varphi) = \mathcal{V}^{i_\varphi}_w(\varphi) \neq 0$, as $\mathcal{M}^{i_\varphi}$ is $\Phi_{\leq i_\varphi}$-determined. Also, for every $\varphi \in \mathrm{supp}(\mathcal{M})$ and $w \in W$, $\mathcal{V}'_w(\varphi) = \mathcal{V}^{i_\varphi}_w(\varphi) = \mathcal{V}_w(\varphi)$, as $\mathcal{M}^{i_\varphi}$ determines $\mathcal{M}$. Finally, let $S \,/\, s$ be a rule of $\mathbf{G}$ and σ a substitution. Let $k = \min\{i \mid \mathit{frm}[\sigma(S \cup \{s\})] \subseteq \Phi_{\leq i}\}$. Then $\mathcal{V}'_w(\varphi) = \mathcal{V}^k_w(\varphi)$ for every $\varphi \in \mathit{frm}[\sigma(S \cup \{s\})]$ and $w \in W$. Suppose that $\mathcal{V}'_w \not\models \sigma(s)$ for some $w \in W$. Then we must also have $\mathcal{V}^k_w \not\models \sigma(s)$. If $S \,/\, s$ is positive, then either $\mathcal{V}^k_w \not\models \sigma(s')$ for some $s' \in S$ with an empty left side (and hence $\mathcal{V}'_w \not\models \sigma(s')$), or $\mathcal{V}^k_u \not\models \sigma(s')$ for some $s' \in S$ with a non-empty left side and $u \in R(w)$ (and hence $\mathcal{V}'_u \not\models \sigma(s')$). Otherwise, $\mathcal{V}^k_w \not\models \sigma(s')$ for some $s' \in S$, and hence $\mathcal{V}'_w \not\models \sigma(s')$. □

As in the case of pure calculi, Theorem 5.4 is now obtained as a simple corollary of Lemma 5.20 and corollary 5.17.

6 Conclusion

We identified two general families of propositional sequent calculi, in which a generalized subformula property is equivalent to cut-admissibility. The first is the family of pure calculi that are $\prec$-directed for some prefinite and structural order $\prec$. The second is a family of *intuitionistic calculi*, obtained by considering *intuitionistic derivations* in pure calculi that are $\prec$-directed. This result sheds light on the relation between these two fundamental properties.

We conclude by outlining several directions for further research. A first interesting question concerns the weak versions of cut-admissibility and analyticity. Similarly to the notions of strong and weak cut-admissibility (see Section 2), one can distinguish between strong and weak Int-cut-admissibility, $\prec$-analyticity, and Int-$\prec$-analyticity. (Note that what we call here $\prec$-analyticity corresponds to strong $\prec$-analyticity; where a calculus $\mathbf{G}$ is weakly $\prec$-analytic if $\vdash_{\mathbf{G}} s$ implies $\vdash^{\prec}_{\mathbf{G}} s$.) Now, it is known that for pure calculi, weak and strong cut-admissibility coincide, and that the same holds for $\prec$-analyticity (see [21]). This is not the case, however, for intuitionistic calculi. For example, the calculus consisting of the rules $p_1 \Rightarrow \,/\, \Rightarrow \circ p_1$ and $p_1 \Rightarrow \,/\, \circ p_1 \Rightarrow$ is weakly Int-$\prec_0$-analytic, but not Int-$\prec_0$-analytic. Also, it admits weak but not strong Int-cut-admissibility. A question that is left for further research is whether weak Int-$\prec$-analyticity guarantees weak Int-cut-admissibility.

Second, the following questions regarding the relations between derivations and intuitionistic derivations are currently left open: Does $\prec$-analyticity imply Int-$\prec$-analyticity? Does cut-admissibility imply Int-cut-admissibility? Do either of the converses hold?

Finally, our approach should be further developed for more expressive languages, which include quantifiers and modalities. For the former, the three-valued semantics should be elevated to three-valued first-order structures. The main obstacle, however, is the fact that the usual subformula relation in a first-order language is not prefinite, while our main construction requires enumerating formulas in a way that respects the subformula relation. For modalities, we expect that the Kripke semantics used here for intuitionistic calculi could be adapted for calculi with modalities. We note, however, that such an approach will have certain limitations, as some analytic calculi for modal logics (e.g., $S5$ and B [40, 33]) do not admit cut-admissibility.

