

# Bringing closure to theory combination properties

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**Abstract.** We consider the closure of three classical combination properties, namely, stable infiniteness, gentleness and shininess (or, equivalently for decidable theories, strong politeness), under intersection and combinability. We compute every possible intersection, and then compute the maximal set of theories that can be combined with each resulting intersection. We iterate this process until no new sets are identified. How many properties will we end up with?

## 1 Introduction

Theory combination in Satisfiability Modulo Theories [1,2] studies how to combine decision procedures of two theories into a decision procedure for their combination (or, axiomatically, their union). Most research in this field focuses on *disjoint* combination, where the theories share no symbols other than equality.

Nelson and Oppen proved [8,9] that any pair of decidable stably infinite<sup>1</sup> theories can be disjointly combined. Tinelli and Zarba [12] proved that every decidable shiny theory can be disjointly combined with any decidable theory.<sup>2</sup> Fontaine [5] proved that every decidable gentle theory can be disjointly combined with a general class of theories that includes gentle theories.

Recently, we studied the other directions of these results [10]. We have shown that the first two results are *sharp*: every decidable theory that can be disjointly combined with all decidable stably infinite theories must be stably infinite; and every decidable theory that can be disjointly combined with all decidable theories must be shiny. As for gentleness, we proved in [14,10] that the result of [5] can be made sharp, by introducing the notion of theories with *computable finite spectra*.

All these results are summarized in Table 1 and Figure 1a, using the notations of Table 2. They also employ a function, denoted  $G$ , that takes a set  $X$  of decidable theories, and returns the set  $G(X)$  of all decidable theories that can be disjointly combined with every theory of  $X$ . For example, the Nelson–Oppen theorem is phrased as  $\mathfrak{T}_{\text{SI}} \subseteq G(\mathfrak{T}_{\text{SI}})$  and its sharpened version as  $\mathfrak{T}_{\text{SI}} = G(\mathfrak{T}_{\text{SI}})$ .

In Figure 1a, an edge between two sets of theories means that the lower is a subset of the upper (all inclusions were proven in [10]).  $G(X)$  is the mirror of  $X$  against the dashed line. This structure is contained in the lattice of sharp theory combination properties, introduced in [10]: the order is  $\subseteq$ , the meet is the intersection, and the join (which is not as useful in our context) is defined in

<sup>1</sup> All technical notions in the introduction will be formally defined in Section 2.

<sup>2</sup> Following [3,11,6], the same can be said about decidable strongly polite theories.

Table 1: Classical theory combination results and their sharpened versions

| Source | Original Result                                                    | Sharpness Results [10]                                                                                               |
|--------|--------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| [8]    | $\mathfrak{T}_{\text{SI}} \subseteq G(\mathfrak{T}_{\text{SI}})$   | $\mathfrak{T}_{\text{SI}} = G(\mathfrak{T}_{\text{SI}})$                                                             |
| [12]   | $\mathfrak{T}_{\text{shiny}} \subseteq G(\mathfrak{T})$            | $\mathfrak{T}_{\text{shiny}} = G(\mathfrak{T}), G(\mathfrak{T}_{\text{shiny}}) = \mathfrak{T}$                       |
| [5]    | $\mathfrak{T}_{\text{gen}} \subseteq G(\mathfrak{T}_{\text{gen}})$ | $\mathfrak{T}_{\text{CFS}} = G(\mathfrak{T}_{\text{gen}}), G(\mathfrak{T}_{\text{CFS}}) = \mathfrak{T}_{\text{gen}}$ |

