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Combining Stable Infiniteness and (Strong) Politeness

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Abstract Polite theory combination is a method for obtaining a solver for a combination of two (or more) theories using the solvers of each individual theory as black boxes. Unlike the earlier Nelson-Oppen method, which is usable only when both theories are stably infinite, only one of the theories needs to be strongly polite in order to use the polite combination method. In its original presentation, politeness was required from one of the theories rather than strong politeness, which was later proven to be insufficient. The first contribution of this paper is a proof that indeed these two notions are different, obtained by presenting a polite theory that is not strongly polite. We also study several variants of this question.

The cost of the generality afforded by the polite combination method, compared to the Nelson-Oppen method, is a larger space of arrangements to consider, involving variables that are not necessarily shared between the purified parts of the input formula. The second contribution of this paper is a hybrid method (building on both polite and Nelson-Oppen combination), which aims to reduce the number of considered variables when a theory is stably infinite with respect to some of its sorts but not all of them. The time required to reason about arrangements is exponential in the worst case, so reducing the number of variables considered has

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the potential to improve performance significantly. We show preliminary evidence for this by demonstrating significant speed-up on a smart contract verification benchmark.[1]

1 Introduction

In 1979, Nelson and Oppen [18] proposed a general framework for combining theories with disjoint signatures. Using this framework, it is possible, under certain conditions, to obtain a decision procedure for a combined theory (e.g., the theory of arrays of integers) using decision procedures for each theory in the combination as black boxes. In this framework, the problem of checking the satisfiability of a quantifier-free formula in the combined theory is reduced to that of checking the satisfiability of a conjunction of *pure* formulas, one for each component theory. Each pure formula is then sent to a a *theory solver*, a satisfiability solver specialized on the corresponding theory, along with a guessed arrangement (a set of equalities and disequalities that capture an equivalence relation) of the variables shared among the pure formulas. The main requirement needed for the completeness of this technique [17] is that each theory involved be stably infinite. While many important theories are stably infinite, some are not, including the widely-used theory of fixed-length bit-vectors. In order to be able to combine a larger class of theories, the polite combination method was introduced by Ranise et al. [19], based on a previous method by Tinelli and Zarba [24], and later refined by Jovanovic and Barrett [13]. In polite combination, one theory must be *polite*, a stronger requirement than stable-infiniteness, but there is no requirement on the other theory: in particular, it does not need to be stably infinite. Unlike the Nelson-Oppen method, however, polite combination requires guessing arrangements over *all* variables of certain sorts, not just the shared ones. At a high level, polite theories have two properties: smoothness and finite witnessability (see Section 2). The polite combination theorem in [19] contained an error, which was identified in [13]. A fix was also proposed in [13], which relies on stronger requirements for finite witnessability. Following Casal and Rasga [9], we call this strengthened version *strong finite witnessability*. A theory that is both smooth and strongly finitely witnessable is called *strongly polite*. Shifting from polite theories to strongly polite theories allowed [13] to fix the proof of correcntess of the polite combination method. It was unclear, however, whether the notions of polite and strongly polite theories were actually different.

This paper makes two contributions. First, we provide an affirmative answer to the question of whether politeness and strong politeness are different notions, by presenting an example of a theory that is polite but not strongly polite. The given theory is over an empty signature and has two sorts, and was originally studied in [9] in the context of shiny theories (the difference between polite and strongly polite theories was not discussed in that paper). Here we state and prove the separation of politeness and strong politeness, without using shiny theories. Proving that a theory is strongly polite is harder than proving that it is just polite. This result shows that the additional effort is sometimes needed to be able to use

[1] A preliminary version of this work was published in the proceedings of CADE-28 [21]. The current article incorporates some updates to the text, adds detailed proofs to all claims, and is accompanied by an artifact that can be used to reproduce the case study reported in Section 5.

the combination theorem from [13]. Our separation result is further refined, as we also show that for empty signatures, at least two sorts are needed to present a polite theory that is not strongly polite. Further, we show that for the empty signature with only one sort, there is a finitely witnessable theory that is not strongly finite witnessable. Such a theory cannot be smooth.

Second, we show that the number of variables that need to be considered in arrangements can be reduced in the presence of more information about the combined theories. In particular, we study the case where one theory is strongly polite w.r.t. a set S of sorts and the other is stably infinite w.r.t. a subset $S' \subseteq S$ of the sorts. For such cases, we show that it is possible to perform Nelson-Oppen combination for S' and polite combination for $S \setminus S'$. This means that for the sorts in S', only shared variables need to be considered for the guessed arrangement, which can considerably reduce its size. We also show that the set of shared variables can be reduced for a couple of other variations of conditions on the theories. Finally, we present a preliminary case study using a challenge benchmark from a smart contract verification application. We show that the reduction of shared variables is substantial and significantly improves the solving time. Verification of smart contracts using SMT (and the analyzed benchmark in particular) is the main motivation behind the second contribution of this paper.

Related Work: Polite combination is part of a more general effort to replace the stable infiniteness symmetric condition in the Nelson-Oppen approach with a weaker condition. Other examples of this effort include the notions of *shiny* [24], *parametric* [15], and *gentle* [12] theories. Gentle, shiny, and polite theories can be combined à la Nelson-Oppen with any arbitrary theory. Shiny theories were introduced by Tinelli and Zarba [24] as a class of mono-sorted theories. Based on the same principles as shininess, politeness is particularly well-suited to deal with theories expressed in many-sorted logic. Polite theories were introduced by Ranise et al. [19] to provide a more effective combination approach compared to parametric and shiny theories, the former requiring solvers to reason about cardinalities and the latter relying on expensive computations of minimal cardinalities of models. Shiny theories were extended to many-sorted signatures in [19], where there is a sufficient condition for their equivalence with polite theories. For the mono-sorted case, a sufficient condition for the equivalence of shiny theories and strongly polite theories was given by Casal and Rasga [8]. In later work [9], the same authors proposed a generalization of shiny theories to many-sorted signatures different from the one in [19], and proved that it is equivalent to strongly polite theories with a decidable quantifier-free fragment. We discuss the connection between these results and the present paper in Remarks 2 and 3. The strong politeness of the theory of algebraic datatypes [6] was proven by Sheng et al. [20] who also introduced *additive witnesses*, which provide a sufficient condition for a polite theory to be also strongly polite. In this paper we present a theory that is polite but not strongly polite. In accordance with [20], the witness that we provide for this theory is not additive.

The paper is organized as follows. Section 2 introduces the necessary notions from first-order logic and polite theories, as well as examples that will be used throughout the paper. Section 3 discusses the difference between politeness and strong politeness and shows they are not equivalent. Section 4 gives the improvements for the combination process under certain conditions, and Section 5 demonstrates the effectiveness of these improvements for a challenge benchmark. We conclude with Section 6

2 Many-sorted Theories and Theory Combination

We review in this section the relevant notions from many-sorted first-order logic and theory combination. This section also includes several examples that will be used throughout the article.

2.1 Signatures and Structures

We begin with a review of many-sorted first-order logic with equality (see [11,22] for more details). A *signature* Σ consists of a set $\mathcal{S}_\Sigma$ (of *sorts*), a set $\mathcal{F}_\Sigma$ of function symbols, and a set $\mathcal{P}_\Sigma$ of predicate symbols. We assume that $\mathcal{S}_\Sigma$, $\mathcal{F}_\Sigma$ and $\mathcal{P}_\Sigma$ are countable. Function symbols have *arities* of the form $\sigma_1 \times \cdots \times \sigma_n \to \sigma$, and predicate symbols have arities of the form $\sigma_1 \times \cdots \times \sigma_n$, with $\sigma_1, \ldots, \sigma_n, \sigma \in \mathcal{S}_\Sigma$. For each sort $\sigma \in \mathcal{S}_\Sigma$, $\mathcal{P}_\Sigma$ includes an *equality symbol* $=_\sigma$ of arity $\sigma \times \sigma$. We denote it by $=$ when σ is clear from context. When $=_\sigma$ are the only symbols in Σ, we say that Σ is *empty*. If two signatures share no symbols except $=_\sigma$ we call them *disjoint* (they may share sorts). We assume an underlying countably infinite set of variables for each sort. For each signature Σ, well-sorted (Σ-)terms, (Σ-)formulas, and (Σ-)literals are defined in the usual way. We include the universally valid formula `true` in the set of (Σ-)formulas.

For a Σ-formula ϕ and a sort σ, we denote the set of free variables in ϕ of sort σ by $vars_\sigma(\phi)$. This notation naturally extends to $vars_S(\phi)$ where S is a set of sorts. We denote by $vars(\phi)$ the set of all free variables in ϕ. We denote by $QF(\Sigma)$ the set of quantifier-free Σ-formulas.

A Σ-*structure* $\mathcal{A}$ is a many-sorted structure that provides semantics for the symbols in Σ (but not for variables). It consists of a non-empty *domain* $\sigma^\mathcal{A}$ for each sort $\sigma \in \mathcal{S}_\Sigma$, an interpretation $f^\mathcal{A}$ for every $f \in \mathcal{F}_\Sigma$, as well as an interpretation $P^\mathcal{A}$ for every $P \in \mathcal{P}_\Sigma$. We further require that $=_\sigma$ be interpreted as the identity relation over $\sigma^\mathcal{A}$ for every $\sigma \in \mathcal{S}_\Sigma$. A Σ-*interpretation* $\mathcal{I}$ is an extension of a Σ-structure with an interpretation for some set V of variables. We will not specify the set V when it is clear from the context or not important. Interpretations extend to Σ-terms in the usual way. For any Σ-term t, $t^\mathcal{I}$ denotes the interpretation of t in $\mathcal{I}$. When α is a set of Σ-terms, $\alpha^\mathcal{I} = \left\{ t^\mathcal{I} \mid t \in \alpha \right\}$. Given a Σ-interpretation $\mathcal{I}$ and a sub-signature Σ' of Σ, the *reduct* of $\mathcal{I}$ to Σ' is obtained from $\mathcal{I}$ by restricting it to the sorts and symbols of Σ'. Satisfaction is defined as usual: $\mathcal{I} \models \phi$ denotes that $\mathcal{I}$ satisfies ϕ, and given any set Θ of formulas, $\mathcal{I} \models \Theta$ if $\mathcal{I} \models \phi$ for every $\phi \in \Theta$.

2.2 Theories

A Σ-*theory* $\mathcal{T}$ is the class of all Σ-structures that satisfy some set Ax of Σ-*sentences*, i.e., Σ-formulas with no free variables. For each such set Ax, we say that $\mathcal{T}$ is *axiomatized* by Ax. A $\mathcal{T}$-*interpretation* is a Σ-interpretation whose underlying structure is in $\mathcal{T}$. A Σ-formula ϕ is $\mathcal{T}$-*satisfiable* if $\mathcal{A} \models \phi$ for some $\mathcal{T}$-interpretation

$\mathcal{A}$. A set Θ of Σ-formulas is $\mathcal{T}$*-satisfiable* if $\mathcal{A} \models \Theta$ for some $\mathcal{T}$-interpretation $\mathcal{A}$. Two formulas ϕ and ψ are $\mathcal{T}$*-equivalent* if they are satisfied by the same $\mathcal{T}$-interpretations.