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A Proof outline of Theorems 4.8, 4.13 and 4.14

A.1 Soundness

Call an inference step *valid* in a set of trivaluations if every trivaluation of this set that satisfies the premises of the step, also satisfies its conclusion. Clearly, (ID) and (WEAK) are valid in the set of all trivaluations, and (CUT) is valid whenever the cut formula φ satisfies $v(\varphi) \in \{-1, 1\}$. Finally, every application of a rule $s_1, \ldots, s_n \,/\, s$ of $\mathbf{G}$ must be valid in the set of $\mathbf{G}$-legal trivaluations. Indeed, let $\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n \,/\, \sigma(s) \cup c_1 \cup \ldots \cup c_n$ be an application of this rule and v a $\mathbf{G}$-legal trivaluation such that $v \models \sigma(s_i) \cup c_i$ for every $1 \leq i \leq n$. Then either $v \models c_i$ for some $1 \leq i \leq n$, or $v \models \sigma(s_i)$ for every $1 \leq i \leq n$. In both cases, $v \models \sigma(s) \cup c_1 \cup \ldots \cup c_n$ (in the first case this trivially holds, and in the second it holds since v is $\mathbf{G}$-legal). Using the above, the proof proceeds by usual induction on the lengths of derivations.

A.2 Completeness

An *ω-sequent* is a pair $\langle L, R \rangle$ of (possibly infinite) sets of formulas, denoted $L \Rightarrow R$. For a set S of (ordinary) sequents and an ω-sequent $L \Rightarrow R$, we write $S \vdash_{\mathbf{G}} L \Rightarrow R$ if there are finite $\Gamma \subseteq L$ and $\Delta \subseteq R$ such that $S \vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$. The notations $\vdash^{\text{cf}}_{\mathbf{G}}$ and $\vdash^{\preceq}_{\mathbf{G}}$, as well as other definitions regarding sequents, are adapted to ω-sequents in the obvious way.

Let $\vdash \in \left\{\vdash_{\mathbf{G}}, \vdash^{\text{cf}}_{\mathbf{G}}, \vdash^{\preceq}_{\mathbf{G}}\right\}$, S a set of sequents and s a sequent such that $S \nvdash s$. Call an ω-sequent $L \Rightarrow R$ *$\langle \vdash, S \rangle$-underivable* if $S \nvdash L \Rightarrow R$. Call it *$\langle \vdash, S \rangle$-maximal* if $S \vdash L, \varphi \Rightarrow R$ for every $\varphi \in \mathcal{L} \setminus L$ and $S \vdash L \Rightarrow \varphi, R$ for every $\varphi \in \mathcal{L} \setminus R$. It is routine to extend s to a $\langle \vdash, S \rangle$-maximal underivable ω-sequent $L \Rightarrow R$. Define a trivaluation v as follows: $v(\varphi) = 1$ if $\varphi \in L$, $v(\varphi) = -1$ if $\varphi \in R$, and $v(\varphi) = 0$ otherwise.

First note that v is well defined, as $L \cap R = \emptyset$ (otherwise, $S \vdash L \Rightarrow R$). Clearly, $v \not\models s$, and for every $\Gamma \Rightarrow \Delta \in S$ we have that $S \vdash \Gamma \Rightarrow \Delta$, which means that either $\Gamma \not\subseteq L$ or $\Delta \not\subseteq R$, and so $v \models \Gamma \Rightarrow \Delta$.

We show that v is $\mathbf{G}$-legal. Let $S_0 \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a rule of $\mathbf{G}$ and σ a substitution. Suppose that $v \models \sigma(S_0)$. Let us show that for all $\Gamma \Rightarrow \Delta \in S_0$, we have $S \vdash L, \sigma(\Gamma) \Rightarrow \sigma(\Delta), R$. Let $\Gamma \Rightarrow \Delta \in S_0$. By our assumption, there exists either $\varphi \in \Gamma$ such that $v(\sigma(\varphi)) < 1$ (and then $\sigma(\varphi) \notin L$) or $\psi \in \Delta$ such that $v(\sigma(\psi)) > -1$ (and then $\sigma(\psi) \notin R$). Since $L \Rightarrow R$ is $\langle \vdash, S \rangle$-maximal underivable, $S \vdash L, \sigma(\Gamma) \Rightarrow \sigma(\Delta), R$. By applying $S_0 \,/\, s_0$, we obtain $S \vdash L, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), R$.[4] $L \Rightarrow R$ is $\langle \vdash, S \rangle$-underivable, and hence there must be either $\varphi \in \Gamma_0$ such that $\sigma(\varphi) \notin L$ (and then $v(\sigma(\varphi)) < 1$) or $\psi \in \Delta_0$ such that $\sigma(\psi) \notin R$ (and then $v(\sigma(\psi)) > -1$). In other words, $v \models \sigma(\Gamma_0 \Rightarrow \Delta_0)$.

