Non-deterministic Matrices in Action: Expansions, Refinements, and Rexpansions

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***Abstract*—The operations of expansion and refinement on non-deterministic matrices (Nmatrices) are composed to form a new operation called rexpansion. Properties of this operation are investigated, together with their effects on the induced consequence relations. A semantic method for obtaining conservative extensions of matrix-defined logics is introduced and applied to fragments of the classical two-valued matrix, as well as to other well-known many-valued matrices. The central application of rexpansion that we present is the construction of truth-preserving paraconsistent conservative extensions of Gödel fuzzy logic.**

I. Introduction

Since its introduction in [8], the framework of *non-deterministic matrices*[1] (Nmatrices) has proven to be very useful, since it has almost all the advantages of the framework of ordinary matrices, while capturing logics that are practically left out by it. Accordingly, Nmatrices have been widely investigated and utilized in various areas, like many-valued logics [17], paraconsistent logics [7], and proof theory [18].

In [2] and [7] two fundamental operations which are peculiar to Nmatrices were introduced: expansion and refinement. Both of them transform a given Nmatrix (that may be an ordinary matrix) to another one. In this paper we show that these operations are most useful when they are combined into one operation, which we call *rexpansion* (refined expansion). We investigate rexpansion as a powerful tool for generating new Nmatrices from existing ones. Properties of this combined operation are presented, along with its effects on the consequence relations which are induced by the operated Nmatrices.

The main application of rexpansion in this paper is for the problem of conservatively extending a given logic $\mathbf{L}$ with new connectives which have some desirable properties. The method is to apply appropriate rexpansion to a matrix (or an Nmatrix) that is known to be characteristic for the logic, getting by this alternative semantics for $\mathbf{L}$ for which the addition of the desired connectives is an easier task. We demonstrate this method with several examples. The most important of them provides a new (and as we show, significantly better) solution for the problem of constructing *paraconsistent fuzzy logics*. A first solution to this problem was given in [15], using a completely different approach. However, we show that the solution in [15] has some serious drawbacks, which are overcome in the solution proposed here.

[1] For a survey of Nmatrices, see [9].

II. Preliminaries

A. Non-deterministic Matrices

A *propositional language* $\mathcal{L}$ consists of a countably infinite set of atomic variables $At = \{p_1, p_2, \ldots\}$ and a finite set $\diamondsuit_{\mathcal{L}}$ of propositional connectives. The set of all n-ary connectives of $\mathcal{L}$ is denoted by $\diamondsuit^n_{\mathcal{L}}$, and its set of well-formed formulas by $\mathcal{W}(\mathcal{L})$. We sometimes identify $\mathcal{L}$ with its set of connectives (e.g. when speaking about "the language $\{\wedge, \vee, \neg\}$").

A *propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ such that $\mathcal{L}$ is a propositional language and $\vdash_{\mathbf{L}}$ is a structural (Tarskian) consequence relation for $\mathcal{L}$.

In what follows, $\mathcal{L}$ denotes an arbitrary propositional language. We assume that the reader is familiar with the notion of a *matrix* for $\mathcal{L}$ and the logic induced by it (see e.g. [20]).

Matrices are *truth-functional*, that is, the truth value of a compound formula is uniquely determined by the truth values of its immediate subformulas. Many non-classical logics, however, do not have effective truth-functional semantics. In [9], matrices are generalized to allow non-deterministic assignments of truth values to compound formulas. This extended framework provides effective semantics for many non-classical logics that are left out in the framework of ordinary matrices:

Definition 1 A *non-deterministic matrix (Nmatrix)* for $\mathcal{L}$ is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that $\mathcal{V}$ is a non-empty set (of *truth values*), $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (of *designated* truth values), and $\mathcal{O} : \diamondsuit_{\mathcal{L}} \to \bigcup_{i=0}^{\infty} (\mathcal{V}^i \to P^+(\mathcal{V}))$ is a function assigning a non-deterministic "truth-table" $\mathcal{O}(\diamond) : \mathcal{V}^i \to P^+(\mathcal{V})$ for every connective $\diamond$ of arity i (where $P^+(\mathcal{V}) = P(\mathcal{V}) \setminus \{\emptyset\}$).

To be considered as a particular instance of Nmatrices, we take matrices to be Nmatrices in which $\mathcal{O}(\diamond)(x_1, \ldots, x_n)$ is a singleton for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $x_1, \ldots, x_n \in \mathcal{V}$. In matrices mentioned below we freely interchange truth values with their singletons, whenever there is no danger of confusion.

Nmatrices induce consequence relations in a similar way to matrices, that is, through the use of *valuations*. This is formally defined as follows:

Definition 2 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$-*valuation* is a function v from $\mathcal{W}(\mathcal{L})$ to $\mathcal{V}$ such that for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \mathcal{O}(\diamond)(v(\psi_1), \ldots, v(\psi_n))$. v is an $\mathcal{M}$-*model* of a formula ψ (in symbols: $v \vDash^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. It is an $\mathcal{M}$-*model* of a set $\mathcal{T}$ of formulas (in symbols: $v \vDash^{\mathcal{M}} \mathcal{T}$) if $v \vDash^{\mathcal{M}} \psi$ for every $\psi \in \mathcal{T}$. A formula ψ is an $\mathcal{M}$*-consequence* of a set $\mathcal{T}$ of formulas (in symbols: $\mathcal{T} \vdash_{\mathcal{M}} \psi$) if every $\mathcal{M}$-model of $\mathcal{T}$ is an $\mathcal{M}$-model of ψ. We say that $\mathcal{M}$ *induces* a logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ (or that $\mathcal{M}$ is *characteristic* for $\mathcal{L}$) if $\vdash_{\mathcal{M}} = \vdash_{\mathbf{L}}$.