Example 1 Let Σ_{List} be a signature of finite lists containing the sorts elem_1, elem_2, and list, as well as the function symbols cons of arity $\mathsf{elem}_1 \times \mathsf{elem}_2 \times \mathsf{list} \to \mathsf{list}$, car_1 of arity $\mathsf{list} \to \mathsf{elem}_1$, car_2 of arity $\mathsf{list} \to \mathsf{elem}_2$, cdr of arity $\mathsf{list} \to \mathsf{list}$, and nil of arity list. The Σ_{List}-theory $\mathcal{T}_{\text{List}}$ corresponds to an SMT-LIB 2 theory of algebraic datatypes [3,6], where elem_1 and elem_2 are interpreted as some sets (of *elements*), and list is interpreted as finite lists of pairs of elements, one from elem_1 and the other from elem_2; cons denotes a list constructor that takes two elements and a list, and inserts the pair of those two elements at the head of the list; nil denotes the empty list. The pair $(\mathsf{car}_1(l), \mathsf{car}_2(l))$ denotes the first entry in l, and $\mathsf{cdr}(l)$ denotes the list obtained from l by removing its first entry.

Example 2 The signature Σ_{Int} includes a single sort int; all numerals $0, 1, \ldots$, all of sort int; the function symbols $+$, $-$ and $\cdot$ of arity $\mathsf{int} \times \mathsf{int} \to \mathsf{int}$; and the predicate symbols $<$ and $\leq$ of arity $\mathsf{int} \times \mathsf{int}$. The Σ_{Int}-theory $\mathcal{T}_{\text{Int}}$ corresponds to integer arithmetic in SMT-LIB 2, and the interpretation of the symbols is the same as in the standard structure of the integers.

Example 3 The signature Σ_{BV4} includes a single sort $\mathsf{BV4}$ and various function and predicate symbols for reasoning about bit-vectors of length 4 (such as & for bit-wise *and*, constants of the form 0110, etc.). The Σ_{BV4}-theory $\mathcal{T}_{\text{BV4}}$ corresponds to SMT-LIB 2 bit-vectors of size 4, with the expected semantics of constants and operators.

Let Σ_1, Σ_2 be signatures, $\mathcal{T}_1$ a Σ_1-theory, and $\mathcal{T}_2$ a Σ_2-theory. The *combination* of $\mathcal{T}_1$ and $\mathcal{T}_2$, denoted $\mathcal{T}_1 \oplus \mathcal{T}_2$, consists of all $\Sigma_1 \cup \Sigma_2$-structures $\mathcal{A}$, such that $\mathcal{A}^{\Sigma_1}$ is in $\mathcal{T}_1$ and $\mathcal{A}^{\Sigma_2}$ is in $\mathcal{T}_2$, where $\mathcal{A}^{\Sigma_i}$ is the reduct of $\mathcal{A}$ to Σ_i for $i \in \{1, 2\}$.

Example 4 Let $\mathcal{T}_{\text{IntBV4}}$ be $\mathcal{T}_{\text{Int}} \oplus \mathcal{T}_{\text{BV4}}$. It is the combined theory of integers and bit-vectors from Examples 2 and 3. It has all the sorts and operators from both theories. If we rename the sorts elem_1 and elem_2 of Σ_{List} to int and $\mathsf{BV4}$, respectively, we can obtain a theory $\mathcal{T}_{\text{ListIntBV4}}$ defined as $\mathcal{T}_{\text{IntBV4}} \oplus \mathcal{T}_{\text{List}}$. This is the theory of lists of pairs, where each pair consists of an integer and a bit-vector of size 4. Note that the theories $\mathcal{T}_{\text{Int}}$, $\mathcal{T}_{\text{BV4}}$, and $\mathcal{T}_{\text{List}}$ are pairwise disjoint.

The following definitions and theorems will be useful in the sequel. The first is a generalization of the Löwenheim-Skolem theorem for many-sorted languages.

Theorem 1 (Theorem 9 of [22]) *Let Σ be a signature, and Θ a set of Σ-formulas that is satisfiable. Then there exists an interpretation $\mathcal{A}$ that satisfies Θ, in which $\sigma^{\mathcal{A}}$ is countable whenever it is infinite.*[2]

Next, we formally define arrangements.

[2] In [22] this was proven more generally, for order-sorted logics which extend many-sorted logics with subsorts.

Definition 1 (Arrangement) Let S be a set of sorts, V a finite set of variables whose sorts are in S, and $\{V_\sigma \mid \sigma \in S\}$ a partition of V such that V_σ is the set of variables of sort σ in V. A formula δ is an *arrangement of* V if

$$\delta = \bigwedge_{\sigma\in S} (\bigwedge_{(x,y)\in E_\sigma} (x = y) \ \wedge \bigwedge_{x,y\in V_\sigma,(x,y)\notin E_\sigma} (x \neq y)) \ ,$$

where E_σ is some equivalence relation over V_σ for each $\sigma \in S$.

For any set S, let $|S|$ denote the cardinality of S. The following theorem from [13] is a variant of a theorem from [23] and is often used to help prove theory combination theorems.

Theorem 2 (Theorem 2.5 of [13]) *For $i = 1, 2$, let Σ_i be disjoint signatures, $S_i = \mathcal{S}_{\Sigma_i}$, $\mathcal{T}_i$ be a Σ_i-theory, and φ_i be a conjunction of Σ_i-literals. Let $S = S_1 \cap S_2$ and $V = vars(\varphi_1) \cap vars(\varphi_2)$. If there exist a $\mathcal{T}_1$-interpretation $\mathcal{A}$, a $\mathcal{T}_2$ interpretation $\mathcal{B}$, and an arrangement δ_V of V such that:*

1. $\mathcal{A} \models \varphi_1 \wedge \delta_V$;
2. $\mathcal{B} \models \varphi_2 \wedge \delta_V$; *and*
3. $|\sigma^{\mathcal{A}}| = |\sigma^{\mathcal{B}}|$ *for every* $\sigma \in S$,

then $\varphi_1 \wedge \varphi_2$ is $\mathcal{T}_1 \oplus \mathcal{T}_2$-satisfiable.

2.3 Polite Theories

We now give the background definitions necessary for both Nelson-Oppen and polite combination. In what follows, Σ is an arbitrary (many-sorted) signature, $S \subseteq \mathcal{S}_\Sigma$ is a set of sorts, and $\mathcal{T}$ is a Σ-theory. We start with stable infiniteness and smoothness.

Definition 2 (Stably Infinite) $\mathcal{T}$ is *stably infinite with respect to* S if every quantifier-free Σ-formula that is $\mathcal{T}$-satisfiable is also satisfied by a $\mathcal{T}$-interpretation $\mathcal{A}$ in which $\sigma^{\mathcal{A}}$ is infinite for every $\sigma \in S$.

Definition 3 (Smooth) $\mathcal{T}$ is *smooth w.r.t.* S if for every quantifier-free formula ϕ, $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ, and function κ from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$ for every $\sigma \in S$, there exists a $\mathcal{T}$-interpretation $\mathcal{A}'$ that satisfies ϕ with $\left|\sigma^{\mathcal{A}'}\right| = \kappa(\sigma)$ for every $\sigma \in S$.

We identify singleton sets with their single elements when there is no ambiguity (e.g., when saying that a theory is smooth w.r.t. a sort σ).

It is easy to show that every smooth theory is also stably infinite. The most noticeable difference between the two notions, however, concerns finite and uncountable cardinalities. For example, if we require gaps between finite cardinalities of formulas (e.g., by requiring only even cardinalities), then the resulting theory cannot be smooth (see, e.g., Section 3.4).

We next define politeness and related concepts, following the presentation in [20].

Definition 4 (Finite Witness) Let ϕ be a quantifier-free Σ-formula. A Σ-interpretation $\mathcal{A}$ *finitely witnesses* ϕ *for* $\mathcal{T}$ *w.r.t.* S (or, is a *finite witness of* ϕ *for* $\mathcal{T}$ *w.r.t.* S), if $\mathcal{A} \models \phi$ and $\sigma^{\mathcal{A}} = vars_{\sigma}(\phi)^{\mathcal{A}}$ for every $\sigma \in S$. We say that ϕ is *finitely witnessed for* $\mathcal{T}$ *w.r.t.* S if it is either $\mathcal{T}$-unsatisfiable or has a finite witness for $\mathcal{T}$ w.r.t. S. We say that ϕ is *strongly finitely witnessed for* $\mathcal{T}$ *w.r.t.* S if for every[3] finite set V of variables whose sorts are in S, and every arrangement δ_V of V, we have that $\phi \wedge \delta_V$ is finitely witnessed for $\mathcal{T}$ w.r.t. S.

A function $wit : QF(\Sigma) \rightarrow QF(\Sigma)$ is a *(strong) witness for* $\mathcal{T}$ *w.r.t.* S if for every $\phi \in QF(\Sigma)$ we have that:

1. ϕ and $\exists \overrightarrow{w}.\, wit(\phi)$ are $\mathcal{T}$-equivalent for $\overrightarrow{w} = vars(wit(\phi)) \setminus vars(\phi)$; and
2. $wit(\phi)$ is (strongly) finitely witnessed for $\mathcal{T}$ w.r.t. S.

$\mathcal{T}$ is *(strongly) finitely witnessable* w.r.t. S if there exists a computable (strong) witness for $\mathcal{T}$ w.r.t. S.

The main difference between finite witnessability and strong finite witnessability is that the latter notion takes into account arbitrary arrangements over arbitrary (yet finite) sets of variables. This difference is highlighted, for example, in Section 3.1.

Definition 5 (Polite) $\mathcal{T}$ is *(strongly) polite w.r.t.* S if it is smooth and (strongly) finitely witnessable w.r.t. S.

2.4 Theories vs. Classes of Structures

In papers about theory combination, theories are often defined in terms of some set Ax of sentences (*axioms*) (see, e.g., [9,22,13]). Specifically, a theory is defined as the set of all sentences entailed by Ax or, interchangeably, as the class of all structures that satisfy Ax. The latter is the approach we take in this paper. The main reason for this is that the combination theorems we prove and cite here rely on some forms of the Löwenheim-Skolem theorem, which do not hold for arbitrary classes of structures, but do hold when defining theories this way. On the other hand, theories in the SMT-LIB 2 standard, as well as in many SMT papers about individual theories, are defined more generally as classes of structures without reference to a set of axioms.

We point out that this discrepancy is not substantial since the two notions of a theory as a class of structures are easily interreducible: every theory T in the second, more general sense induces a theory in the first sense that is equivalent to T for all of our intents and purposes since it entails exactly the same (first-order) sentences as T. To be more precise, the combination theorems that we prove and cite only hold when considering theories as classes of structures satisfying a given set of axioms, a restriction also present in other papers on theory combination. They can be used, however, when designing solvers for satisfiability of formulas, because the transformation between the two notions of a theory preserves entailment and hence satisfiability. For sake of completeness, we prove that indeed satisfiability is preserved.

[3] For the results proven below, this full generality regarding V is not needed (e.g., it is sufficient to consider only variables in ϕ) . However, for the validity of other results in polite theory combination, an arbitrary (finite) V is required. For more details, see Footnote 4 of [14].

Statement	[13]	This Paper
f.w. formulas $\neq$ s.f.w. formulas	Example 3 of [13]	Example 5
witness $\neq$ strong witness	Example 3 of [13]	Example 6
polite $\neq$ strongly polite	No Answer	Section 3.2
1-sort, empty sig: polite $=$ strongly polite	No Answer	Section 3.3
f.w. theories $\neq$ s.f.w. theories	No Answer	Section 3.4

Fig. 1 A summary of the results regarding politeness and strong politeness. The abbreviation (s.f.w) f.w. stands for (strong) finite witnessability.