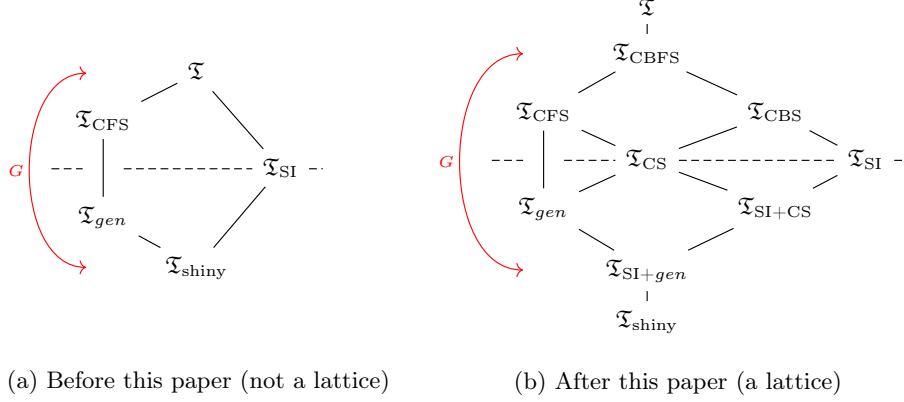


Fig. 1: Diagrams of theory combination properties

terms of the meet and  $G$ . But, Figure 1a itself is not a lattice. For example, the intersection of gentleness and stable infiniteness is not there. The main motivating question of this paper is: **what is the smallest lattice, closed under  $G$ , containing Figure 1a?** The answer, as we prove in Section 3, is Figure 1b.

As a practical outcome, we introduce two new combination methods in Section 4, that allow combinations absent from Figure 1a. We improve the understanding of theory combination properties at a general, abstract level; in the future, when theories are considered for implementation, researchers will have a larger suite of tests they can perform to determine the theories' combinability properties. And, if a pair of theories cannot be combined, researchers will have a more precise understanding of how the expressiveness of one of the theories needs to be weakened to allow combination. Following [10], we focus on one-sorted logic, leaving the generalization to many-sorted logic to future work.

**Outline.** In Section 2 we present preliminaries, from first-order logic, through theory combination, to the Galois connection of theory combination. Section 3 presents the process through which we close the lattice; Section 3.2 gives relationships between properties and may be skipped in a first reading, being quite technical, much like the new theories of Table 6, which are useful only for proofs. Section 4 gives further details on new combination methods. Section 4.1 shows sharpness. Section 5 introduces a separating property that we call *self-combinability*, roughly meaning the ability of a theory to be combined with (a copy of) itself. Section 6 concludes and sketches directions for future work.

$$\begin{aligned}
\neq(x_1, \dots, x_n) &= \bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^n \neg(x_i = x_j) & \psi_{\leq n} &= \exists x_1, \dots, x_n. \forall y. \bigvee_{i=1}^n y = x_i \\
\psi_{\geq n} &= \exists x_1, \dots, x_n. \neq(x_1, \dots, x_n) & \psi_{=n} &= \psi_{\geq n} \wedge \psi_{\leq n}
\end{aligned}$$

Fig. 2: Cardinality formulas

## 2 Preliminaries

### 2.1 First-order logic

We review standard definitions regarding first-order logic [4]. A **signature**  $\Sigma$  is a set of function and predicate symbols, each equipped with an arity, that includes  $=$  as a binary predicate symbol. We only consider signatures that are at most countably infinite. Two signatures are **disjoint** if they share only the equality symbol. We define terms, formulas, literals, and sentences (formulas with no free variables) in the standard way;  $QF(\Sigma)$  is the set of quantifier-free  $\Sigma$ -formulas.

A  $\Sigma$ -**structure**  $\mathbb{A}$  is a non-empty set  $dom(\mathbb{A})$  (its domain) equipped with functions  $f^{\mathbb{A}} : dom(\mathbb{A})^n \rightarrow dom(\mathbb{A})$  and predicates  $P^{\mathbb{A}} \subseteq dom(\mathbb{A})^m$ , for all function symbols  $f$  and predicate symbols  $P$  in  $\Sigma$ , of arities  $n$  and  $m$ , respectively. A  $\Sigma$ -**interpretation**  $\mathcal{A}$  is an extension of a  $\Sigma$ -structure, where each variable  $x$  is assigned a value  $x^{\mathcal{A}} \in dom(\mathcal{A})$ . We define the value  $\tau^{\mathcal{A}} \in dom(\mathcal{A})$  of a term, and the truth-value  $\varphi^{\mathcal{A}}$  of a formula in an interpretation  $\mathcal{A}$  in the usual way; if  $\varphi$  is true in  $\mathcal{A}$  we write  $\mathcal{A} \models \varphi$ . In Figure 2 the formula  $\neq(x_1, \dots, x_n)$  for  $n \geq 2$  states the variables  $x_1$  through  $x_n$  all take different values (for  $n = 1$  it is simply *true*). The formulas  $\psi_{\geq n}$ ,  $\psi_{=n}$  and  $\psi_{\leq n}$  state that there are at least, exactly, and at most  $n$  elements in the domain, respectively.