Like matrices, Nmatrices provide an *analytic* semantic framework, in the sense that for every Nmatrix $\mathcal{M}$, every partial $\mathcal{M}$-valuation can be extended to a full $\mathcal{M}$-valuation.[2] A useful consequence of this property concerns the notion of *conservative extensions*:

Definition 3 *A logic* $\mathbf{L}_2 = \langle \mathcal{L}_2, \vdash_{\mathbf{L}_2} \rangle$ *is* conservative over *a logic* $\mathbf{L}_1 = \langle \mathcal{L}_1, \vdash_{\mathbf{L}_1} \rangle$ *(or in other words:* $\mathbf{L}_2$ *is a* conservative extension *of* $\mathbf{L}_1$*) if* $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$*, and* $\mathcal{T} \vdash_{\mathbf{L}_1} \varphi$ *iff* $\mathcal{T} \vdash_{\mathbf{L}_2} \varphi$ *for every* $\mathcal{T} \subseteq \mathcal{W}(\mathcal{L}_1)$ *and* $\varphi \in \mathcal{W}(\mathcal{L}_1)$*.*

The framework of Nmatrices exhibits a modular character, that that ties the extension of an Nmatrix with the extension of its induced logic.

Definition 4 Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, and $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. $\mathcal{M}_2$ is an *extension of* $\mathcal{M}_1$ *to* $\mathcal{L}_2$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$, and $\mathcal{O}_1(\diamond) = \mathcal{O}_2(\diamond)$ for every $\diamond \in \diamondsuit_{\mathcal{L}_1}$.

Proposition 1 *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be propositional languages such that* $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$*, and* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be Nmatrices for* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, respectively. If* $\mathcal{M}_1$ *is an extension of* $\mathcal{M}_2$ *to* $\mathcal{L}_2$ *then* $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ *is conservative over* $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$*.*

B. Expansions and Refinements

Next we present two basic operations from [2] and [7], that can be performed on Nmatrices: *expansions* and *refinements*. Loosely speaking, an expansion of an Nmatrix is obtained by making several distinct copies of each truth value, so that the new designated values are the copies of the original ones, and each value in the interpretation of the connectives is replaced by all of its copies. This is formally defined as follows:

Definition 5 A function F is called an *expansion function* if for every $x \in dom(F)$, $F(x)$ is a non-empty set, and $F(x) \cap F(y) = \emptyset$ whenever $x \neq y$. F is an *expansion function for an Nmatrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\mathcal{L}$ if $dom(F) = \mathcal{V}$. For every $y \in \bigcup Im(F)$ we denote by $\widetilde{F}[y]$ the unique element $x \in dom(F)$ such that $y \in F(x)$.

Definition 6 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and F an expansion function for $\mathcal{M}$. The F*-expansion of* $\mathcal{M}$ is the Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$, where $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F(x)$, $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F(x)$, and $\mathcal{O}_F(\diamond)(y_1, \ldots, y_n) = \bigcup_{z \in \mathcal{O}(\diamond)(\widetilde{F}[y_1], \ldots, \widetilde{F}[y_n])} F(z)$ for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $y_1, \ldots, y_n \in \mathcal{V}_F$. $\mathcal{M}_2$ is an *expansion* of $\mathcal{M}_1$ if it is the F-expansion of it for some F.

Nothing but uniformly duplicating all truth values is done in expansions, and hence the consequence relation remains the same, as was shown in [2]:

[2]Following [9], we use the term *analytic* for this property.

Proposition 2 *Let* $\mathcal{M}_2$ *be an expansion of* $\mathcal{M}_1$*. Then* $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$*.*

Example 1 Two Nmatrices are isomorphic to one another if and only if one is the F-expansion of the other for some expansion function F (in which $F(x)$ is always a singleton).

Example 2 Consider the usual matrix for classical logic, where the truth values are t and f. By assigning $\{t, \top\}$ to t and $\{f\}$ to f, we obtain an expansion function. The outcome of this expansion would be a non-deterministic matrix for classical logic, in which, for example, the interpretation of negation is $\mathcal{O}(\neg)(t) = \mathcal{O}(\neg)(\top) = \{f\}$ and $\mathcal{O}(\neg)(f) = \{t, \top\}$.

Example 3 The classical matrix can be further expanded by assigning $[0, \frac{1}{2})$ to f and $\left[\frac{1}{2}, 1\right]$ to t. The outcome of this expansion would be another non-deterministic matrix for classical logic. The interpretation of negation would then be $\mathcal{O}(\neg)(x) = [0, \frac{1}{2})$ whenever $x \geq \frac{1}{2}$ and $\mathcal{O}(\neg)(x) = \left[\frac{1}{2}, 1\right]$ whenever $x < \frac{1}{2}$.