Lemma 1 *Let Σ be a signature, $\mathcal{C}$ a class of Σ-structures, Ax the set of Σ-sentences satisfied by all structures of $\mathcal{C}$, and $\mathcal{T}_{\mathcal{C}}$ the class of all Σ-structures that satisfy all sentences of Ax. Then, for every Σ-formula φ, φ is $\mathcal{T}_{\mathcal{C}}$-satisfiable iff φ is satisfied by some Σ-interpretation whose underlying structure is in $\mathcal{C}$.*

Proof Every interpretation whose underlying structure is in $\mathcal{C}$ is, by construction of $\mathcal{T}_{\mathcal{C}}$, a $\mathcal{T}_{\mathcal{C}}$-interpretation, and so the right-to-left direction trivially holds. Now, suppose φ is not satisfied by any Σ-interpretation whose underlying structure is in $\mathcal{C}$. Then its existential closure $\exists \overline{x}.\varphi$ is not satisfied by any structure of $\mathcal{C}$, and hence $\neg\exists \overline{x}.\varphi \in Ax$. *Ad absurdum*, suppose that φ is $\mathcal{T}_{\mathcal{C}}$-satisfiable. Then there is a $\mathcal{T}_{\mathcal{C}}$-interpretation $\mathcal{A}$ such that $\mathcal{A} \models \varphi$. In particular, $\mathcal{A} \models \exists \overline{x}.\varphi$. But since $\mathcal{A}$ is a $\mathcal{T}_{\mathcal{C}}$-interpretation, we must also have $\mathcal{A} \models \neg\exists \overline{x}.\varphi$, which is a contradiction. □

3 Politeness and Strong Politeness

In this section, we study the difference between politeness and strong politeness. Since the introduction of strong politeness in [13], it has been unclear whether it is strictly stronger than politeness, that is, whether there exists a theory that is polite but not strongly polite. We present an example of such a theory, answering the open question affirmatively. This result is followed by further analysis of notions related to politeness. The section is organized as follows. In Section 3.1 we reformulate an example given in [13], showing that there are witnesses that are not strong witnesses. We then present a polite theory that is not strongly polite in Section 3.2. The theory is over an empty signature (i.e., containing no symbols except for equality) with two sorts. We show in Section 3.3 that politeness and strong politeness are equivalent for empty signatures with a single sort. Finally, we show in Section 3.4 that this equivalence does not hold for finite witnessability alone. Figure 1 summarizes the results of this section and compares them to what was already known in [13].

3.1 Witnesses vs. Strong Witnesses

In [13], an example was given for a witness that is not strong. We reformulate this example in terms of the notions that are defined in the current paper, that is, witnessed formulas are not the same as strongly witnessed formulas (Example 5), and witnesses are not the same as strong witnesses (Example 6).

$$distinct(x_1, \ldots, x_n) := \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

$$\psi^{\sigma}_{\geq n} := \exists x_1, \ldots, x_n . distinct(x_1, \ldots, x_n)$$

$$\psi^{\sigma}_{\leq n} := \exists x_1, \ldots, x_n . \forall y . \bigvee_{i=1}^{n} y = x_i$$

$$\psi^{\sigma}_{=n} := \psi^{\sigma}_{\geq n} \wedge \psi^{\sigma}_{\leq n}$$

Fig. 2 Cardinality formulas for sort σ. All variables are assumed to have sort σ.

Example 5 Let Σ_0 be an empty signature with a single sort σ, and let $\mathcal{T}_0$ be a Σ_0-theory consisting of all Σ_0-structures with at least two elements. Let ϕ be the formula $x = x \wedge w = w$ where both x and w are variables. This formula is finitely witnessed for $\mathcal{T}_0$ w.r.t. σ, but not strongly. Indeed, for $\delta_V \equiv (x = w)$, $\phi \wedge \delta_V$ is not finitely witnessed for $\mathcal{T}_0$ w.r.t. σ: a finite witness would be required to have only a single element and would therefore not be a $\mathcal{T}_0$-interpretation.

The next example shows that witnesses and strong witnesses are not equivalent.

Example 6 Take Σ_0, σ, and $\mathcal{T}_0$ as in Example 5, and for every ϕ, define $wit(\phi)$ to be $(\phi \ \wedge \ w_1 = w_1 \ \wedge \ w_2 = w_2)$ for some variables $w_1, w_2 \notin vars_{\sigma}(\phi)$. Function wit is a witness for $\mathcal{T}_0$ w.r.t. σ. However, it is not a strong witness for $\mathcal{T}$ w.r.t. σ.

Although the theory $\mathcal{T}_0$ in the above examples does serve to distinguish formulas and witnesses that are and are not strong, it cannot be used to do the same for theories themselves. This is because $\mathcal{T}_0$ is, in fact, strongly polite, via a different witness function.

Example 7 The function $wit'(\phi) = (\phi \wedge w_1 \neq w_2)$, for some $w_1, w_2 \notin vars_{\sigma}(\phi)$, *is* a strong witness for $\mathcal{T}_0$ w.r.t. S, as proved in [13].

Remark 1 Notice that Example 6 is quite typical for proofs of finite witnessability. Indeed, it is often enough to just add enough variables, with trivial assertions regarding the new variables. With strong finite witnessability, things are usually more complicated. For example, the witness of Example 7 introduces a disequality, thus incorporating some of the properties of the theory (namely, having at least two elements in the domain) into the witness. In some cases, more involved strong witnesses are needed (see e.g., [20]).

A natural question, then, is whether there is a theory that can separate the two notions of politeness. The following subsection provides an affirmative answer.

3.2 A Polite Theory that is not Strongly Polite

Let Σ_2 be a signature with just two sorts σ_1 and σ_2 and no function or predicate symbols (except $=$). Let $\mathcal{T}_{2,3}$ be the Σ_2-theory from [9], consisting of all Σ_2-structures $\mathcal{A}$ such that either $\left|\sigma_1^{\mathcal{A}}\right| = 2 \wedge \left|\sigma_2^{\mathcal{A}}\right| \geq \aleph_0$ or $\left|\sigma_1^{\mathcal{A}}\right| \geq 3 \wedge \left|\sigma_2^{\mathcal{A}}\right| \geq 3$ [9], where $\aleph_0$ is the cardinality of the set of natural numbers.[4] Notice that $\mathcal{T}_{2,3}$ can be axiomatized using the following set of axioms, given the definitions in Figure 2:

$$\left\{\psi^{\sigma_1}_{\geq 2}, \psi^{\sigma_2}_{\geq 3}\right\} \cup \{\psi^{\sigma_1}_{=2} \rightarrow \neg\psi^{\sigma_2}_{=n} \mid n \geq 3\}$$

We show that $\mathcal{T}_{2,3}$ is polite, but is not strongly polite. Its smoothness is shown by extending any given structure with new elements as needed.

Lemma 2 *$\mathcal{T}_{2,3}$ is smooth w.r.t.* $\{\sigma_1, \sigma_2\}$.

Proof Let ϕ be a quantifier-free Σ_2-formula, $\mathcal{A}$ a $\mathcal{T}_{2,3}$-interpretation that satisfies ϕ, and κ a function from $\{\sigma_1, \sigma_2\}$ to the class of cardinals such that $\kappa(\sigma_1) \geq \left|\sigma_1^{\mathcal{A}}\right|$ and $\kappa(\sigma_2) \geq \left|\sigma_2^{\mathcal{A}}\right|$. We construct a Σ_2-interpretation $\mathcal{A}'$ as follows. For $i \in \{1, 2\}$, we let $\sigma_i^{\mathcal{A}'} := \sigma_i^{\mathcal{A}} \uplus B$ for some set $\mathcal{B}$ of cardinality $\kappa(\sigma_i)$ if the latter is infinite, or of cardinality $\kappa(\sigma_i) - |\sigma_i|^{\mathcal{A}}$ otherwise. Notice that this is well defined because $\kappa(\sigma_i) \geq \left|\sigma_i^{\mathcal{A}}\right|$. As for variables, $x^{\mathcal{A}'} := x^{\mathcal{A}}$ for each variable x in $vars(\phi)$. This is well defined because the domains of σ_1 and σ_2 were only possibly extended, not reduced. First, we prove that $\mathcal{A}'$ is a $\mathcal{T}_{2,3}$-interpretation. If $\kappa(\sigma_1) = 2$, then since $\kappa(\sigma_1) \geq \left|\sigma_1^{\mathcal{A}}\right|$, we must have that $\left|\sigma_1^{\mathcal{A}}\right| = 2$, which means that $|\sigma_2|^{\mathcal{A}}$ is infinite, which in turn means that $\kappa(\sigma_2)$ is infinite as well. Hence in this case we have $\left|\sigma_1^{\mathcal{A}'}\right| = \kappa(\sigma_1) = 2$ and $\left|\sigma_2^{\mathcal{A}'}\right| = \kappa(\sigma_2) \geq \aleph_0$. Otherwise, $\kappa(\sigma_1) \geq 3$, and hence $\left|\sigma_1^{\mathcal{A}'}\right| = \kappa(\sigma_1) \geq 3$ and also $\left|\sigma_2^{\mathcal{A}'}\right| = \kappa(\sigma_2) \geq \left|\sigma_2^{\mathcal{A}}\right| \geq 3$. Clearly, $\mathcal{A}'$ satisfies ϕ as the interpretations of variables did not change and ϕ is quantifier-free. Finally, $\left|\sigma_1^{\mathcal{A}'}\right| = \kappa(\sigma_1)$ and $\left|\sigma_2^{\mathcal{A}'}\right| = \kappa(\sigma_2)$ by construction.

□

We now show that $\mathcal{T}_{2,3}$ is finitely witnessable, but there is no strong witness for it.

Lemma 3 *$\mathcal{T}_{2,3}$ is finitely witnessable w.r.t.* $\{\sigma_1, \sigma_2\}$.

Proof Define a function wit by $wit(\phi) := \phi \wedge x_1 = x_1 \wedge x_2 = x_2 \wedge x_3 = x_3 \wedge y_1 = y_1 \wedge y_2 = y_2 \wedge y_3 = y_3$ for fresh variables x_1, x_2 and x_3 of sort σ_1 and y_1, y_2 and y_3 of sort σ_2. We prove that wit is a witness for $\mathcal{T}_{2,3}$ w.r.t. $\{\sigma_1, \sigma_2\}$. The formulas ϕ and $\exists x_1, x_2, x_3, y_1, y_2, y_3.wit(\phi)$ are trivially logically equivalent and in particular $\mathcal{T}_{2,3}$-equivalent. We prove that $wit(\phi)$ is finitely witnessed for $\mathcal{T}_{2,3}$ w.r.t. $\{\sigma_1, \sigma_2\}$. Suppose that $wit(\phi)$ is $\mathcal{T}_{2,3}$-satisfiable and let $\mathcal{A}$ be a satisfying $\mathcal{T}_{2,3}$-interpretation. Define a Σ_2-interpretation $\mathcal{B}$ simply by $\sigma_1^{\mathcal{B}} = vars_{\sigma_1}(\phi)^{\mathcal{A}} \uplus \{a_1, a_2, a_3\}$ and $\sigma_2^{\mathcal{B}} = vars_{\sigma_2}(\phi)^{\mathcal{A}} \uplus \{b_1, b_2, b_3\}$ for $a_1, a_2, a_3 \notin \sigma_1^{\mathcal{A}}$ and $b_1, b_2, b_3 \notin \sigma_2^{\mathcal{A}}$. The interpretations of variables from ϕ are the same as in $\mathcal{A}$. As for the fresh variables $x_i^{\mathcal{B}} := a_i$ and $y_i^{\mathcal{B}} := b_i$ for $i \in \{1, 2, 3\}$. We prove that $\mathcal{B}$ finitely witnesses $wit(\phi)$ for $\mathcal{T}_{2,3}$ w.r.t. $\{\sigma_1, \sigma_2\}$. First, $\mathcal{B}$ is a $\mathcal{T}_{2,3}$-interpretation, as by construction $\left|\sigma_1^{\mathcal{B}}\right|, \left|\sigma_2^{\mathcal{B}}\right| \geq 3$. Second, $\mathcal{B} \models \phi$ as the interpretations of variables from ϕ did not change, and trivially satisfies the new identities, and so $\mathcal{B} \models wit(\phi)$. Third, by construction $\sigma_1^{\mathcal{B}} = vars_{\sigma_1}(\phi)^{\mathcal{A}} \uplus \{a_1, a_2, a_3\} = vars_{\sigma_1}(\phi)^{\mathcal{B}} \uplus \left\{x_1^{\mathcal{B}}, x_2^{\mathcal{B}}, x_3^{\mathcal{B}}\right\} = vars_{\sigma_1}(wit(\phi))^{\mathcal{B}}$, and similarly for σ_2.