A  $\Sigma$ -**theory** is a set of  $\Sigma$ -sentences (**axioms**). A  $\Sigma$ -interpretation that satisfies all axioms of  $\mathcal{T}$  is called a  $\mathcal{T}$ -**interpretation**, or a **model of  $\mathcal{T}$** . A formula is  $\mathcal{T}$ -**satisfiable** if there is a  $\mathcal{T}$ -interpretation that satisfies it, and a set of formulas is  $\mathcal{T}$ -satisfiable if there is a  $\mathcal{T}$ -interpretation that satisfies all formulas in the set simultaneously. A  $\Sigma$ -theory is **decidable** when there is an algorithm that takes a quantifier-free  $\Sigma$ -formula and returns whether or not it is  $\mathcal{T}$ -satisfiable.  $\mathcal{T}_{Eq}$  is the theory over the empty signature (that only has  $=$  as a symbol) having no axioms. Given signatures  $\Sigma_1$  and  $\Sigma_2$ ,  $\Sigma_1 \sqcup \Sigma_2$  is their **disjoint union**, that is, if there are any symbols in common between the two, we rename them in  $\Sigma_2$  before adding them to the symbols of  $\Sigma_1$ . Given a  $\Sigma_1$ -theory  $\mathcal{T}_1$  and a  $\Sigma_2$ -theory  $\mathcal{T}_2$ , their **disjoint combination**  $\mathcal{T}_1 \sqcup \mathcal{T}_2$  is the  $\Sigma_1 \sqcup \Sigma_2$ -theory whose axiomatization is the union of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  (after renaming the symbols in  $\Sigma_2$  occurring in  $\mathcal{T}_2$ ).

We denote  $\mathbb{N} \cup \{\aleph_0\}$  by  $\mathbb{N}_\omega$ , and  $\{n \in \mathbb{N}_\omega : p \leq n \leq q\}$  by  $[p, q]$ . The **spectrum** of a formula  $\varphi$  in a theory  $\mathcal{T}$ , denoted  $Spec_{\mathcal{T}}(\varphi)$ , is the subset of  $\mathbb{N}_\omega$  of those elements  $n$  such that there exists a  $\mathcal{T}$ -interpretation  $\mathcal{A}$  that satisfies  $\varphi$  with  $|dom(\mathcal{A})| = n$ . Our restriction from the class of all cardinals to  $\mathbb{N}_\omega$  in this definition is possible thanks to the downward Löwenheim–Skolem theorem, and the assumption that all signatures we deal with are countable. We state the Löwenheim–Skolem theorem as well as the compactness theorem:

Table 2: Notations for sets of theories

| Symbol                         | Description (decidable and ...)        | Definition   |
|--------------------------------|----------------------------------------|--------------|
| $\mathfrak{T}$                 | —                                      | Section 2.1  |
| $\mathfrak{T}_{\text{SI}}$     | Stably infinite                        | Section 2.2  |
| $\mathfrak{T}_{\text{shiny}}$  | Shiny                                  |              |
| $\mathfrak{T}_{\text{gen}}$    | Gentle                                 |              |
| $\mathfrak{T}_{\text{SI+gen}}$ | Stably infinite and Gentle             |              |
| $\mathfrak{T}_{\text{CFS}}$    | Computable finite spectra              |              |
| $\mathfrak{T}_{\text{CS}}$     | Computable spectra                     |              |
| $\mathfrak{T}_{\text{SI+CS}}$  | Stably infinite and computable spectra | Definition 1 |
| $\mathfrak{T}_{\text{CBFS}}$   | Computable bounded finite spectra      |              |
| $\mathfrak{T}_{\text{CBS}}$    | Computable bounded spectra             | Definition 2 |