Next, we define the *refinement* operation on Nmatrices. Loosely speaking, refining an Nmatrix means dismissing some of its truth values, and then possibly reducing the amount of non-determinism. This is formally defined as follows:

Definition 7 Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$. $\mathcal{M}_2$ is a *refinement* of $\mathcal{M}_1$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1$, $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$, and $\mathcal{O}_2(\diamond)(x_1, \ldots, x_n) \subseteq \mathcal{O}_1(\diamond)(x_1, \ldots, x_n)$ for every $\diamond \in \diamondsuit^n_{\mathcal{L}}$ and $x_1, \ldots, x_n \in \mathcal{V}_2$. $\mathcal{M}_2$ is a *simple refinement* of $\mathcal{M}_1$ if in addition, $\mathcal{V}_2 = \mathcal{V}_1$.

Example 4 The infinite characteristic Nmatrix for classical logic from Example 3 can be (simply) refined by e.g. redefining $\mathcal{O}(\neg)$ in the following way: $\mathcal{O}(\neg)(x) = \{0\}$ whenever $x \geq \frac{1}{2}$ and $\mathcal{O}(\neg)(x) = \{1\}$ whenever $x < \frac{1}{2}$.

Refining an Nmatrix $\mathcal{M}$ can reduce the set of $\mathcal{M}$-valuations, and so we have the following proposition from [7]:

Proposition 3 *Let* $\mathcal{M}_2$ *be a refinement of* $\mathcal{M}_1$*. Then* $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$*.*

III. Refined Expansions

In this section we combine the two basic operations defined above and obtain *refined expansions* (in short: *rexpansions*). In what follows, $\mathcal{L}$ continues to denote a fixed propositional language, and by an Nmatrix we mean an Nmatrix for $\mathcal{L}$, unless stated otherwise.

A. Definition and Properties

We start by explicitly defining the combined operation and exploring its properties.

Definition 8 Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices and F an expansion function for $\mathcal{M}_1$. $\mathcal{M}_2$ is an F*-rexpansion of* $\mathcal{M}_1$ if it is a refinement of the F-expansion of $\mathcal{M}_1$. It is called: (i) *simple* if it is a simple refinement of the F-expansion of $\mathcal{M}_1$; (ii) *preserving* if $F(x) \cap \mathcal{V}_2 \neq \emptyset$ for every $x \in \mathcal{V}_1$; and (iii) *strongly preserving* if it is preserving, and for every $x_1, \ldots, x_n \in \mathcal{V}_2$,

$\diamond \in \Diamond^n_{\mathcal{L}}$, and $y \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1],\ldots,\widetilde{F}[x_n])$, it holds that $F(y) \cap \mathcal{O}_2(\diamond)(x_1,\ldots,x_n) \neq \emptyset$. $\mathcal{M}_2$ is a *rexpansion of* $\mathcal{M}_1$ if it is an F-rexpansion of it for some F.

If $\mathcal{M}_2$ is a rexpansion of $\mathcal{M}_1$, we may call $\mathcal{M}_2$ "preserving", "strongly preserving" or "simple" (without the suffix "rexpansion of $\mathcal{M}_1$") whenever that is clear from the context.

First, let us elaborate on the connections between the different properties of rexpansions:

Lemma 1 *Every simple rexpansion is preserving, every expansion is a strongly preserving rexpansion, and every preserving rexpansion of a matrix is strongly preserving.*

Proof: We prove the third property. Let $\mathcal{M}_1$ be a matrix and $\mathcal{M}_2$ a preserving F-rexpansion of $\mathcal{M}_1$. For every $x_1,\ldots,x_n \in \mathcal{V}_2$, $\diamond \in \Diamond^n_{\mathcal{L}}$, and $y \in \mathcal{O}_1(\diamond)(\widetilde{F}[x_1],\ldots,\widetilde{F}[x_1])$, we have $\mathcal{O}_2(\diamond)(x_1,\ldots,x_n) \subseteq \bigcup_{z\in\mathcal{O}_1(\diamond)(\widetilde{F}[x_1],\ldots,\widetilde{F}[x_1])} F(z)$. Since $\mathcal{M}_1$ is a matrix, $\mathcal{O}_2(\diamond)(x_1,\ldots,x_n) \subseteq F(y)$, and hence $F(y) \cap \mathcal{O}_2(\diamond)(x_1,\ldots,x_n) = \mathcal{O}_2(\diamond)(x_1,\ldots,x_n) \neq \emptyset$. ∎

Example 5 The Nmatrix from Example 4 is a rexpansion of the classical matrix, which is simple and strongly preserving.

Next we provide a necessary and sufficient condition for an Nmatrix to be a rexpansion of another Nmatrix.

Proposition 4 *$\mathcal{M}_2 = \langle\mathcal{V}_2,\mathcal{D}_2,\mathcal{O}_2\rangle$ is a rexpansion of $\mathcal{M}_1 = \langle\mathcal{V}_1,\mathcal{D}_1,\mathcal{O}_1\rangle$ iff there exists $f : \mathcal{V}_2 \to \mathcal{V}_1$ such that: (i) For every $x \in \mathcal{V}_2$, $x \in \mathcal{D}_2$ iff $f(x) \in \mathcal{D}_1$; and (ii) For every $x_1,\ldots,x_n \in \mathcal{V}_2$ and $y \in \mathcal{O}_2(\diamond)(x_1,\ldots,x_n)$, it holds that $f(y) \in \mathcal{O}_1(\diamond)(f(x_1),\ldots,f(x_n))$.*

Proof: Suppose such a function f exists. Let $\mathcal{V}$ be some set such that $\mathcal{V} \cap \mathcal{V}_2 = \emptyset$ and $|\mathcal{V}| = |\mathcal{V}_1|$, and let $g : \mathcal{V}_1 \to \mathcal{V}$ be a bijection. Using properties (i) and (ii) of f, it can be shown that $\mathcal{M}_2$ is an F-rexpansion of $\mathcal{M}_1$ for

$$F = \lambda x \in \mathcal{V}_1. \begin{cases} f^{-1}[\{x\}] & x \in Im(f) \\ \{g(x)\} & otherwise \end{cases}.$$

For the converse, If $\mathcal{M}_2$ is an F-rexpansion of $\mathcal{M}_1$, then the function $\lambda x \in \mathcal{V}_2.\widetilde{F}[x]$ satisfies (i) and (ii). ∎

Remark 1 In [9], the term 'simple refinement' was reserved for what is called here 'refinement', while the term 'refinement' was related to the functions from Proposition 4.