□

[4] In [9], the first condition is written $\left|\sigma_1^{\mathcal{A}}\right| \geq 2$. We use equality as this is equivalent and we believe it makes things clearer.

Lemma 4 *$\mathcal{T}_{2,3}$ is not strongly finitely witnessable w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof Let wit be a witness for $\mathcal{T}_{2,3}$ w.r.t. $\{\sigma_1, \sigma_2\}$. We show that it is not strong. In particular, we show that $wit(v = v)$ is not strongly finitely witnessed for $\mathcal{T}_{2,3}$ w.r.t. $\{\sigma_1, \sigma_2\}$. Consider a $\mathcal{T}_{2,3}$-interpretation $\mathcal{A}$ with $\left|\sigma_1^{\mathcal{A}}\right| = 2$ and $\left|\sigma_2^{\mathcal{A}}\right| = \aleph_0$. Clearly, $\mathcal{A} \models v = v$, and so $\mathcal{A} \models \exists\, \overline{w}.\, wit(v = v)$, with $\overline{w}$ being the variables in $wit(v = v)$ other than v. This in turn means that there is a $\mathcal{T}_{2,3}$-interpretation $\mathcal{A}'$ that satisfies $wit(v = v)$, different from $\mathcal{A}$ only in the interpretations of $\overline{w}$, if anywhere. Let δ be the arrangement over $vars(wit(v = v))$ induced by $\mathcal{A}'$, that is: for each $x, y \in vars(wit(v = v))$, $x = y$ is a literal of δ iff $x^{\mathcal{A}'} = y^{\mathcal{A}'}$; and $x \neq y$ is a literal of δ iff $x^{\mathcal{A}'} \neq y^{\mathcal{A}'}$. Then, δ either asserts that all variables in $vars_{\sigma_1}(wit(v = v))$ are identical, or it partitions them into two equivalence classes. $\mathcal{A}' \models wit(v = v) \wedge \delta$, and so $wit(v = v) \wedge \delta$ is $\mathcal{T}_{2,3}$-satisfiable. We show that it does not have a finite witness for $\mathcal{T}_{2,3}$ w.r.t. S. Suppose for contradiction the existence of $\mathcal{B}$, a finite witness of $wit(v = v) \wedge \delta$ for $\mathcal{T}_{2,3}$ w.r.t. S. Then $\left|\sigma_1^{\mathcal{B}}\right| = \left|vars_{\sigma_1}(wit(v = v) \wedge \delta)^{\mathcal{B}}\right|$. Now, $\mathcal{B} \models \delta$ and $\mathcal{B}$ is a $\mathcal{T}_{2,3}$-interpretation, meaning $\left|\sigma_1^{\mathcal{B}}\right| \geq 2$, so if δ requires all variables of sort σ_1 to be equal, we already have a contradiction. On the other hand, if δ partitions the variables into two equivalence classes, we get that $\left|\sigma_1^{\mathcal{B}}\right| = 2$. But since $\mathcal{B}$ finitely witnesses $wit(v = v) \wedge \delta$ for $\mathcal{T}_{2,3}$ w.r.t. $\{\sigma_1, \sigma_2\}$, we also get that $\sigma_2^{\mathcal{B}}$ is finite, meaning $\mathcal{B}$ is not a $\mathcal{T}_{2,3}$-interpretation. $\square$

Lemmas 2 to 4 have shown that $\mathcal{T}_{2,3}$ is polite but is not strongly polite. Indeed, using the polite combination method from [13] with this theory can cause problems. Consider the theory $\mathcal{T}_{1,1}$ that consists of all Σ_2-structures $\mathcal{A}$ such that $\left|\sigma_1^{\mathcal{A}}\right| = \left|\sigma_2^{\mathcal{A}}\right| = 1$. Clearly, $\mathcal{T}_{1,1} \oplus \mathcal{T}_{2,3}$ contains no structures, and hence no formula is $\mathcal{T}_{1,1} \oplus \mathcal{T}_{2,3}$-satisfiable. However, denote the formula $\mathtt{true}$ by φ_1 and the formula $x = x$ by φ_2 for some variable x of sort σ_1. Then $wit(\varphi_2)$ is $x = x \wedge \bigwedge_{i=1}^{3} x_i = x_i \wedge y_i = y_i$. Let δ be the arrangement $x = x_1 = x_2 = x_3 \wedge y_1 = y_2 = y_3$. It can be shown that $wit(\varphi_2) \wedge \delta$ is $\mathcal{T}_{2,3}$-satisfiable and $\varphi_1 \wedge \delta$ is $\mathcal{T}_{1,1}$-satisfiable. Hence, the combination method of [13] would consider $\varphi_1 \wedge \varphi_2$ to be $\mathcal{T}_{1,1} \oplus \mathcal{T}_{2,3}$-satisfiable, which is impossible. Thus, the fact that $\mathcal{T}_{2,3}$ is not strongly polite propagates all the way to the polite combination method.

Remark 2 An alternative way to separate politeness from strong politeness using $\mathcal{T}_{2,3}$ can be obtained through shiny theories, as follows. Shiny theories were introduced in [24] for the mono-sorted case, and were generalized to many-sorted signatures in two different ways in [9] and [19]. In [9], $T_{2,3}$ was introduced as a theory that is shiny according to [19], but not according to [9]. Theorem 1 of [9] states that their notion of shininess is equivalent to strong politeness for theories in which the satisfiability problem for quantifier-free formulas is decidable. It can be shown that this is the case for $T_{2,3}$. Since it is not shiny according to [9], we get that $T_{2,3}$ is not strongly polite. Furthermore, Proposition 18 of [19] states that every shiny theory (according to their definition) is polite. Hence we get that $T_{2,3}$ is polite but not strongly polite.

We have (and prefer) a direct proof based only on politeness, without a detour through shininess. That proof is provided next. Note also that [9] dealt only with strongly polite theories and did not study the weaker notion of polite theories. In particular, the fact that strong politeness is different from politeness was not stated nor proved there.

3.3 Mono-sorted Politeness

Theory $\mathcal{T}_{2,3}$ includes two sorts but is otherwise empty. In this section, we show that requiring two sorts is essential for separating politeness from strong politeness in otherwise empty signatures.[5] That is, we prove that politeness implies strong politeness otherwise. Let Σ_0 be the signature with a single sort σ and no function or predicate symbols (except $=$). We show that smooth Σ_0-theories have a certain form and conclude strong politeness from politeness.

Lemma 5 *Let $\mathcal{T}$ be a Σ_0-theory. If $\mathcal{T}$ is smooth w.r.t. σ and includes a finite structure, then $\mathcal{T}$ is axiomatized by $\psi^{\sigma}_{\geq n}$ from Figure 2 for some $n > 0$.*

Proof Let $\mathcal{A}$ be the $\mathcal{T}$-structure with a minimal number of elements, and let $n = \left|\sigma^{\mathcal{A}}\right|$. To show that every Σ_0-structure that satisfies $\psi^{\sigma}_{\geq n}$ belongs to $\mathcal{T}$, let $\mathcal{B}$ be a Σ_0-structure that satisfies $\psi^{\sigma}_{\geq n}$ and let m be the cardinality of $\sigma^{\mathcal{B}}$. Then $m \geq n$. Clearly, $\mathcal{A} \models x = x$ and has n elements. Since $\mathcal{T}$ is smooth w.r.t. σ, there exists a $\mathcal{T}$-interpretation (that satisfies $x = x$) with cardinality m. This interpretation must be isomorphic to $\mathcal{B}$, as the lack of any symbols means that the only thing that distinguishes between Σ_0-structures is their cardinality. For the converse, note that by the choice of n as minimal, every $\mathcal{T}$-structure satisfies $\psi^{\sigma}_{\geq n}$.

□

Proposition 1 *If $\mathcal{T}$ is a Σ_0-theory that is polite w.r.t. σ, then it is strongly polite w.r.t. σ.*

Proof The formula $x = x$ is clearly $\mathcal{T}$-satisfiable. Since $\mathcal{T}$ is finitely witnessable (say with witness wit), there is a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit(x = x)$ such that $\sigma^{\mathcal{A}}$ is finite. $\mathcal{T}$ is smooth, and hence, by Lemma 5, axiomatized by $\psi^{\sigma}_{\geq n}$ for some n. Define $wit'(\phi) := \phi \wedge distinct(x_1, \ldots, x_n)$ for fresh $x_1, \ldots, x_n$. Since $\mathcal{T}$ is axiomatized by $\psi^{\sigma}_{\geq n}$, ϕ is $\mathcal{T}$-equivalent to $\exists \overline{x}. wit'(\phi)$. Further, for any arrangement δ over some set of variables, and any $\mathcal{T}$-interpretation $\mathcal{A}'$ that satisfies $wit'(\phi) \wedge \delta$, if the domain of $\mathcal{A}'$ is reduced to contain only the elements in $vars(wit'(\phi) \wedge \delta)^{\mathcal{A}'}$, the result is still a $\mathcal{T}$-interpretation since $wit'(\phi)$ contains $distinct(x_1, \ldots, x_n)$. We therefore get that wit' is a strong witness for $\mathcal{T}$ w.r.t. σ.

□

Remark 3 We again point out, as we did in Remark 2, that an alternative way to obtain this result is via shiny theories, using results in [19], which introduced polite theories, as well as [8], which compared strongly polite theories to shiny theories in the mono-sorted case. Specifically, in the presence of a single sort, Proposition 19 of [19] states that:

($*$) if the question of whether a polite theory over a finite signature contains a given finite structure is decidable, then the theory is shiny.

In turn, Proposition 1 of [8] states that:

($**$) every shiny theory over a mono-sorted signature with a decidable satisfiability problem for quantifier-free formulas is also strongly polite.

It can be shown that the question of whether a polite Σ_0-theory contains a finite structure is decidable. It can also be shown that the satisfiability of quantifier-free

[5] The case of non-empty signatures is addressed in a recent paper by Toledo et al. [25].

formulas is decidable for such theories. Using (∗) and (∗∗), we get that in Σ_0-theories, politeness implies strong politeness. As above (Remark 2), we prefer a direct route for showing this result, without going through shiny theories.

3.4 Mono-sorted Finite Witnessability

We have seen that for Σ_0-theories, politeness and strong politeness are the same. Now we show that smoothness is crucial for this equivalence, i.e., that there is no such equivalence between finite witnessability and strong finite witnessability.

Let $\mathcal{T}^{\infty}_{\text{Even}}$ be the Σ_0-theory of all Σ_0-structures $\mathcal{A}$ such that $\left|\sigma^{\mathcal{A}}\right|$ is even or infinite.[6] Clearly, this theory is not smooth.