**Theorem 1 ([7, Theorems 2.3.4 and 2.3.7]).** *Let  $\Delta$  be a set of formulas over a countable signature, and let  $\kappa \geq \aleph_0$ . Then,  $\Delta$  is satisfied by an interpretation of size  $\kappa$  if and only if it is satisfied by an interpretation of size  $\aleph_0$ .*

**Theorem 2 ([7, Theorem 2.1.4]).** *A set of formulas is satisfiable if and only if every finite subset of it is satisfiable.*

**Corollary 1.** *Let  $\Sigma$  be a countable signature,  $\mathcal{T}$  a  $\Sigma$ -theory, and  $\phi$  a  $\Sigma$ -formula. Suppose  $\phi$  is  $\mathcal{T}$ -satisfiable but  $\aleph_0 \notin \text{Spec}_{\mathcal{T}}(\phi)$ . Then  $\text{Spec}_{\mathcal{T}}(\phi)$  is finite.*

## 2.2 Theory combination properties

In Table 2, we summarize the theory combination properties of the paper: essentially, we shall denote the set of decidable theories with property  $X$  by  $\mathfrak{T}_X$ .

Let  $\mathcal{T}$  be a theory, and let us denote  $\mathbb{N} \setminus \{0\}$  by  $\mathbb{N}^*$ .  $\mathcal{T}$  is **stably infinite** [8] if  $\text{Spec}_{\mathcal{T}}(\phi) \neq \emptyset$  implies  $\aleph_0 \in \text{Spec}_{\mathcal{T}}(\phi)$ .  $\mathcal{T}$  has **computable finite spectra** [14] if there is an algorithm that takes a quantifier-free formula  $\phi$  and  $n \in \mathbb{N}^*$ , and returns whether  $n \in \text{Spec}_{\mathcal{T}}(\phi)$ ;  $\mathcal{T}$  is **infinitely decidable** [14] if there is an algorithm that takes a quantifier-free formula  $\phi$  and returns whether  $\aleph_0 \in \text{Spec}_{\mathcal{T}}(\phi)$ ; a theory with both these properties is said to have **computable spectra**.

$\mathcal{T}$  is **smooth** when, given a quantifier-free formula  $\phi$ , a  $\mathcal{T}$ -interpretation  $\mathcal{A}$  that satisfies  $\phi$ , and any cardinal  $\kappa \geq |\text{dom}(\mathcal{A})|$ , there exists a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  that satisfies  $\phi$  with  $|\text{dom}(\mathcal{B})| = \kappa$ .  $\mathcal{T}$  has the **finite model property** when,  $\text{Spec}_{\mathcal{T}}(\phi) \neq \emptyset$  implies  $\text{Spec}_{\mathcal{T}}(\phi) \cap \mathbb{N}^* \neq \emptyset$ , for every quantifier-free formula  $\phi$ . And, if  $\mathcal{T}$  is decidable, its **minimal model function** is the function  $\text{minmod}_{\mathcal{T}}$  that takes a quantifier-free formula  $\phi$  and returns, if  $\phi$  is  $\mathcal{T}$ -satisfiable, the least element in  $\text{Spec}_{\mathcal{T}}(\phi)$ .  $\mathcal{T}$  is **shiny** [12] when it is smooth, has the finite model property, and its minimal model function is computable.

Denote  $\mathbb{N}_{\omega} \setminus \{0\}$  by  $\mathbb{N}_{\omega}^*$ .  $\mathcal{T}$  is **gentle** [5,14] when there is an algorithm that takes a quantifier-free formula  $\phi$  and returns a pair  $(S, b)$ , where  $S \subset \mathbb{N}^*$  is finite and  $b$  is a Boolean, such that: if  $b$  is true,  $\text{Spec}_{\mathcal{T}}(\phi) = S$ ; and if  $b$  is false,  $\text{Spec}_{\mathcal{T}}(\phi) = \mathbb{N}_{\omega}^* \setminus S$ . The original definition of gentleness, however, is a bit