Next, we distinguish the following cases:

1. $\vdash = \vdash_{\mathbf{G}}$: we prove that v is fully determined. Otherwise, we would have some $\varphi \in \mathcal{L} \setminus (L \cup R)$. Since $L \Rightarrow R$ is $\langle S, \vdash_{\mathbf{G}} \rangle$-maximal, we would have $S \vdash_{\mathbf{G}} L \Rightarrow \varphi, R$ and $S \vdash_{\mathbf{G}} L, \varphi \Rightarrow R$. Applying (CUT), we would then obtain $S \vdash_{\mathbf{G}} L \Rightarrow R$, which is a contradiction.

2. $\vdash = \vdash^{\text{cf}}_{\mathbf{G}}$: we show that v is $\mathit{frm}[S]$-determined in a similar manner, by taking φ from $\mathit{frm}[S]$.

3. $\vdash = \vdash^{\preceq}_{\mathbf{G}}$: we show that v is $\preceq[S \cup \{s\}]$-determined. This is also shown similarly, but requires some additional arguments: Assume otherwise, and let $\varphi \in \preceq[S \cup \{s\}] \setminus (L \cup R)$. Since $L \Rightarrow R$ is $\langle S, \vdash^{\preceq}_{\mathbf{G}} \rangle$-maximal, we have $S \vdash^{\preceq}_{\mathbf{G}} L \Rightarrow \varphi, R$ and $S \vdash^{\preceq}_{\mathbf{G}} L, \varphi \Rightarrow R$. Thus, there are finite $\Gamma_1, \Gamma_2 \subseteq L$ and $\Delta_1, \Delta_2 \subseteq R$, such that the sequent $\Gamma_1 \Rightarrow \varphi, \Delta_1$ is derivable from S in $\mathbf{G}$ using only formulas from $\preceq[S \cup \{\Gamma_1 \Rightarrow \varphi, \Delta_1\}]$, and $\Gamma_2, \varphi \Rightarrow \Delta_2$ is derivable from S in $\mathbf{G}$ using only formulas from $\preceq[S \cup \{\Gamma_2, \varphi \Rightarrow \Delta_2\}]$. Using (CUT), we get that the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is derivable from S in $\mathbf{G}$ using only formulas from $\preceq[S \cup \{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\}] \cup \preceq[\varphi]$. Using weakening, together with the fact that $\varphi \in \prec[S \cup \{s\}]$, we get that the sequent $(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) \cup s$ is derivable from S in $\mathbf{G}$, using only formulas from $\preceq[S \cup \{(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) \cup s\}]$. In other words, we have $S \vdash^{\preceq}_{\mathbf{G}} (\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) \cup s$. $L \Rightarrow R$ extends $(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) \cup s$, and thus we get a contradiction to the fact that $S \nvdash^{\preceq}_{\mathbf{G}} L \Rightarrow R$. □

[4] When $\vdash = \vdash^{\preceq}_{\mathbf{G}}$, this requires the assumption that $\mathbf{G}$ is $\prec$-ordered.

B Proof outline of Theorem 5.15

B.1 Soundness

Like the above, the key point is to show that applications of rules of $\mathbf{G}$ are valid in $\mathbf{G}$-legal trimodels. Let

$$\frac{\Gamma_1', \sigma(\Gamma_1) \Rightarrow \sigma(\Delta_1), \Delta_1', \ldots, \Gamma_n', \sigma(\Gamma_n) \Rightarrow \sigma(\Delta_n), \Delta_n'}{\Gamma_1', \ldots, \Gamma_n', \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), \Delta_1', \ldots, \Delta_n'}$$