Lemma 6 *$\mathcal{T}^{\infty}_{\text{Even}}$ is not smooth w.r.t. σ.*

Proof Let ϕ be $x = x$ and $\mathcal{A}$ be a Σ-interpretation with $\sigma^{\mathcal{A}} = \{a_1, a_2\}$ for some distinct elements a_1, a_2 and with $x^{\mathcal{A}} = a_1$. Then $\mathcal{A}$ is a $\mathcal{T}^{\infty}_{\text{Even}}$-interpretation that satisfies ϕ. Let κ defined by $\kappa(s) = 3$. Then $3 = \kappa(s) \geq \left|\sigma^{\mathcal{A}}\right| = 2$. However, there is no Σ-interpretation $\mathcal{A}'$ with $\left|\sigma^{\mathcal{A}'}\right| = 3$.

□

Next, we show that the theory is finitely witnessable, but not strongly so.

Lemma 7 *$\mathcal{T}^{\infty}_{\text{Even}}$ is finitely witnessable w.r.t. σ.*

Proof For a quantifier-free Σ_0-formula ϕ, define $wit(\phi)$ as follows. Let E be the set of all equivalence relations over $vars(\phi) \cup \{w\}$ for some fresh variable w. Let $even(E)$ be the set of all equivalence relations in E for which the number of equivalence classes is even. Then, $wit(\phi)$ is $\phi \wedge \bigvee_{e \in even(E)} \delta_e$, where for an equivalence relation $e \in even(E)$, δ_e is the arrangement induced by e:

$$\bigwedge_{(x,y) \in e} x = y \wedge \bigwedge_{x,y \in vars(\phi) \cup \{w\} \wedge (x,y) \notin e} x \neq y$$

We prove that wit is a witness. Let ϕ be a Σ-formula. We first prove that it is $\mathcal{T}^{\infty}_{\text{Even}}$-equivalent to $\exists\, w.\ wit(\phi)$. Since ϕ is a conjunct of $wit(\phi)$ that does not include w, every $\mathcal{A}$-interpretation that satisfies $wit(\phi)$ also satisfies ϕ. For the other direction, let $\mathcal{A}$ be a $\mathcal{T}^{\infty}_{\text{Even}}$-interpretation satisfying ϕ. Even though $\mathcal{A}$ may have infinitely many elements, the number of elements in $vars(\phi)^{\mathcal{A}}$ must be finite. If the number of elements in $vars(\phi)^{\mathcal{A}}$ is even, then let a be some arbitrary element of $vars(\phi)^{\mathcal{A}}$. Otherwise, let a be an element in $\mathcal{A}$ different from all the elements in $vars(\phi)^{\mathcal{A}}$ (there must be such an element since $\mathcal{A}$ has an even or infinite number of elements). In either case, the number of elements in $vars(\phi)^{\mathcal{A}} \cup \{a\}$ is even. Thus, if we modify $\mathcal{A}$ to map w to a, then it must satisfy one of the disjuncts in $\bigvee_{e \in even(E)} \delta_e$. Hence, $\mathcal{A}$ satisfies $\exists\, w.\ wit(\phi)$.

Next, if $wit(\phi)$ is $\mathcal{T}^{\infty}_{\text{Even}}$-satisfiable, then there is a satisfying $\mathcal{T}^{\infty}_{\text{Even}}$-interpretation $\mathcal{A}$ satisfying it. $\mathcal{A}$ must satisfy one of the disjuncts in $wit(\phi)$, which means $\left|vars(wit(\phi))^{\mathcal{A}}\right|$ is even. The restriction of $\mathcal{A}$ to $vars(wit(\phi))^{\mathcal{A}}$ is a $\mathcal{T}^{\infty}_{\text{Even}}$-interpretation that finitely witnesses $wit(\phi)$.

□

[6] Notice that $\mathcal{T}^{\infty}_{\text{Even}}$ can be axiomatized using the set $\left\{\neg\psi^{\sigma}_{=2n+1} \mid n \in \mathbb{N}\right\}$ (see Figure 2).

Lemma 8 $\mathcal{T}^{\infty}_{\text{Even}}$ *is not strongly finitely witnessable w.r.t.* σ*.*

Proof Let $wit : QF(\Sigma_0) \rightarrow QF(\Sigma_0)$ be a witness for $\mathcal{T}^{\infty}_{\text{Even}}$ w.r.t. σ. We prove that wit is not a strong witness for $\mathcal{T}^{\infty}_{\text{Even}}$ w.r.t. σ, by showing that $wit(x = x)$ is not strongly finitely witnessed for $\mathcal{T}^{\infty}_{\text{Even}}$ w.r.t. σ. Consider a $\mathcal{T}^{\infty}_{\text{Even}}$-interpretation $\mathcal{A}$ with 2 elements, which interprets all the variables in $vars(wit(x = x))$. Clearly, $\mathcal{A} \models x = x$, and therefore, $\mathcal{A} \models \exists \overline{w}.\, wit(x = x)$, where $\overline{w}$ is $vars(wit(x = x)) \setminus \{x\}$. Hence, there exists a $\mathcal{T}^{\infty}_{\text{Even}}$-interpretation $\mathcal{A}'$, identical to $\mathcal{A}$, except possibly in its interpretation of variables in $vars(wit(x = x)) \setminus \{x\}$, that satisfies $wit(x = x)$. In particular, $\mathcal{A}'$ has two elements. Let $\delta_{\mathcal{A}'}$ be the arrangement over $vars(wit(x = x))$ satisfied by $\mathcal{A}'$. Then $\delta_{\mathcal{A}'}$ induces an equivalence relation with either 1 or 2 equivalence classes. Let v be a variable not in $vars(wit(x = x))$. Define an arrangement δ over $vars(wit(x = x)) \cup \{v\}$ as follows: If $\delta_{\mathcal{A}'}$ induces one equivalence class, $\delta := \delta_{\mathcal{A}'} \wedge \bigwedge_{u \in vars(wit(x=x))} v = u$. Otherwise, $\delta := \delta_{\mathcal{A}'} \wedge \bigwedge_{u \in vars(wit(x=x))} v \neq u$. In the first case, δ induces one equivalence class, and in the second, three. $wit(x = x) \wedge \delta$ is clearly $\mathcal{T}^{\infty}_{\text{Even}}$-satisfiable, but it does not have a finite witness for $\mathcal{T}^{\infty}_{\text{Even}}$ w.r.t. σ, as any interpretation $\mathcal{B}$ that finitely witnesses it has either 1 or 3 elements, and hence it is not in $\mathcal{T}^{\infty}_{\text{Even}}$.

□

4 A Blend of Polite and Stably-Infinite Theories

In this section, we show that the polite combination method can be optimized to reduce the search space of possible arrangements. In what follows, Σ_1 and Σ_2 are disjoint signatures, $S = \mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2} \neq \emptyset$, $\mathcal{T}_1$ is a Σ_1-theory, $\mathcal{T}_2$ is a Σ_2-theory, φ_1 is a conjunction of Σ_1-literals, and φ_2 is a conjunction of Σ_2-literals. When both theories are stably-infinite, the Nelson-Oppen procedure reduces the $\mathcal{T}_1 \oplus \mathcal{T}_2$-satisfiability of $\varphi_1 \wedge \varphi_2$ to the existence of an arrangement δ over the set $V = vars_S(\varphi_1) \cap vars_S(\varphi_2)$, such that $\varphi_1 \wedge \delta$ is $\mathcal{T}_1$-satisfiable and $\varphi_2 \wedge \delta$ is $\mathcal{T}_2$-satisfiable. The correctness of this reduction relies on the fact that both theories are stably infinite w.r.t. S.

In contrast, the polite combination method only requires a condition (namely strong politeness) from one of the theories, while the other theory is unrestricted and, in particular, not necessarily stably infinite. Thus, when $\mathcal{T}_2$ is strongly polite, polite combination reduces the $\mathcal{T}_1 \oplus \mathcal{T}_2$-satisfiability of $\varphi_1 \wedge \varphi_2$ to the existence of an arrangement δ such that $\varphi_1 \wedge \delta$ is $\mathcal{T}_1$-satisfiable and $wit(\varphi_2) \wedge \delta$ is $\mathcal{T}_2$-satisfiable, where wit is a strong witness for $\mathcal{T}_2$ w.r.t. S. The difference with the Nelson-Oppen procedure is that the arrangement δ in this case is not over V above but over a different set $V' = vars_S(wit(\varphi_2))$, Thus, the flexibility offered by polite combination comes with a price. The set V' is potentially larger than V as it contains *all* variables with sorts in S that occur in $wit(\varphi_2)$, not just those that also occur in φ_1. Since the search space of arrangements over a set grows exponentially with its size, this difference can become crucial. If $\mathcal{T}_1$ happens to be stably infinite w.r.t. S, however, we can fall back to Nelson-Oppen combination and only consider variables that are shared by the two formulas. But what if $\mathcal{T}_1$ is stably infinite only w.r.t. to some proper subset $S' \subset S$? Can this knowledge about $\mathcal{T}_1$ help in finding some set V'' of variables between V and V', such that we need only consider arrangements of V''? In this section we prove that this is possible by taking V'' to include only the variables of sorts in S' that are shared between φ_1 and $wit(\varphi_2)$, and all the variables of sorts in $S \setminus S'$ that occur in $wit(\varphi_2)$. We also identify several weaker conditions on $\mathcal{T}_2$ that are sufficient for the combination theorem to hold.

4.1 Refined Combination Theorem

To put the discussion above in formal terms, we recall the following theorem.

Theorem 3 ([13]) *If $\mathcal{T}_2$ is strongly polite w.r.t. S with a witness wit, then the following are equivalent:*

1. *$\varphi_1 \wedge \varphi_2$ is $(\mathcal{T}_1 \oplus \mathcal{T}_2)$-satisfiable;*
2. *there exists an arrangement δ_V over V, such that $\varphi_1 \wedge \delta_V$ is $\mathcal{T}_1$-satisfiable and $wit(\varphi_2) \wedge \delta_V$ is $\mathcal{T}_2$-satisfiable,*

where $V = \bigcup_{\sigma \in S} V_\sigma$, and $V_\sigma = vars_\sigma(wit(\varphi_2))$ for each $\sigma \in S$.

Our goal is to identify general cases in which information regarding $\mathcal{T}_1$ can help reduce the size of the set V. To this end, we extend the definitions of stably infinite, smooth, and strongly finitely witnessable to two sets of sorts rather than one. Roughly speaking, in this extension, the usual definition is taken for the first set, and some cardinality-preserving constraints are enforced on the second set.

Definition 6 Let Σ be a signature, S_1, S_2 two disjoint subsets of $\mathcal{S}_\Sigma$, and $\mathcal{T}$ a Σ-theory.