Another useful property of the rexpansion operation is that it induces some forms of transitivity:

Theorem 1
1) If $\mathcal{M}_2$ is a preserving rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a (preserving) rexpansion of $\mathcal{M}_2$, then $\mathcal{M}_3$ is a (preserving) rexpansion of $\mathcal{M}_1$.
2) If $\mathcal{M}_2$ is a strongly preserving rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a strongly preserving rexpansion of $\mathcal{M}_2$, then $\mathcal{M}_3$ is a strongly preserving rexpansion of $\mathcal{M}_1$.

Proof Outline: Let F and G be expansion functions such that $\mathcal{M}_2$ is a preserving F-rexpansion of $\mathcal{M}_1$ and $\mathcal{M}_3$ is a G-rexpansion of $\mathcal{M}_2$. For every $1 \leq i \leq 3$, assume that $\mathcal{M}_i = \langle\mathcal{V}_i,\mathcal{D}_i,\mathcal{O}_i\rangle$. Consider the function $H = \lambda x \in \mathcal{V}_1. \bigcup_{y\in F(x)\cap\mathcal{V}_2} G(y)$. Using the fact that $\mathcal{M}_2$ is preserving, it can be shown that H is an expansion function for $\mathcal{M}_1$. Next, one needs to verify that $\mathcal{M}_3$ is indeed a H-rexpansion of $\mathcal{M}_1$, and also that the preserving and strongly preserving conditions survive this construction. ∎

Corollary 1 *For every sequence $\mathcal{M}_1,\ldots,\mathcal{M}_n$ of Nmatrices such that $\mathcal{M}_{i+1}$ is an expansion or a simple refinement of $\mathcal{M}_i$, we have that $\mathcal{M}_n$ is a preserving rexpansion of $\mathcal{M}_1$.*

B. Consequence Relations

In this section we investigate the effect rexpansions induce on semantically defined consequence relations. Our main theorem is the following:

Theorem 2 *If $\mathcal{M}_2$ is a rexpansion of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Moreover, if $\mathcal{M}_2$ is strongly preserving then $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$.*

Proof Outline: The first part follows directly from Propositions 2 and 3 above. Next, suppose $\mathcal{M}_2$ is a strongly preserving F-rexpansion of $\mathcal{M}_1$. We prove that $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$. For this, it suffices to prove that for every $\mathcal{M}_1$-valuation v there exists an $\mathcal{M}_2$-valuation v' such that $v \vDash^{\mathcal{M}_1} \psi$ iff $v' \vDash^{\mathcal{M}_2} \psi$ for every $\psi \in \mathcal{W}(\mathcal{L})$. Let $c : P(\mathcal{V}_2) \setminus \{\emptyset\} \to \mathcal{V}_2$ and suppose that for every $X \in P(\mathcal{V}_2) \setminus \{\emptyset\}$, $c(X) \in X$.[3] Let $\psi_1,\psi_2,\ldots$ be an enumeration of $\mathcal{W}(\mathcal{L})$ such that if ψ_i is a subformula of ψ_j then $i < j$.

Now let v be an $\mathcal{M}_1$-valuation. For the construction of v', we first define a sequence $v_0,v_1,\ldots$ of *partial functions* from $\mathcal{W}(\mathcal{L})$ to $\mathcal{V}_2$: v_0 is the empty function, and for every $i > 0$, v_i is defined as follows. For every $\psi \in dom(v_{i-1})$, $v_i(\psi) = v_{i-1}(\psi)$. If $\psi_i \notin dom(v_{i-1})$, then: (i) If ψ_i is atomic and $F(v(\psi_i)) \cap \mathcal{V}_2$ is not empty, $v_i(\psi_i) = c(F(v(\psi_i)) \cap \mathcal{V}_2)$; and (ii) If ψ_i has the form $\diamond(\varphi_1,\ldots,\varphi_n)$ for $\varphi_1,\ldots,\varphi_n \in dom(v_{i-1})$ and $F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1),\ldots,v_{i-1}(\varphi_n))$ is not empty, $v_i(\psi_i) = c(F(v(\psi_i)) \cap \mathcal{O}_2(\diamond)(v_{i-1}(\varphi_1),\ldots,v_{i-1}(\varphi_n)))$.