be an application of a rule $r = \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ of $\mathbf{G}$, and suppose that $\mathcal{V}_w \not\models \Gamma_1', \ldots, \Gamma_n', \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), \Delta_1', \ldots, \Delta_n'$ for some $\mathbf{G}$-legal trimodel $\mathcal{M} = \langle W, R, \mathcal{V} \rangle$ and $w \in W$. We show that $\mathcal{M} \not\models \Gamma_i', \sigma(\Gamma_i) \Rightarrow \sigma(\Delta_i), \Delta_i'$ for some $1 \leq i \leq n$. If r is not positive, then this is shown similarly to the soundness proof in Appendix A.1. Suppose that r is positive. Since $\mathcal{V}_w \not\models \Gamma_1', \ldots, \Gamma_n', \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), \Delta_1', \ldots, \Delta_n'$, we have that $\mathcal{V}_w \not\models \Gamma_i' \Rightarrow \Delta_i'$ for every $1 \leq i \leq n$. Since $\mathcal{M}$ is $\mathbf{G}$-legal, we have some $1 \leq i \leq n$ such that either $\Gamma_i = \emptyset$ and $\mathcal{V}_w \not\models \sigma(\Gamma_i \Rightarrow \Delta_i)$, or $\Gamma_i \neq \emptyset$ and $\mathcal{V}_{w'} \not\models \sigma(\Gamma_i \Rightarrow \Delta_i)$ for some $w' \in R(w)$. In the first case we get that $\mathcal{V}_w \not\models \Gamma_i', \sigma(\Gamma_i) \Rightarrow \sigma(\Delta_i), \Delta_i'$. In the second, we have $\Delta_i' = \emptyset$ (since r is positive). Since $\mathcal{V}_w \not\models \Gamma_i' \Rightarrow$, we have $\mathcal{V}_w(\varphi) = 1$ for every $\varphi \in \Gamma_i'$. $\mathcal{M}$ is a trimodel, and thus $\mathcal{V}_{w'}(\varphi) = 1$ for every $\varphi \in \Gamma_i'$. We therefore have $\mathcal{V}_{w'} \not\models \Gamma_i', \sigma(\Gamma_i) \Rightarrow \sigma(\Delta_i), \Delta_i'$.

B.2 Completeness for $\vdash_{\mathbf{G_{Int}}}$ and $\vdash_{\mathbf{G_{Int}}}^{\mathrm{cf}}$

Recall the definitions concerning ω-sequents (see appendix A.2), and adapt them to $\vdash_{\mathbf{G_{Int}}}$ and $\vdash_{\mathbf{G_{Int}}}^{\mathrm{cf}}$. Let $\vdash \in \left\{\vdash_{\mathbf{G_{Int}}}, \vdash_{\mathbf{G_{Int}}}^{\mathrm{cf}}\right\}$, S a set of sequents and s a sequent such that $S \not\vdash s$. It is routine to extend s to a $\langle \vdash, S \rangle$-maximal-underivable ω-sequent $L_s \Rightarrow R_s$.

Define a trimodel $\mathcal{M} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ as follows: W is the set of all $\langle \vdash, S \rangle$-maximal-underivable ω-sequents; $(L \Rightarrow R)\mathcal{R}(L' \Rightarrow R')$ iff $L \subseteq L'$; and for every $L \Rightarrow R \in W$ and $\varphi \in \mathcal{L}$, $\mathcal{V}_{L \Rightarrow R}(\varphi) = 1$ if $\varphi \in L$, $\mathcal{V}_{L \Rightarrow R}(\varphi) = -1$ if $\varphi \in R$, and $\mathcal{V}_{L \Rightarrow R}(\varphi) = 0$ otherwise.

$\mathcal{M}$ is indeed a trimodel, as $\mathcal{R}$ is transitive and reflexive, and if $\varphi \in L$ and $(L \Rightarrow R)\mathcal{R}(L' \Rightarrow R')$ then $\varphi \in L'$. Also, $\mathcal{M} \not\models s$, and $\mathcal{M} \models S$.