Table 3: Analogy between lattice-theoretic notions and theory combination

| Notion                             | Lattice Theory                                      | Theory Combination                                                                       |
|------------------------------------|-----------------------------------------------------|------------------------------------------------------------------------------------------|
| $G$                                | A Galois connection                                 | A function that maps $X$ to the set of all decidable theories combinable with $X$        |
| $G$ is antitone                    | $X \subseteq Y \Rightarrow G(Y) \subseteq G(X)$     | If $\mathcal{T}$ is combinable with $Y$ , then it is combinable with every subset of $Y$ |
| $G$ is a Galois connection         | $X \subseteq G(Y) \Leftrightarrow Y \subseteq G(X)$ | Combinability is symmetric                                                               |
| $G \circ G$                        | $X \subseteq G(G(X))$                               | There is always a combination theorem between $X$ and $G(X)$                             |
| $X$ is closed                      | $X = G(G(X))$                                       | The combination theorem between $X$ and $G(X)$ is sharp                                  |
| Closing a set under $\cap$ and $G$ | Finding the minimal containing lattice              | Finding the minimal set of properties with sharp combination theorems                    |

different. It says that, for every quantifier-free formula  $\phi$ :  $\text{Spec}_{\mathcal{T}}(\phi)$  is either a finite subset of  $\mathbb{N}^*$  or a cofinite subset of  $\mathbb{N}_{\omega}^*$  containing  $\aleph_0$ ; there is an algorithm that tells us which is the case (corresponding to our Boolean  $b$ ); and there is an algorithm that, given  $\phi$ , returns  $\text{Spec}_{\mathcal{T}}(\phi)$  if this set is finite, and  $\mathbb{N}_{\omega}^* \setminus \text{Spec}_{\mathcal{T}}(\phi)$  otherwise. We prefer the former one, as it merges both algorithms into one.

**Proposition 1 ([10]).**  $\mathfrak{T}_{shiny} \subseteq \mathfrak{T}_{gen} \cap \mathfrak{T}_{SI}$  and  $\mathfrak{T}_{gen} \subseteq \mathfrak{T}_{CFS}$ ; the theories of  $\mathfrak{T}_{SI}$  are all infinitely decidable; and  $\mathfrak{T}_{SI} \cap \mathfrak{T}_{CS} = \mathfrak{T}_{SI} \cap \mathfrak{T}_{CFS}$ .

We call the following result *Fontaine’s lemma*. Its “if” direction was proved in [5], while its “only if” direction was proved in [10]. We could state it for arbitrary quantifier-free formulas, but we follow its original formulation.

**Lemma 1 (Fontaine’s lemma).** *The disjoint combination  $\mathcal{T}_1 \sqcup \mathcal{T}_2$  is decidable if and only if the following problem is decidable: given conjunctions of literals  $\phi_1$  and  $\phi_2$ , determine whether  $\text{Spec}_{\mathcal{T}_1}(\phi_1) \cap \text{Spec}_{\mathcal{T}_2}(\phi_2) = \emptyset$ .*

### 2.3 The lattice of theory combination properties

We review the main definitions of [10]. Two theories are **combinable** if their disjoint combination is decidable. Two sets of theories are combinable if every theory of one is combinable with every theory of the other. We may identify an element with the singleton containing it. Let  $\mathfrak{T}$  be the set of decidable theories.<sup>3</sup> For each  $X \subseteq \mathfrak{T}$ ,  $G(X)$  is the set of all decidable theories combinable with  $X$ .

<sup>3</sup> This, in principle, would not constitute a set, but a proper class, but we can just select all symbols for our (countable) signatures from a countable pool of symbols.