1. $\mathcal{T}$ is *(strongly) stably infinite w.r.t.* (S_1, S_2) if for every quantifier-free Σ-formula ϕ and $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying ϕ, there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ such that $\mathcal{B} \models \phi$, $|\sigma^{\mathcal{B}}|$ is infinite for every $\sigma \in S_1$, and $|\sigma^{\mathcal{B}}| \leq |\sigma^{\mathcal{A}}|$ ($|\sigma^{\mathcal{B}}| = |\sigma^{\mathcal{A}}|$) for every $\sigma \in S_2$.
2. $\mathcal{T}$ is *smooth w.r.t.* (S_1, S_2) if for every quantifier-free Σ-formula ϕ, $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying ϕ, and function κ from S_1 to the class of cardinals such that $\kappa(\sigma) \geq \left|\sigma^{\mathcal{A}}\right|$ for each $\sigma \in S_1$, there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies ϕ, with $\left|\sigma^{\mathcal{B}}\right| = \kappa(\sigma)$ for each $\sigma \in S_1$, and with $\left|\sigma^{\mathcal{B}}\right|$ infinite whenever $\left|\sigma^{\mathcal{A}}\right|$ is infinite for each $\sigma \in S_2$.
3. $\mathcal{T}$ is *strongly finitely witnessable w.r.t.* (S_1, S_2) if there is a computable function $wit : QF(\Sigma) \rightarrow QF(\Sigma)$ such that for every quantifier-free Σ-formula ϕ:
   (a) ϕ and $\exists\, \overrightarrow{w}.\, wit(\phi)$ are $\mathcal{T}$-equivalent for $\overrightarrow{w} = vars(wit(\phi)) \setminus vars(\phi)$; and
   (b) for every $\mathcal{T}$-interpretation $\mathcal{A}$ and arrangement δ of any set of variables whose sorts are in S_1, if $\mathcal{A}$ satisfies $wit(\phi) \wedge \delta$, then there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ that finitely witnesses $wit(\phi) \wedge \delta$ w.r.t. S_1 and for which $\left|\sigma^{\mathcal{B}}\right|$ is infinite whenever $\left|\sigma^{\mathcal{A}}\right|$ is infinite, for each $\sigma \in S_2$.

Example 8 Consider the signature Σ_2 from Section 3.2 with two sorts, σ_1 and σ_2, and no symbols other than equalities.

1. Let $\mathcal{T}_{\mathrm{inf,inf}}$ be the Σ_2-theory whose structures $\mathcal{A}$ are those in which $\sigma_2^{\mathcal{A}}$ is infinite whenever $\sigma_1^{\mathcal{A}}$ is infinite. Then, $\mathcal{T}_{\mathrm{inf,inf}}$ is stably infinite w.r.t. $\{\sigma_1\}$, but not w.r.t. $(\{\sigma_1\}, \{\sigma_2\})$. Indeed, consider the structure $\mathcal{A}$ in which $\sigma_1^{\mathcal{A}} = \sigma_2^{\mathcal{A}} = \{1\}$. Then $\mathcal{A}$ can be extended to a Σ_2-structure $\mathcal{B}$ such that $\sigma_1^{\mathcal{B}}$ is infinite, but then $\sigma_2^{\mathcal{B}}$ is infinite as well, and so we cannot have $\left|\sigma_2^{\mathcal{B}}\right| \leq \left|\sigma_2^{\mathcal{A}}\right|$.

2. Let $\mathcal{T}_{\text{inf,fin}}$ be the Σ_2-theory whose structures $\mathcal{A}$ are those in which $\sigma_2^{\mathcal{A}}$ is finite whenever $\sigma_1^{\mathcal{A}}$ is infinite. Then, $\mathcal{T}_{\text{inf,fin}}$ is smooth w.r.t. $\{\sigma_1\}$ (if σ_1 has to be interpreted as an infinite domain, we can make the interpretation of σ_2 finite, since the signature is empty). But, $\mathcal{T}_{\text{inf,fin}}$ is not smooth w.r.t. $(\{\sigma_1\}, \{\sigma_2\})$: consider the structure $\mathcal{A}$ in which $\sigma_1^{\mathcal{A}} = \{1\}$ and $\sigma_2^{\mathcal{A}} = \mathbb{N}$. Then $\mathcal{A}$ can be extended to a Σ_2-structure $\mathcal{B}$ such that $\sigma_1^{\mathcal{B}}$ is infinite, but then $\sigma_2^{\mathcal{B}}$ must be finite for $\mathcal{B}$ to be in $\mathcal{T}_{\text{inf,fin}}$, even though $\sigma_2^{\mathcal{A}}$ is infinite.
3. Let $\mathcal{T}_{\text{fin,fin}}$ be the Σ_2-theory whose structures $\mathcal{A}$ are those in which $\sigma_2^{\mathcal{A}}$ is finite whenever $\sigma_1^{\mathcal{A}}$ is finite. $\mathcal{T}_{\text{fin,fin}}$ is strongly finitely witnessable w.r.t. $\{\sigma_1\}$, as when forcing the interpretation of σ_1 to be finite, we can do the same for σ_2 because the signature is empty: if the formula has variables of sort σ_2, we can restrict the domain of σ_2 to be the interpretations of these variables. Otherwise, we can just have a single element as the domain of σ_2. However, $\mathcal{T}_{\text{fin,fin}}$ is not strongly finitely witnessable w.r.t. $(\{\sigma_1\}, \{\sigma_2\})$: consider the structure $\mathcal{A}$ in which $\sigma_1^{\mathcal{A}} = \sigma_2^{\mathcal{A}} = \mathbb{N}$. Then, for any structure $\mathcal{B}$ with a finite $\sigma_1^{\mathcal{B}}$, we must also have that $\sigma_2^{\mathcal{B}}$ is finite in order for $\mathcal{B}$ to be in the theory, even though $\sigma_2^{\mathcal{A}}$ is infinite.

Our main result is the following.

Theorem 4 *Let $S^{\text{si}} \subseteq S$ and $S^{\text{nsi}} = S \setminus S^{\text{si}}$. Suppose $\mathcal{T}_1$ is stably infinite w.r.t. S^{si} and one of the following holds:*

1. *$\mathcal{T}_2$ is strongly stably infinite w.r.t. $(S^{\text{si}}, S^{\text{nsi}})$ and strongly polite w.r.t. S^{nsi} with a witness wit.*
2. *$\mathcal{T}_2$ is stably infinite w.r.t. $(S^{\text{si}}, S^{\text{nsi}})$, smooth w.r.t. $(S^{\text{nsi}}, S^{\text{si}})$, and strongly finitely witnessable w.r.t. S^{nsi} with a witness wit.*
3. *$\mathcal{T}_2$ is stably infinite w.r.t. S^{si} while smooth and strongly finitely-witnessable w.r.t. $(S^{\text{nsi}}, S^{\text{si}})$ with a witness wit.*

Then the following are equivalent:

1. *$\varphi_1 \wedge \varphi_2$ is $(\mathcal{T}_1 \oplus \mathcal{T}_2)$-satisfiable;*
2. *There exists an arrangement δ_V over V such that $\varphi_1 \wedge \delta_V$ is $\mathcal{T}_1$-satisfiable, and $wit(\varphi_2) \wedge \delta_V$ is $\mathcal{T}_2$-satisfiable,*

where $V = \bigcup_{\sigma \in S} V_\sigma$, with $V_\sigma = vars_\sigma(wit(\varphi_2))$ for every $\sigma \in S^{\text{nsi}}$ and $V_\sigma = vars_\sigma(\varphi_1) \cap vars_\sigma(wit(\varphi_2))$ for every $\sigma \in S^{\text{si}}$.

All three items of Theorem 4 include assumptions guaranteeing that the two theories agree on cardinalities of shared sorts. For example, in the first item, we first shrink the S^{nsi}-domains of the T_2-model using strong finite witnessability, and then expand them using smoothness. But then, to obtain infinite domains for the S^{si} sorts, stable infiniteness is not enough, as we need to maintain the cardinalities of the S^{nsi} domains while making the domains of the S^{si} sorts infinite. For this, the stronger property of strong stable infiniteness is used.

A proof of this theorem is provided in Section 4.2, below. Figure 3 is a visualization of the claims in Theorem 4. The theorem considers two variants of strong finite witnessability (Definition 4 and Item 3 of Definition 6), two variants of smoothness (Definition 3 and Item 2 of Definition 6), and three variants of stable infiniteness (Definition 2 and the two new variants from Item 1 of Definition 6). For each of the three cases of Theorem 4, Figure 3 shows which variant of each

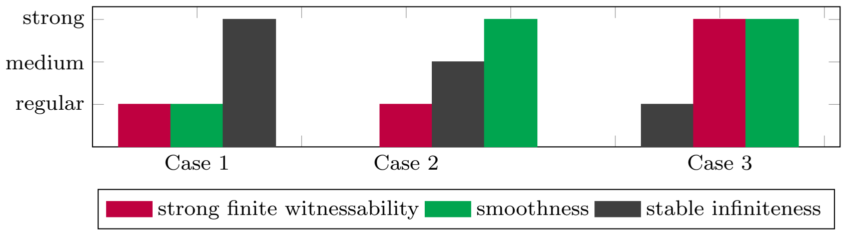

Fig. 3 Theorem 4. The height of each bar corresponds to the strength of the property. The bars are ordered according to their usage in the proof.

property is assumed. The height of each bar corresponds to the strength of the property. In the first case, we use ordinary strong finite witnessability and smoothness, but the strongest variant of stable infiniteness; in the second, we use ordinary strong finite witnessability with the new variants of smoothness and (non-strong) stable infiniteness; and for the third, we use ordinary stable infiniteness and the stronger variants of strong finite witnessability and smoothness. The order of the bars corresponds to the order of their usage in the proof of each case. (This is evident in the proof of Lemma 10.) The stage at which stable infiniteness is used determines the required strength of the other properties: whatever is used before is taken in ordinary form, and whatever is used after requires a stronger form.

Going back to the standard definitions of stable infiniteness, smoothness, and strong finite witnessability, we get the following corollary.

Corollary 1 *Let $S^{\mathrm{si}} \subseteq S$ and $S^{\mathrm{nsi}} = S \setminus S^{\mathrm{si}}$. Suppose $\mathcal{T}_1$ is stably infinite w.r.t. S^{si} and $\mathcal{T}_2$ is strongly finitely witnessable w.r.t. S^{nsi} with witness wit and smooth w.r.t. S. Then, the following are equivalent:*

1. *$\varphi_1 \wedge \varphi_2$ is $(\mathcal{T}_1 \oplus \mathcal{T}_2)$-satisfiable;*
2. *there exists an arrangement δ_V over V such that $\varphi_1 \wedge \delta_V$ is $\mathcal{T}_1$-satisfiable and $wit(\varphi_2) \wedge \delta_V$ is $\mathcal{T}_2$-satisfiable,*

where $V = \bigcup_{\sigma \in S} V_\sigma$, with $V_\sigma = vars_\sigma(wit(\varphi_2))$ for $\sigma \in S^{\mathrm{nsi}}$ and $V_\sigma = vars_\sigma(\varphi_1) \cap vars_\sigma(wit(\varphi_2))$ for $\sigma \in S^{\mathrm{si}}$.

Proof $\mathcal{T}_2$ is smooth w.r.t. $S^{\mathrm{si}} \cup S^{\mathrm{nsi}}$. In particular, it is smooth w.r.t. S^{nsi}, and so it is strongly polite w.r.t. S^{nsi}. We show that it is also strongly stably infinite w.r.t. $(S^{\mathrm{si}}, S^{\mathrm{nsi}})$, and then the result follows from case 1 of Theorem 4. Let ϕ be a Σ-formula and $\mathcal{A}$ a $\mathcal{T}$-interpretation that satisfies ϕ. Define $\kappa(\sigma)$ to be $\aleph_0$ for every $\sigma \in S^{\mathrm{si}}$ such that $\sigma^{\mathcal{A}}$ is finite, $\kappa(\sigma) = |\sigma^{\mathcal{A}}|$ for every $\sigma \in S^{\mathrm{si}}$ such that $\sigma^{\mathcal{A}}$ is infinite, and $\kappa(\sigma) = |\sigma^{\mathcal{A}}|$ for every $\sigma \in S^{\mathrm{nsi}}$. Since $\mathcal{T}$ is smooth w.r.t. $S^{\mathrm{si}} \cup S^{\mathrm{nsi}}$, there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies ϕ with $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$ (which is infinite) for every $\sigma \in S^{\mathrm{si}}$ and $|\sigma^{\mathcal{B}}| = \kappa(\sigma) = |\sigma^{\mathcal{A}}|$ for every $\sigma \in S^{\mathrm{nsi}}$.