One proves by induction on i that: (i) $v_i(\psi) \in F(v(\psi))$ for every $\psi \in dom(v_i)$; (ii) $dom(v_i) = \{\psi_1,\ldots,\psi_i\}$; and (iii) v_i satisfies the conditions induced by $\mathcal{M}_2$, that is: $v_i(\diamond(\varphi_1,\ldots,\varphi_n)) \in \mathcal{O}_2(\diamond)(v_i(\varphi_1),\ldots,v_i(\varphi_n))$ whenever $\varphi_1,\ldots,\varphi_n$ and $\diamond(\varphi_1,\ldots,\varphi_n)$ are in $dom(v_i)$. Next, For every $\psi \in \mathcal{W}(\mathcal{L})$, let $i_\psi = \iota i \in \mathbb{N}.\psi = \psi_i$.[4] v' is defined by $v'(\psi) = v_{i_\psi}(\psi)$. It is left to verify that v' is an $\mathcal{M}_2$-valuation, and that $v \vDash^{\mathcal{M}_1} \psi$ iff $v' \vDash^{\mathcal{M}_2} \psi$ for every formula ψ. ∎

The following corollary immediately follows as a consequence of Lemma 1 and Theorem 2:

Corollary 2 *Let $\mathcal{M}_2$ be a preserving rexpansion of $\mathcal{M}_1$. If $\mathcal{M}_1$ is a matrix then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.*

An important consequence of Corollary 2 and Proposition 1 (the usefulness of which is demonstrated in Sections IV and V)

[3] The existence of c relies on the axiom of choice in case $\mathcal{V}_2$ is infinite.
[4] That is, i_ψ is the index of ψ in the enumeration.

is a general method of providing a given logic with an alternative new semantics, and then use it for conservatively augmenting it with new connectives. The following corollary establishes this application.

Corollary 3 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, $\mathcal{M}_1$ a matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some preserving rexpansion of $\mathcal{M}_1$. Then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$.*

We conclude this section with a stronger instance of Corollary 3, that applies only for two-valued matrices:

Corollary 4 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages such that $\mathcal{W}(\mathcal{L}_1) \subseteq \mathcal{W}(\mathcal{L}_2)$, $\mathcal{M}_1 = \langle \{t, f\}, \{t\}, \mathcal{O}_1 \rangle$ a matrix for $\mathcal{L}_1$, and $\mathcal{M}_2$ an extension to $\mathcal{L}_2$ of some rexpansion of $\mathcal{M}_1$. Then $\langle \mathcal{L}_2, \vdash_{\mathcal{M}_2} \rangle$ is conservative over $\langle \mathcal{L}_1, \vdash_{\mathcal{M}_1} \rangle$.*

Proof: By Definition 1, for every F-expansion $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$ of $\mathcal{M}_1$ we must have $F(t) \cap \mathcal{V}' \neq \emptyset$ (as otherwise $\mathcal{D}' = \emptyset$) and $F(f) \cap \mathcal{V}' \neq \emptyset$ (as otherwise $\mathcal{D}' = \mathcal{V}'$). The result then follows from Corollary 3. ∎

IV. Some Basic Applications

In this section we present some examples of applications of rexpansion in non-classical logics. This is done by performing it on fragments of well-known matrices, and thus obtaining conservative extensions of their induced logics.

Let $CL = \{\neg, \vee, \wedge, \supset, \bot\}$. Denote the (propositional) language whose set of connectives is CL by $\mathcal{CL}$. For any $C \subseteq CL$, we denote the C-fragment of $\mathcal{CL}$ by $\mathcal{CL}^C$, the C-fragment of the classical matrix by $\mathcal{M}^C_{\mathbf{CL}}$, and the C-fragment of classical logic by $\mathbf{CL}^C$. We start with a direct consequence of Corollary 4:

Lemma 2 *Let $C \subseteq CL$. If $\mathcal{L}$ is a language such that $\mathcal{W}(\mathcal{CL}^C) \subseteq \mathcal{W}(\mathcal{L})$, and $\mathcal{M}$ is an extension to $\mathcal{L}$ of some rexpansion of $\mathcal{M}^C_{\mathbf{CL}}$, then $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is conservative over $\mathbf{CL}^C$.*

Now we use Lemma 2 to present conservative extensions of fragments of classical logic. Some of the resulting logics are *paraconsistent*, that is: unlike classical logic, they tolerate contradictions. Here is the formal definition from [12]:

Definition 9 A logic $\mathbf{L}$ in $\mathcal{CL}$ is *paraconsistent* if: (i) $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ only if $\mathcal{T} \vdash_{\mathbf{CL}} \varphi$.[5]; and (ii) $\neg p, p \nvdash_{\mathbf{L}} q$. It is *boldly paraconsistent* if, in addition, $\neg p, p \nvdash_{\mathbf{L}} \varphi$ whenever $\nvdash_{\mathbf{L}} \varphi$ and $p \notin At(\varphi)$.

Example 6 Asenjo–Priest's three-valued logic of paradox $\mathbf{LP}$ [5] [19] and Kleene's three-valued logic $\mathbf{KL}$ [16] are both defined by matrices that differ only in the set of designated values. Consider the set $\mathcal{V}_3 = \{t, f, i\}$, and the interpretation function $\mathcal{O}_3$ that is defined by the following tables:

$\mathcal{O}_3(\wedge)$	t	f	i
t	t	f	i
f	f	f	f
i	i	f	i

$\mathcal{O}_3(\vee)$	t	f	i
t	t	t	t
f	t	f	i
i	t	i	i

	$\mathcal{O}_3(\neg)$
t	f
f	t
i	i

[5]We follow [4] and include this requirement, as in practice, most paraconsistent logics are sub-classical.