**Algorithm 1** Pseudocode for closure under  $\cap$  and  $G$ .

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1: function GENERATE( $S$ )
2:   while true do
3:      $S_\cap \leftarrow \{X \cap Y \mid X, Y \in S\}$ 
4:      $S_G \leftarrow \{G(X) \mid X \in S\}$ 
5:     if  $S = S \cup S_\cap \cup S_G$  then return  $S$ 
6:    $S \leftarrow S \cup S_\cap \cup S_G$ 

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Table 4: Execution of Algorithm 1

| Iteration | Line | Computation                                                | Result                         | Proof         |
|-----------|------|------------------------------------------------------------|--------------------------------|---------------|
| 0         | —    | —                                                          | $\mathfrak{T}_{\text{SI}}$     | —             |
|           |      |                                                            | $\mathfrak{T}_{\text{shiny}}$  |               |
|           |      |                                                            | $\mathfrak{T}_{\text{gen}}$    |               |
| 1         | 3    | $\mathfrak{T}_{\text{SI}} \cap \mathfrak{T}_{\text{gen}}$  | $\mathfrak{T}_{\text{SI+gen}}$ | —             |
|           | 4    | $G(\mathfrak{T}_{\text{shiny}})$                           | $\mathfrak{T}$                 | Table 1       |
|           | 4    | $G(\mathfrak{T}_{\text{gen}})$                             | $\mathfrak{T}_{\text{CFS}}$    | Table 1       |
| 2         | 3    | $\mathfrak{T}_{\text{SI}} \cap \mathfrak{T}_{\text{CFS}}$  | $\mathfrak{T}_{\text{SI+CS}}$  | Proposition 1 |
|           | 4    | $G(\mathfrak{T}_{\text{SI+gen}})$                          | $\mathfrak{T}_{\text{CBFS}}$   | Corollary 2   |
| 3         | 4    | $G(\mathfrak{T}_{\text{SI+CS}})$                           | $\mathfrak{T}_{\text{CBS}}$    | Corollary 2   |
| 4         | 3    | $\mathfrak{T}_{\text{CFS}} \cap \mathfrak{T}_{\text{CBS}}$ | $\mathfrak{T}_{\text{CS}}$     | Proposition 4 |

If  $X \subseteq Y$  then every theory combinable with  $Y$  is also combinable with  $X$ . Hence,  $X \subseteq Y$  implies  $G(Y) \subseteq G(X)$ . Further, combinability is symmetric:  $X \subseteq G(Y)$  iff  $Y \subseteq G(X)$ . Thus,  $G$  is an **antitone Galois connection**. A **combination theorem** has the form  $X \subseteq G(Y)$ . We always have a combination theorem between  $X$  and  $Y = G(X)$ . Such a theorem is **sharp** if, moreover,  $X = G(Y)$ . In this case,  $X$  is called **closed under  $G$** , or just **closed**.

Take any set of closed sets of theories and for every  $X$  and  $Y$  in it, add infimum  $G(G(X \cap Y))$  and supremum  $G(G(X \cup Y))$ , until fixpoint. The result is a **lattice** ordered by  $\subseteq$ . If  $X$  and  $Y$  are closed, then the infimum and supremum are  $X \cap Y$  and  $G(G(X) \cap G(Y))$ , resp. Thus it suffices to compute closure under  $\cap$  and  $G$ . Table 3 summarizes the lattice-theoretic view of theory combination.

### 3 So, how many combination properties?

Algorithm 1 takes a set  $S$  of properties, and closes it under  $G$  and intersection. As seen in Section 2.3, this suffices for computing the minimal lattice that contains  $S$ . A priori, this process does not have to terminate.

Remarkably, it terminates when the input set is  $\{\mathfrak{T}_{\text{SI}}, \mathfrak{T}_{\text{shiny}}, \mathfrak{T}_{\text{gen}}\}$ . The resulting set is presented in Figure 1b as a Hasse diagram. It mostly contains properties defined in Section 2, though it also involves two new properties, namely *computable bounded finite spectra* and *computable bounded spectra*. These give rise to two new combination theorems that we introduce in Section 4.

**Theorem 3.**  $\text{GENERATE}(\{\mathfrak{T}_{\text{SI}}, \mathfrak{T}_{\text{shiny}}, \mathfrak{T}_{\text{gen}}\})$  terminates, and its output consists of the 10 properties that appear in Figure 1b.

*Proof.* The proof has two parts: First, we show that each property in Figure 1b must indeed be in the resulting set, by justifying inclusion. Second, we prove that there are no other added properties: after 4 iterations of the loop of Algorithm 1, we reach a fixed-point.