□

Finally, the following result, which is closest to Theorem 3, is directly obtained from Corollary 1.

Corollary 2 *Let $S^{\mathrm{si}} \subseteq S$ and $S^{\mathrm{nsi}} = S \setminus S^{\mathrm{si}}$. If $\mathcal{T}_1$ is stably infinite w.r.t. S^{si} and $\mathcal{T}_2$ is strongly polite w.r.t. S with a witness wit, then the following are equivalent:*

1. *$\varphi_1 \wedge \varphi_2$ is $(\mathcal{T}_1 \oplus \mathcal{T}_2)$-satisfiable;*
2. *there exists an arrangement δ_V over V such that $\varphi_1 \wedge \delta_V$ is $\mathcal{T}_1$-satisfiable and $wit(\varphi_2) \wedge \delta_V$ is $\mathcal{T}_2$-satisfiable,*

where $V = \bigcup_{\sigma \in S} V_\sigma$, with $V_\sigma = vars_\sigma(wit(\varphi_2))$ for each $\sigma \in S^{\mathrm{nsi}}$ and $V_\sigma = vars_\sigma(\varphi_1) \cap vars_\sigma(wit(\varphi_2))$ for each $\sigma \in S^{\mathrm{si}}$.

Proof The strong politeness of $\mathcal{T}_2$ w.r.t. $S^{\mathrm{si}} \cup S^{\mathrm{nsi}}$ implies that it is strongly finitely witnessable w.r.t. S^{nsi} and smooth w.r.t. $S^{\mathrm{si}} \cup S^{\mathrm{nsi}}$.

Compared to Theorem 3, Corollary 2 partitions S into S^{si} and S^{nsi} and requires that $\mathcal{T}_1$ be stably infinite w.r.t. S^{si}. The gain from this requirement is that the set V_σ is potentially reduced for $\sigma \in S^{\mathrm{si}}$. Note that unlike Theorem 4 and Corollary 1, Corollary 2 has the same assumptions regarding $\mathcal{T}_2$ as the original Theorem 3 from [13]. We show its potential impact in the next example.

Example 9 Consider the theory $\mathcal{T}_{\mathrm{ListIntBV4}}$ from Example 4. It is strongly polite w.r.t. list and is stably infinite w.r.t. int. Hence, our approach is applicable to it. Let φ_1 be $x = 5 \wedge v = 0000 \wedge w = w \mathbin{\&} v$, and let φ_2 be $a_0 = cons(x, v, a_1) \wedge \bigwedge_{i=1}^{n} a_i = cons(y_i, w, a_{i+1})$. Using the witness function wit from [20], $wit(\varphi_2) = \varphi_2$. The polite combination approach reduces the $\mathcal{T}_{\mathrm{ListIntBV4}}$-satisfiability of $\varphi_1 \wedge \varphi_2$ to the existence of an arrangement δ over $\{x, v, w\} \cup \{y_1, \ldots, y_n\}$, such that $\varphi_1 \wedge \delta$ is $\mathcal{T}_{\mathrm{IntBV4}}$-satisfiable and $wit(\varphi_2) \wedge \delta$ is $\mathcal{T}_{\mathrm{List}}$-satisfiable. Corollary 2 shows that we can do better. Since $\mathcal{T}_{\mathrm{IntBV4}}$ is stably infinite w.r.t. $\{\mathsf{int}\}$, it is enough to check the existence of an arrangement over the variables of sort BV4 that occur in $wit(\varphi_2)$, together with the variables of sort int that are shared between φ_1 and φ_2. This means that arrangements over $\{x, v, w\}$ are considered, instead of over $\{x, v, w\} \cup \{y_1, \ldots, y_n\}$. As n becomes large, standard polite combination requires considering exponentially more arrangements, while the number of arrangements considered by our combination method remains the same.

Remark 4 We remark that various other theories can be given as examples for being strongly polite w.r.t. some of the sorts and stably infinite w.r.t. other sorts. Roughly speaking, in typical applications, the sorts with respect to which the theory would be strongly polite are *container* sorts, such as lists, arrays, etc. The sorts with respect to which the theory would be stably infinite may be *element* sorts, such as integers, reals, etc.

We further note that, as Example 9 illustrates, we expect that the most useful of the results in this section is Corollary 2. The motivation behind Theorem 4 is that it provides the most general result we were able to prove, and makes the proof of Corollary 2 simpler.

4.2 Proof of Theorem 4

The $1 \to 2$ direction is straightforward, using the reducts of the satisfying interpretation of $\varphi_1 \wedge \varphi_2$ to Σ_1 and Σ_2 and the arrangements induced by the satisfying interpretations. We focus on the $2 \to 1$ direction and begin with the following lemma, which strengthens Theorem 1, obtaining a many-sorted Löwenheim-Skolem Theorem, where the cardinality of the finite sorts remains the same.

Lemma 9 *Let Σ be a signature, $\mathcal{T}$ a Σ-theory, ϕ a Σ-formula, and $\mathcal{A}$ a $\mathcal{T}$-interpretation that satisfies ϕ. Let $\mathcal{S}_\Sigma = S_{\mathcal{A}}^{fin} \uplus S_{\mathcal{A}}^{inf}$, where $\sigma^{\mathcal{A}}$ is finite for every $\sigma \in S_{\mathcal{A}}^{fin}$ and $\sigma^{\mathcal{A}}$ is infinite for every $\sigma \in S_{\mathcal{A}}^{inf}$. Then there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies ϕ such that $\left|\sigma^{\mathcal{B}}\right| = \left|\sigma^{\mathcal{A}}\right|$ for every $\sigma \in S_{\mathcal{A}}^{fin}$ and $\sigma^{\mathcal{B}}$ is countable for every $\sigma \in S_{\mathcal{A}}^{inf}$.*

Proof Let Ax be the set of sentences that are satisfied by every $\mathcal{T}$-structure. Define the following sets, based on formulas that are defined in Figure 2:

$$\begin{aligned}
fin_{\mathcal{A}} &:= \left\{\psi^{\sigma}_{=|\sigma^{\mathcal{A}}|} \mid \sigma \in S_{\mathcal{A}}^{fin}\right\}\\
inf_{\mathcal{A}} &:= \left\{\neg\psi^{\sigma}_{=n} \mid \sigma \in S_{\mathcal{A}}^{inf}, n \in \mathbb{N}\right\}\\
\Theta &:= Ax \cup fin_{\mathcal{A}} \cup inf_{\mathcal{A}} \cup \{\phi\}
\end{aligned}$$

Clearly, $\mathcal{A} \models \Theta$. By Theorem 1, there exists a Σ-interpretation $\mathcal{B}$ that satisfies Θ in which $\sigma^{\mathcal{B}}$ is countable whenever it is infinite, for every $\sigma \in \mathcal{S}_\Sigma$. This in particular holds for every $\sigma \in S_{\mathcal{A}}^{inf}$. Now let $\sigma \in S_{\mathcal{A}}^{fin}$, then since $\mathcal{B} \models fin_{\mathcal{A}}$, $\left|\sigma^{\mathcal{B}}\right| = \left|\sigma^{\mathcal{A}}\right|$. Finally, $\mathcal{B} \models \phi$ and it is a $\mathcal{T}$-interpretation.

□

The proof of Theorem 4 continues with the following main lemma.

Lemma 10 (Main Lemma) *Let $S^{\mathrm{si}} \subseteq S$ and $S^{\mathrm{nsi}} = S \backslash S^{\mathrm{si}}$, Suppose $\mathcal{T}_1$ is stably infinite w.r.t. S^{si} and that one of the three cases of Theorem 4 holds. Further, assume there exists an arrangement δ_V over V such that $\varphi_1 \wedge \delta_V$ is $\mathcal{T}_1$-satisfiable, and $wit(\varphi_2) \wedge \delta_V$ is $\mathcal{T}_2$-satisfiable, where $V = \bigcup_{\sigma \in S} V_\sigma$, with $V_\sigma = vars_\sigma(wit(\varphi_2))$ for each $\sigma \in S^{\mathrm{nsi}}$ and $V_\sigma = vars_\sigma(\varphi_1) \cap vars_\sigma(wit(\varphi_2))$ for each $\sigma \in S^{\mathrm{si}}$. Then, there is a $\mathcal{T}_1$-interpretation $\mathcal{A}$ that satisfies $\varphi_1 \wedge \delta_V$ and a $\mathcal{T}_2$-interpretation $\mathcal{B}$ that satisfies $wit(\varphi_2) \wedge \delta_V$ such that $\left|\sigma^{\mathcal{A}}\right| = \left|\sigma^{\mathcal{B}}\right|$ for all $\sigma \in S$.*

Proof Let $\psi_2 := wit(\varphi_2)$. Since $\mathcal{T}_1$ is stably infinite w.r.t. S^{si}, there is a $\mathcal{T}_1$-interpretation $\mathcal{A}$ satisfying $\varphi_1 \wedge \delta_V$ in which $\sigma^{\mathcal{A}}$ is infinite for each $\sigma \in S^{\mathrm{si}}$. By Theorem 1, we may assume that $\sigma^{\mathcal{A}}$ is countable for each $\sigma \in S^{\mathrm{si}}$, as well as for each $\sigma \in S^{\mathrm{nsi}}$ such that $\sigma^{\mathcal{A}}$ is infinite. We consider the cases of Theorem 4:

Case 1 Suppose $\mathcal{T}_2$ is strongly stably infinite w.r.t. $(S^{\mathrm{si}}, S^{\mathrm{nsi}})$ and strongly polite w.r.t. S^{nsi}. Since $\mathcal{T}_2$ is strongly finitely-witnessable w.r.t. S^{nsi}, there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}$ that satisfies $\psi_2 \wedge \delta_V$ such that $\sigma^{\mathcal{B}} = V_\sigma^{\mathcal{B}}$ for each $\sigma \in S^{\mathrm{nsi}}$. Since $\mathcal{A}$ and $\mathcal{B}$ satisfy δ_V, we have that for every $\sigma \in S^{\mathrm{nsi}}$, $\left|\sigma^{\mathcal{B}}\right| = \left|V_\sigma^{\mathcal{B}}\right| = \left|V_\sigma^{\mathcal{A}}\right| \leq \left|\sigma^{\mathcal{A}}\right|$. $\mathcal{T}_2$ is also smooth w.r.t. S^{nsi}, and so there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}'$ satisfying $\psi_2 \wedge \delta_V$ such that $\left|\sigma^{\mathcal{B}'}\right| = \left|\sigma^{\mathcal{A}}\right|$ for each $\sigma \in S^{\mathrm{nsi}}$. Finally, $\mathcal{T}_2$ is strongly stably infinite w.r.t. $(S^{\mathrm{si}}, S^{\mathrm{nsi}})$, so there is a $\mathcal{T}_2$-interpretation $\mathcal{B}''$ that satisfies $\psi_2 \wedge \delta_V$ such that $\sigma^{\mathcal{B}''}$ is infinite for each $\sigma \in S^{\mathrm{si}}$ and $\left|\sigma^{\mathcal{B}''}\right| = \left|\sigma^{\mathcal{B}'}\right| = \left|\sigma^{\mathcal{A}}\right|$ for each $\sigma \in S^{\mathrm{nsi}}$. By Lemma 9, we may assume that $\sigma^{\mathcal{B}''}$ is countable for each $\sigma \in S^{\mathrm{si}}$. Thus, $\left|\sigma^{\mathcal{B}''}\right| = \left|\sigma^{\mathcal{A}}\right|$ for each $\sigma \in S$.