$\mathbf{LP}$ is characterized by $\mathcal{M}_{\mathbf{LP}} = \langle \mathcal{V}_3, \{t, i\}, \mathcal{O}_3 \rangle$, and $\mathbf{KL}$ by $\mathcal{M}_{\mathbf{KL}} = \langle \mathcal{V}_3, \{t\}, \mathcal{O}_3 \rangle$. The $\{\wedge, \vee\}$-fragments of $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{KL}}$ are rexpansions of $\mathcal{M}^{\{\wedge,\vee\}}_{\mathbf{CL}}$, as can be witnessed by the rexpansion functions $\lambda x \in \{t, f\} . \begin{cases} \{t, i\} & x = t \\ \{f\} & x = f \end{cases}$ and $\lambda x \in \{t, f\} . \begin{cases} \{t\} & x = t \\ \{f, i\} & x = f \end{cases}$, respectively. By Lemma 2, $\mathbf{LP}$ and $\mathbf{KL}$ are conservative over $\mathbf{CL}^{\{\wedge,\vee\}}$.

Example 6 shows that the converse of Theorem 2 fails: the $\{\wedge, \vee\}$-fragments of $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{KL}}$ induce the same logic, and there is no expansion function between them (and so neither is a strongly preserving expansion of the other).

$\mathbf{LP}$ is among the three-valued paraconsistent logics from [4] that we consider in the next example:

Example 7 Theorem 42 of [4] characterises all three-valued paraconsistent logics in the language $\{\neg, \wedge, \vee, \supset\}$ that admit some natural properties. These logics coincide with the $\{\neg, \wedge, \vee, \supset\}$-fragments of the $8K$ conservative extensions of positive classical logic [13]. The three-valued matrices that induce these logics are all simple refinements of the following Nmatrix $\mathcal{M} = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O} \rangle$, where $\mathcal{O}$ is given by:

$\mathcal{O}(\wedge)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t,\top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$

$\mathcal{O}(\vee)$	t	f	$\top$
t	$\{t\}$	$\{t\}$	$\{t,\top\}$
f	$\{t\}$	$\{f\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

$\mathcal{O}(\supset)$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t,\top\}$
f	$\{t\}$	$\{t\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$

	$\mathcal{O}(\neg)$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t,\top\}$

Now, every simple refinement of $\mathcal{M}$ is an extension to $\{\wedge, \vee, \supset, \neg\}$ of some rexpansion of $\mathcal{M}^{\{\wedge,\vee,\supset\}}_{\mathbf{CL}}$. Indeed, for $F(t) = \{t, \top\}$ and $F(f) = \{f\}$, it is easy to see that the $\{\wedge, \vee, \supset\}$-fragment of $\mathcal{M}$ is a simple refinement of $(\mathcal{M}^{\{\wedge,\vee,\supset\}}_{\mathbf{CL}})_F = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O}' \rangle$, where:

$\mathcal{O}'(\wedge)$	t	f	$\top$
t	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$

$\mathcal{O}'(\vee)$	t	f	$\top$
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

$\mathcal{O}'(\supset)$	t	f	$\top$
t	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f\}$	$\{t,\top\}$

The fact that all these logics are conservative over positive classical logic is then obtained as a consequence of Lemma 2. Actually, by Corollary 4, all extensions of simple refinements of $(\mathcal{M}^{\{\wedge,\vee,\supset\}}_{\mathbf{CL}})_F$ have this property. In addition, they also have many of the natural properties demanded in [4].

Example 8 The $\{\wedge, \vee, \supset, \neg\}$-fragment of the four-valued logic of billatices from [3] is characterized by the matrix $\mathcal{M}_4 = \langle \{t, f, \top, \bot\}, \{\top, \bot\}, \mathcal{O}_4 \rangle$, where $\mathcal{O}_4$ is given by:

$\mathcal{O}_4(\wedge)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\mathcal{O}_4(\vee)$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\mathcal{O}_4(\supset)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	t	t	t	t
$\top$	t	f	$\top$	$\bot$
$\bot$	t	t	t	t

	$\mathcal{O}_4(\neg)$
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

By dismissing $\supset$, we obtain a matrix for the logic of first-degree entailment [10]. Define an expansion function F for $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$ by $F(f) = \{f, \bot\}$ and $F(t) = \{t, \top\}$. It is easy to see that $\mathcal{M}_4$ is an extension to $\{\wedge, \vee, \supset, \neg\}$ of a simple refinement of $(\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}})_F = \langle \{t, \top, f, \bot\}, \{t, \top\}, \mathcal{O}\rangle$, where

$\mathcal{O}(\wedge)$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
f	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$
$\top$	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
$\bot$	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$	$\{f,\bot\}$

$\mathcal{O}(\vee)$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\bot$	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$

$\mathcal{O}(\supset)$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f,\bot\}$	$\{t,\top\}$	$\{f,\bot\}$
$\bot$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

[3] provides an analytic sequent calculus for $\vdash_{\mathcal{M}_4}$, and uses it to prove that it is conservative over $\mathbf{CL}^{\{\wedge,\vee,\supset\}}$ (and that fde is conservative over $\mathbf{CL}^{\{\wedge,\vee\}}$). Here we obtain this result as a simple consequence of Lemma 2, by identifying the $\{\wedge, \vee, \supset\}$-fragment of $\mathcal{M}_4$ as a rexpansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\supset\}}$.