We prove that $\mathcal{M}$ is $\mathbf{G}$-legal. Let $S_0 \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a positive rule of $\mathbf{G}$, σ a substitution and $L \Rightarrow R \in W$. Suppose that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma_0 \Rightarrow \Delta_0)$. Then $\mathcal{V}_{L \Rightarrow R}(\sigma(\varphi)) = 1$ for every $\varphi \in \Gamma_0$, and $\mathcal{V}_{L \Rightarrow R}(\sigma(\psi)) = -1$ for every $\psi \in \Delta_0$. This means that $S \not\vdash L, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), R$. Thus, we must either have $S \not\vdash L, \sigma(\Gamma') \Rightarrow \sigma(\Delta')$ for some $\Gamma' \Rightarrow \Delta' \in S_0$ with $\Gamma' \neq \emptyset$, or $S \not\vdash L \Rightarrow \sigma(\Delta'), R$ for some $\Rightarrow \Delta' \in S_0$. In the first case, extend $L, \sigma(\Gamma') \Rightarrow \sigma(\Delta')$ to a $\langle \vdash, S \rangle$-maximal-underivable ω-sequent $L' \Rightarrow R'$. Clearly, $L \subseteq L'$, and $\mathcal{V}_{L' \Rightarrow R'} \not\models \sigma(\Gamma' \Rightarrow \Delta')$. In the second, the $\langle \vdash, S \rangle$-maximality of $L \Rightarrow R$ ensures that $\sigma(\Delta') \subseteq R$, which means that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Rightarrow \Delta')$.

Next, let $S_0 \,/\, \Gamma_0 \Rightarrow$ be a non-positive rule of $\mathbf{G}$. Suppose that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma_0) \Rightarrow$. We prove that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma' \Rightarrow \Delta')$ for some $\Gamma' \Rightarrow \Delta' \in S_0$. By our assumption, $\mathcal{V}_{L \Rightarrow R}(\sigma(\varphi)) = 1$ for every $\varphi \in \Gamma_0$. This means that $\sigma(\Gamma_0) \subseteq L$. Now, since $L \Rightarrow R \in W$, we have $S \not\vdash L, \sigma(\Gamma_0) \Rightarrow R$. It follows that $S \not\vdash L, \sigma(\Gamma') \Rightarrow \sigma(\Delta'), R$ for some $\Gamma' \Rightarrow \Delta' \in S_0$. By the fact that $L \Rightarrow R$ is $\langle \vdash, S \rangle$-maximal-underivable, we have that $\sigma(\Gamma') \subseteq L$ and $\sigma(\Delta') \subseteq R$, which means that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma' \Rightarrow \Delta')$.

Finally, similarly to the proof of Theorems 4.8 and 4.14, If $\vdash = \vdash_{\mathbf{G_{Int}}}$ then $\mathcal{M}$ is fully determined, and if $\vdash = \vdash_{\mathbf{G_{Int}}}^{\mathrm{cf}}$ then $\mathcal{M}$ is *frm*$[S]$-determined. □

B.3 Completeness for $\vdash_{\mathbf{G_{Int}}}^{\preceq}$

Let S be a set of sequents and s a sequent such that $S \not\vdash_{\mathbf{G_{Int}}}^{\preceq} s$. Let $\mathcal{F} = \preceq[S \cup \{s\}]$. For every set Q of sequents and a sequent q, we write $Q \vdash_{\mathbf{G_{Int}}}^{\mathcal{F}} q$ if there is an intuitionistic derivation of q from Q in $\mathbf{G}$

in which only formulas from $\mathcal{F}$ occur. We call an ω-sequent $L \Rightarrow R$ *underivable* if $L \cup R \subseteq \mathcal{F}$ and $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow R$; and *maximal* if $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \varphi \Rightarrow R$ for every $\varphi \in \mathcal{F} \setminus L$ and $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow \varphi, R$ for every $\varphi \in \mathcal{F} \setminus R$. It is routine to extend s to a maximal underivable ω-sequent $L_s \Rightarrow R_s$.

Define a trimodel $\mathcal{M} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ as follows: W is the set of all maximal underivable ω-sequents; $(L \Rightarrow R)\mathcal{R}(L' \Rightarrow R')$ iff $L \subseteq L'$; and for every $L \Rightarrow R \in W$ and $\varphi \in \mathcal{L}$, $\mathcal{V}_{L \Rightarrow R}(\varphi) = 1$ if $\varphi \in L$, $\mathcal{V}_{L \Rightarrow R}(\varphi) = -1$ if $\varphi \in R$, and $\mathcal{V}_{L \Rightarrow R}(\varphi) = 0$ otherwise.