Case 2 : Suppose $\mathcal{T}_2$ is stably infinite w.r.t $(S^{\mathrm{si}}, S^{\mathrm{nsi}})$, smooth w.r.t. $(S^{\mathrm{nsi}}, S^{\mathrm{si}})$, and strongly finitely witnessable w.r.t. S^{nsi}. Then, there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}$ that satisfies $\psi_2 \wedge \delta_V$ such that $\sigma^{\mathcal{B}} = V_\sigma^{\mathcal{B}}$ for every $\sigma \in S^{\mathrm{nsi}}$. Since $\mathcal{A}$ and $\mathcal{B}$ satisfy δ_V, we have that for every $\sigma \in S^{\mathrm{nsi}}$, $\left|\sigma^{\mathcal{B}}\right| = \left|V_\sigma^{\mathcal{B}}\right| = \left|V_\sigma^{\mathcal{A}}\right| \leq \left|\sigma^{\mathcal{A}}\right|$. $\mathcal{T}_2$ is stably infinite w.r.t. $(S^{\mathrm{si}}, S^{\mathrm{nsi}})$, and so there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}'$ that satisfies $\psi_2 \wedge \delta_V$ such that $\sigma^{\mathcal{B}'}$ is infinite for every $\sigma \in S^{\mathrm{si}}$ and $\left|\sigma^{\mathcal{B}'}\right| \leq \left|\sigma^{\mathcal{B}}\right| \leq \left|\sigma^{\mathcal{A}}\right|$ for every $\sigma \in S^{\mathrm{nsi}}$. $\mathcal{T}_2$ is smooth w.r.t. $(S^{\mathrm{nsi}}, S^{\mathrm{si}})$ and so there is a $\mathcal{T}_2$-interpretation $\mathcal{B}''$ satisfying $\psi_2 \wedge \delta_V$ such that $\left|\sigma^{\mathcal{B}''}\right| = \left|\sigma^{\mathcal{A}}\right|$ for every $\sigma \in S^{\mathrm{nsi}}$ and $\left|\sigma^{\mathcal{B}''}\right|$ is infinite for every $\sigma \in S^{\mathrm{si}}$. Using Lemma 9, we may assume $\sigma^{\mathcal{B}''}$ is countable for each $\sigma \in S^{\mathrm{si}}$, and hence $\left|\sigma^{\mathcal{B}''}\right| = \left|\sigma^{\mathcal{A}}\right|$ for every $\sigma \in S$.

Case 3 : Suppose $\mathcal{T}_2$ is stably infinite w.r.t. S^{si}, smooth w.r.t. $(S^{\mathrm{nsi}}, S^{\mathrm{si}})$, and strongly finitely witnessable w.r.t. $(S^{\mathrm{nsi}}, S^{\mathrm{si}})$. Since it is stably infinite w.r.t. S^{si}, there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}$ that satisfies $\psi_2 \wedge \delta_V$ such that $\sigma^{\mathcal{B}}$ is infinite for every $\sigma \in S^{\mathrm{si}}$. $\mathcal{T}_2$ is strongly finitely-witnessable w.r.t. $(S^{\mathrm{nsi}}, S^{\mathrm{si}})$, and hence there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}'$ that satisfies $\psi_2 \wedge \delta_V$ such that $\sigma^{\mathcal{B}'} = V_\sigma^{\mathcal{B}'}$ for every $\sigma \in S^{\mathrm{nsi}}$ and $\left|\sigma^{\mathcal{B}'}\right|$ is infinite for every $\sigma \in S^{\mathrm{si}}$. Since $\mathcal{A}$ and $\mathcal{B}'$ satisfy δ_V, we have that for every $\sigma \in S^{\mathrm{nsi}}$, $\left|\sigma^{\mathcal{B}'}\right| = \left|V_\sigma^{\mathcal{B}'}\right| = \left|V_\sigma^{\mathcal{A}}\right| \leq \left|\sigma^{\mathcal{A}}\right|$. $\mathcal{T}_2$ is smooth w.r.t. $(S^{\mathrm{nsi}}, S^{\mathrm{si}})$, and so there exists a $\mathcal{T}_2$-interpretation $\mathcal{B}''$ that satisfies $\psi_2 \wedge \delta_V$ such that $\left|\sigma^{\mathcal{B}''}\right| = \left|\sigma^{\mathcal{A}}\right|$ for every $\sigma \in S^{\mathrm{nsi}}$ and $\left|\sigma^{\mathcal{B}''}\right|$ is infinite for every $\sigma \in S^{\mathrm{si}}$. By Lemma 9, we may assume that $\sigma^{\mathcal{B}''}$ is countable for every $\sigma \in S^{\mathrm{si}}$, with the same cardinalities for sorts of S^{nsi}, and so we have $\left|\sigma^{\mathcal{B}''}\right| = \left|\sigma^{\mathcal{A}}\right|$ also for every $\sigma \in S$.

□

We now conclude the proof of Theorem 4. Lemma 10 gives us a $\mathcal{T}_1$ interpretation $\mathcal{A}$ with $\mathcal{A} \models \varphi_1 \wedge \delta_V$ and a $\mathcal{T}_2$ interpretation $\mathcal{B}$ with $\mathcal{B} \models \psi_2 \wedge \delta_V$, and $\left|\sigma^{\mathcal{A}}\right| = \left|\sigma^{\mathcal{B}}\right|$ for $\sigma \in S$. Set $\varphi_1' := \varphi_1 \wedge \delta_V$ and $\varphi_2' := \psi_2 \wedge \delta_V$. Then, $V_\sigma = \mathit{vars}_\sigma(\varphi_1') \cap \mathit{vars}_\sigma(\varphi_2')$ for $\sigma \in S$. Now, $\mathcal{A} \models \varphi_1' \wedge \delta_V$ and $\mathcal{B} \models \varphi_2' \wedge \delta_V$. Also, $|\sigma^{\mathcal{A}}| = |\sigma^{\mathcal{B}}|$ for $\sigma \in S$. By Theorem 2, $\varphi_1' \wedge \varphi_2'$ is $\mathcal{T}_1 \oplus \mathcal{T}_2$-satisfiable. In particular, $\varphi_1 \wedge \{\psi_2\}$ is $\mathcal{T}_1 \oplus \mathcal{T}_2$-satisfiable, and hence also $\varphi_1 \wedge \{\exists \overline{w}.\psi_2\}$, with $\overline{w} = \mathit{vars}(\mathit{wit}(\varphi_2)) \setminus \mathit{vars}(\varphi_2)$. Finally, $\exists \overline{w}.\mathit{wit}(\varphi_2)$ is $\mathcal{T}_2$-equivalent to φ_2, hence $\varphi_1 \wedge \varphi_2$ is $\mathcal{T}_1 \oplus \mathcal{T}_2$-satisfiable.

5 Preliminary Case Study

The results presented in Section 4 were motivated by a set of smart contract verification benchmarks. We obtained these benchmarks by applying the open-source Move Prover verifier [26] to smart contracts found in the open-source Diem project [10]. The Move prover is a formal verifier for smart contracts written in the Move language [7] and was designed to target smart contracts used in the Diem blockchain [1]. It works via a translation to the Boogie verification framework [16], which in turn produces SMT-LIB 2 benchmarks that are dispatched to SMT solvers. The benchmarks we obtained involve datatypes, integers, Booleans, and quantifiers. Our case study began by running cvc5 [2] (the successor of CVC4 [5]) on the benchmarks. For most of the benchmarks that were solved by cvc5, theory combination took a small percentage of the overall runtime of the solver, accounting for 10% or less in all but 1 benchmark. However, solving that benchmark took 81 seconds, of which 20 seconds was dedicated to theory combination.

Remark 5 This paper, as most of the combination literature, considers for simplicity but without loss of generality only *mixed* quantifier-free formulas that are conjunctions of pure subformulas. For such mixed formulas, the only symbols that two pure subformulas from different theories may share are variables. However, all combination results can be lifted to more general mixed quantifier-free formulas by using a suitable notion of *shared term* [4]. This is convenient in practice, if not in theory, since it does not require a conversion of mixed formulas to equisatisfiable conjunctions of pure formulas. Since cvc5 follows this approach, in the following we will talk about shared terms, and arrangements over them, instead of shared variables.

We implemented an optimization to the datatype solver of cvc5 based on Corollary 2. With the original polite combination method, every term that originates from the theory of datatypes with another sort is shared with the other theories, triggering an analysis of the arrangements of these terms. In our optimization, we limit the sharing of such terms to those of Boolean sort. In the language of Corollary 2, $\mathcal{T}_1$ is the combined theory of Booleans, uninterpreted functions, and integers, which is stably infinite w.r.t. the uninterpreted sorts and integer sorts. $\mathcal{T}_2$ is an instance of the theory of datatypes, which is strongly polite w.r.t. its element sorts, which in this case are the sorts of $\mathcal{T}_1$.

A comparison of an original and optimized run on the difficult benchmark is shown in Figure 4.[7] The experiment was run on a machine running Ubuntu with a 3.5GHz Intel Xeon E5-2636 processor and 32GB of memory. As shown, the optimization reduces the total running time by 82%, and the time spent on theory combination in particular by 95%. To further isolate the effectiveness of our optimization, we report the number of terms that each theory solver considered. Each theory solver maintains its own data structure for tracking equality information. These data structures contain terms belonging to the theory that either come from the input assertions or are shared with another theory. A data structure is also maintained that contains all shared terms belonging to any theory.

The last 4 columns of Figure 4 count the number of times (in thousands) a term was added to the equality data structure for the theory of datatypes (DT), integers (INT), and uninterpreted functions and Booleans (UFB), as well as to the the shared term data structure (shared). With the optimization, the datatype solver keeps more inferred assertions internally, which leads to an increase in the number of additions of terms to its data structure. However, sharing fewer terms, reduces the number of terms in the data structures for the other theories. Moreover, the number of shared terms decreases by 55%. This suggests that although the workload on the datatypes theory solver is similar, a decrease in the number of

[7] An artifact which includes the compiled binary of the implementation, the benchmark, the raw results, as well as reproduction instructions is available at `https://doi.org/10.5281/zenodo.6538824`.

	total (s)	comb (s)	DT	INT	UFB	shared
optimized	19.7	1.5	203.5	111.9	46.0	108.7
original	111.6	37.2	192.2	352.5	67.6	242.5

Fig. 4 Runtimes (in seconds) and number of terms (in thousands) added to the data structures of DT, INT, UFB, and the number of shared terms (shared).

shared terms originating from the theory of datatypes in the optimized run results in a significant improvement in the overall runtime. Although our evidence is only anecdotal at the moment, we believe this benchmark is highly representative of the potential benefits of our optimization.

6 Conclusion

In this paper, we have made two contributions to the study of theory combination. First, we separated politeness and strong politeness, which shows that sometimes, the (typically harder) task of finding a strong witness is not a waste of effort. Then, we introduced an optimization to the polite combination method, which applies when one of the theories in the combination is stably infinite w.r.t. a subset of the sorts.

We envision several directions for future work. First, the separation of politeness from strong politeness demonstrates a need to identify sufficient criteria for the equivalence of these notions — such as, for instance, the *additivity* criterion introduced by Sheng et al. [20]. Finding other similar conditions for equivalence would provide additional opportunities for reducing proofs of strong politeness for a theory to simpler proofs of politeness. We also plan to extend the initial implementation in cvc5 and evaluate its impact on more benchmarks.

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