Example 9 Gödel fuzzy logic $\mathbf{G}$ [14] is characterized by the following matrix $\mathcal{M}_{\mathbf{G}} = \langle \mathcal{V}_{\mathbf{G}}, \mathcal{D}_{\mathbf{G}}, \mathcal{O}_{\mathbf{G}}\rangle$ for $\{\wedge, \vee, \supset, \bot\}$, where $\mathcal{V}_{\mathbf{G}} = [0, 1]$, $\mathcal{D}_{\mathbf{G}} = \{1\}$, and $\mathcal{O}_{\mathbf{G}}$ is given by: $\mathcal{O}_{\mathbf{G}}(\bot) = 0$, $\mathcal{O}_{\mathbf{G}}(\vee)(a, b) = \max\{a, b\}$, $\mathcal{O}_{\mathbf{G}}(\wedge)(a, b) = \min\{a, b\}$, and $\mathcal{O}_{\mathbf{G}}(\supset)(a, b) = \begin{cases} 1 & a \leq b \\ b & a \not\leq b \end{cases}$. $\mathcal{M}_{\mathbf{G}}$ is an extension to $\{\wedge, \vee, \supset, \bot\}$ of a simple refinement of the F-expansion of $\mathcal{M}_{\mathbf{CL}}^{\{\wedge,\vee,\bot\}}$, for $F(f) = [0, 1)$ and $F(t) = \{1\}$. By Lemma 2, it is conservative over $\mathbf{CL}^{\{\wedge,\vee,\bot\}}$. This argument does not survive the addition of implication: $\mathcal{O}_{\mathbf{G}}(\supset)(0.5, 0.25) = 0.25$, while $\mathcal{O}_{\mathbf{CL}}(\supset)(0, 0) = 1$ and $0.25 \notin F(1)$. Indeed, $\mathbf{G}$ is not conservative over $\mathbf{CL}^{\{\wedge,\vee,\supset\}}$.

The process described in the above examples need not start with classical logic, as can be seen by the following example:

Example 10 Consider $\mathcal{M} = \langle \{t, f, \top, \bot\}, \{t\}, \mathcal{O}\rangle$, where:

$\mathcal{O}(\wedge)$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	$\bot$
$\bot$	$\bot$	f	$\bot$	$\bot$

$\mathcal{O}(\vee)$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	$\top$
$\bot$	t	$\bot$	$\top$	$\bot$

Negation is defined by $\mathcal{O}(\neg)(t) = f$ and $\mathcal{O}(\neg)(x) = t$ if $x \neq t$. $\wedge$ and $\vee$ are interpreted as minimum and maximum (respectively), for $f \leq \bot \leq \top \leq t$. Its $\{\wedge, \vee\}$-fragment is a simple F-rexpansion of the $\{\wedge, \vee\}$-fragment of $\mathcal{M}_{\mathbf{KL}}$, for $F(t) = \{t\}$, $F(f) = \{f, \bot\}$ and $F(i) = \{\top\}$. By Corollary 3, the logic it induces is conservative over the $\{\wedge, \vee\}$-fragment of $\mathbf{KL}$. Unlike $\mathbf{KL}$, it has tautologies (e.g. $p\vee\neg p$).

V. Negations for Gödel Logic

In this section we combine two properties of logics that were discussed in Section IV: paraconsistency and fuzziness. As an artifact of their definition via matrices with a single designated value, ordinary fuzzy logics, and in particular, Gödel logic (Example 9), preserve *absolute truth*, that is, a formula follows from a set of formulas if whenever the set is *completely true* (i.e. assigned with 1), so is the formula. As a result, none of the standard fuzzy logics is paraconsistent (it is well known [4] that at least two designated valued are required for paraconsistency). In order to develop logics that are both paraconsistent and fuzzy, it is therefore necessary to replace this consequence relation by a less strict one.

The paper [15] does exactly this, by considering a recent approach [11] to fuzzy logic consequence relations, that, instead of preserving absolute truth, preserves *degrees of truth*. In such a consequence relation, a formula φ follows from a set of formulas $\mathcal{T}$ if there is a finite subset $T \subseteq \mathcal{T}$ whose minimal truth value is never greater than that of φ. This change in definition allows [15] to add an involutive negation to standard fuzzy logics and obtain logics that are both fuzzy and paraconsistent. For example, degree-preserving Łukasiewicz-logic (that already includes its own involutive negation) admits the following properties: (i) it is paraconsistent; (ii) De Morgan laws and double negation principle are valid: $\varphi \equiv \neg\neg\varphi$, $\neg(\varphi\vee\psi) \equiv (\neg\varphi\wedge\neg\psi)$ and $\neg(\varphi\wedge\psi) \equiv (\neg\varphi\vee\neg\psi)$; (iii) natural classically valid rules for manipulating negation and implication are recovered: $\neg(\varphi \supset \psi) \supset \neg\psi$, $(\varphi \supset \psi) \supset (\neg(\varphi \supset \psi) \supset \varphi)$; and also (iv) $\varphi \supset (\neg\psi \supset \neg(\varphi \supset \psi))$. However, it does *not* admit the following: (v) being boldly paraconsistent; (vi) M.P. for $\supset$; (vii) deduction theorem for $\supset$; (viii) validity of $\varphi\vee\neg\varphi$; (ix) being conservative over the original truth-preservation logic. Taking Gödel logic rather than Łukasiewicz logic solves (v), (vi), (vii), and (ix), but loses (iv).