The proof that $\mathcal{M}$ is a trimodel that satisfies S but not s is the same as in the completeness proof for $\vdash_{\mathbf{G_{Int}}}$ and $\vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}}$. We prove that $\mathrm{supp}(\mathcal{V}_{L \Rightarrow R}) = \mathcal{F}$ for every $L \Rightarrow R \in W$, that is, $L \cup R = \mathcal{F}$ (in particular, this means $\mathcal{M}$ is $\mathcal{F}$-determined). Let $L \Rightarrow R \in W$. By definition, we have $L \cup R \subseteq \mathcal{F}$. Assume for contradiction that there exists $\varphi \in \mathcal{F} \setminus (L \cup R)$. Since $L \Rightarrow R$ is maximal and underivable, we have both $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \varphi \Rightarrow R$ and $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow \varphi, R$. Using (CUT) on φ, we get $S \vdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow R$, a contradiction to $L \Rightarrow R \in W$.

We prove that $\mathcal{M}$ is $\mathbf{G}$-legal. Let $S_0 \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a positive rule of $\mathbf{G}$, σ a substitution and $L \Rightarrow R \in W$. Suppose that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma_0 \Rightarrow \Delta_0)$. Then $\sigma(\Gamma_0) \subseteq L$ and $\sigma(\Delta_0) \subseteq R$, and in particular, $\sigma(\Gamma_0), \sigma(\Delta_0) \subseteq \mathcal{F}$. Since $\mathcal{F}$ is closed under $\prec$ (that is, $\preceq[\mathcal{F}] = \mathcal{F}$), we also have $\prec[\sigma(\Gamma_0 \Rightarrow \Delta_0)] \subseteq \mathcal{F}$. Since $\mathbf{G}$ is $\prec$-ordered, $\mathit{frm}[\Gamma' \Rightarrow \Delta'] \subseteq \prec[\Gamma_0 \Rightarrow \Delta_0]$ for every $\Gamma' \Rightarrow \Delta' \in S_0$, and so $\sigma(\mathit{frm}[\Gamma' \Rightarrow \Delta']) \subseteq \sigma(\prec[\Gamma_0 \Rightarrow \Delta_0]) \subseteq \prec[\sigma(\Gamma_0 \Rightarrow \Delta_0)] \subseteq \mathcal{F}$ for every such $\Gamma' \Rightarrow \Delta'$, by the structurality of $\prec$. Now, since $\sigma(\Gamma_0) \subseteq L$ and $\sigma(\Delta_0) \subseteq R$, we have $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), R$. Thus, we must either have $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma') \Rightarrow \sigma(\Delta')$ for some $\Gamma' \Rightarrow \Delta' \in S_0$ with $\Gamma' \neq \emptyset$, or $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L \Rightarrow \sigma(\Delta'), R$ for some $\Rightarrow \Delta' \in S_0$. In the first case, extend $L, \sigma(\Gamma') \Rightarrow \sigma(\Delta')$ to a maximal underivable ω-sequent $L' \Rightarrow R'$. Clearly, $L \subseteq L'$, and $\mathcal{V}_{L' \Rightarrow R'} \not\models \sigma(\Gamma' \Rightarrow \Delta')$. In the second, the maximality of $L \Rightarrow R$ ensures that $\sigma(\Delta') \subseteq R$, which means that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Rightarrow \Delta')$.

Next, let $S_0 \,/\, \Gamma_0 \Rightarrow$ be a non-positive rule of $\mathbf{G}$. Suppose that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma_0) \Rightarrow$. We prove that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma' \Rightarrow \Delta')$ for some $\Gamma' \Rightarrow \Delta' \in S_0$. By our assumption, $\sigma(\Gamma_0) \subseteq L$. Since $L \Rightarrow R \in W$, we have $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma_0) \Rightarrow R$. Similarly to the above, it follows that $S \nvdash^{\mathcal{F}}_{\mathbf{G_{Int}}} L, \sigma(\Gamma') \Rightarrow \sigma(\Delta'), R$ for some $\Gamma' \Rightarrow \Delta' \in S_0$. By the fact that $L \Rightarrow R$ is maximal and underivable, we have that $\sigma(\Gamma') \subseteq L$ and $\sigma(\Delta') \subseteq R$, which means that $\mathcal{V}_{L \Rightarrow R} \not\models \sigma(\Gamma' \Rightarrow \Delta')$. $\square$