The method of rexpansions allows us to present a better approach to the construction of paraconsistent conservative extensions of Gödel logic, which stays within the framework of truth-preservation. This is done by relaxing the principle of truth-functionality, and the preservation of absolute truth. The former is done by basing our construction on Nmatrices, and the latter by replacing "completely true" with "true enough", that is, taking a larger set of designated truth values:

Definition 10 Let $0 < t \leq 1$. $\mathcal{M}_{\mathbf{G}}{}^t$ is the Nmatrix for $\{\wedge, \vee, \supset, \bot\}$ obtained from $\mathcal{M}_{\mathbf{G}}$ by: (i) Taking $[t, 1]$ as the designated values. (ii) Changing $\mathcal{O}(\supset)$ to $\mathcal{O}(\supset)(a, b) = \begin{cases} [t, 1] & a \leq b \text{ or } b \geq t \\ \{b\} & a > b \text{ and } b < t \end{cases}$. $\mathcal{M}_{\mathbf{G}}{}^t_{\neg}$ is the extension of $\mathcal{M}_{\mathbf{G}}{}^t$ to $\{\wedge, \vee, \supset, \bot, \neg\}$, in which $\mathcal{O}(\neg)(a) = 1 - a$.[6]

[6]This is Łukasiewicz involutive negation.

Theorem 3 *Let $0 < t \leq 1$ and let $\mathcal{M}$ be a simple refinement of $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$. Then: 1) $\vdash_{\mathcal{M}}$ satisfies (ii), (vi), and (ix). 2) If $t > \frac{1}{2}$ then $\vdash_{\mathcal{M}}$ does not satisfy (i) nor $(viii)$. 3) If $t \leq \frac{1}{2}$ then $\vdash_{\mathcal{M}}$ satisfies (i), (v) and $(viii)$.*

Proof: It is straight-forward to verify that $\mathcal{M}_{\mathbf{G}}{}^{t}$ is a simple F-expansion of $\mathcal{M}_{\mathbf{G}}$, for $F = \lambda x \in [0,1] \,.\begin{cases} [t,1] & x = 1 \\ \{t \cdot x\} & x < 1 \end{cases}$. By Corollary 3, every simple refinement of $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$ induces a logic that is conservative over $\mathbf{G}$. The other properties are verified in a routine manner. ■

An interesting simple refinement of $\mathcal{M}_{\mathbf{G}}{}^{\frac{1}{2}}_{\neg}$ is the matrix for the logic $RM^{\supset}$ [6], where implication is interpreted by: $\mathcal{O}(\supset)(a,b) = \begin{cases} \{1-a\} & a \leq b \leq 1-a \\ \{b\} & else \end{cases}$.[7] This logic satisfies all of the properties discussed above. In fact, it can be shown that it is the only such logic.

The proof of Theorem 3 actually provides another interesting result regarding the Gödel matrix, that it, that the same logic would result if the designated values were taken to be any interval of the form $[t,1]$ for any $0 < t < 1$.

Proposition 5 *Let $0 < t < 1$ and $\mathcal{M}^t = \langle \mathcal{V}^t, \mathcal{D}^t, \mathcal{O}^t \rangle$, where $\mathcal{V}^t = [0,1]$, $\mathcal{D}^t = [t,1]$, and $\mathcal{O}^t = \mathcal{O}_{\mathbf{G}}$. Then $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathcal{M}^t}$.*

Proof: $\mathcal{M}^t$ is a simple refinement of $\mathcal{M}_{\mathbf{G}}{}^{t}$, which is a simple rexpansion of $\mathcal{M}_{\mathbf{G}}$. By Corollary 2, $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathcal{M}^t}$. ■

Finally, we note that other negations can be considered for $\mathbf{G}$, and rexpansions (and in particular Corollary 2) can be used in order to prove that the result is conservative over $\mathbf{G}$.

Lemma 3 *Let A be a set of axioms in $\mathcal{CL}$. If A is valid in $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$ then $\mathbf{G}^A$, the axiomatic extension of $\mathbf{G}$ with A, is conservative over $\mathbf{G}$.*

Proof: $\mathbf{G}^A$ cannot cancel any existing consequences in $\mathbf{G}$. For the converse, any consequence in $\mathbf{G}^A$ is valid in $\mathcal{M}_{\mathbf{G}}{}^{t}_{\neg}$, and therefore also in $\mathbf{G}$. ■

Note that finding a new semantics for the augmented logic is not required, as only *soundness* is needed for the proof.

Example 11 By taking A to consist of the axioms from (ii) above, we obtain an axiomatic extension of $\mathbf{G}$ with a negation that satisfies the usual double negation and De Morgan rules, and is conservative over $\mathbf{G}$.

VI. Conclusion and Further Research

We have investigated rexpansions – compositions of expansions and refinements. Properties of this operation were proved, as well as their effects on consequence relations. Examples of applications of these results were also given, including the construction of conservative extensions for many logics from the literature. Theorem 2 provides a sufficient condition for two Nmatrices to induce the same consequence relation, while Example 6 shows that this condition is not necessary. An interesting direction for further research is to characterize general cases in which the condition it suggests is also necessary. Future work would also include more applications of rexpansions, in the spirit of Sections IV and V. In particular, Section V should be extended beyond Gödel logic, to provide a *general* method for the construction of paraconsistent fuzzy logics, based on rexpansions.

[7] $RM^{\supset}$ is shown in [6] to be equivalent to the famous Dunn-Meyer semi-relevant logic $\mathbf{RM}$ (see e.g. [1]).

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