**Addition of properties:** We justify the addition of each property in Figure 1b to the resulting set, by specifying in which step of Algorithm 1 it is added, and explaining why. A tabular representation of this process is available in Table 4.

**Iteration 0:** Clearly,  $\mathfrak{T}_{\text{SI}}$ ,  $\mathfrak{T}_{\text{shiny}}$  and  $\mathfrak{T}_{\text{gen}}$  must be included, as they are the input to the algorithm, and the algorithm never removes a property.

**Iteration 1:** The intersections of  $\mathfrak{T}_{\text{SI}}$ ,  $\mathfrak{T}_{\text{shiny}}$  and  $\mathfrak{T}_{\text{gen}}$  are added in line 3 of the first iteration of the loop. Every shiny theory is stably infinite and gentle from Proposition 1, and thus we only have to add  $\mathfrak{T}_{\text{SI}} \cap \mathfrak{T}_{\text{gen}}$ , denoted  $\mathfrak{T}_{\text{SI+gen}}$ .

Also,  $G(X)$  must be added for each property  $X$  in the input set, which is done in line 4. By Table 1,  $G(\mathfrak{T}_{\text{shiny}}) = \mathfrak{T}$ ,  $G(\mathfrak{T}_{\text{SI}}) = \mathfrak{T}_{\text{SI}}$ , and  $G(\mathfrak{T}_{\text{gen}}) = \mathfrak{T}_{\text{CFS}}$ . Thus,  $\mathfrak{T}$  and  $\mathfrak{T}_{\text{CFS}}$  are added ( $\mathfrak{T}_{\text{SI}}$  is already included).

**Iteration 2:** Now we close the 6 properties collected so far under intersection. Most of the resulting intersections are already within these 6 properties:  $\mathfrak{T}_{\text{shiny}} \subseteq \mathfrak{T}_{\text{SI+gen}} \subseteq \mathfrak{T}_{\text{gen}} \subseteq \mathfrak{T}_{\text{CFS}} \subseteq \mathfrak{T}$ , and also  $\mathfrak{T}_{\text{shiny}} \subseteq \mathfrak{T}_{\text{SI+gen}} \subseteq \mathfrak{T}_{\text{SI}} \subseteq \mathfrak{T}$  (see Proposition 1), and the intersections of every two properties in the same sequence of inclusions is just the smaller one. Thus, we only add a single property, namely the intersection of  $\mathfrak{T}_{\text{SI}}$  and  $\mathfrak{T}_{\text{CFS}}$ , that is,  $\mathfrak{T}_{\text{SI}} \cap \mathfrak{T}_{\text{CFS}}$ . By Proposition 1,  $\mathfrak{T}_{\text{SI}} \cap \mathfrak{T}_{\text{CFS}} = \mathfrak{T}_{\text{SI}} \cap \mathfrak{T}_{\text{CS}}$ , and we choose to denote this property by  $\mathfrak{T}_{\text{SI+CS}}$  because, as will be seen below, in the last iteration of the loop we will add  $\mathfrak{T}_{\text{CS}}$ , and the notation  $\mathfrak{T}_{\text{SI+CS}}$  will then better reflect the placing of this property in Figure 1b.

The next addition is due to the closure under  $G$  computed in iteration 2. We have already closed the three initial properties under  $G$  in iteration 1. Also,  $G(\mathfrak{T}) = \mathfrak{T}_{\text{shiny}}$  and  $G(\mathfrak{T}_{\text{CFS}}) = \mathfrak{T}_{\text{gen}}$ , as shown in Table 1. Thus in this iteration, we only add  $G(\mathfrak{T}_{\text{SI+gen}})$ . This requires a new combination theorem, based on a new property of theories, that we call *computable bounded finite spectra*, denoted  $\mathfrak{T}_{\text{CBFS}}$ : a theory has this property when it has an algorithm that, given a quantifier-free  $\phi$  and  $m, n \in \mathbb{N}^*$ , returns whether  $n$  is in  $\text{Spec}_{\mathcal{T}}(\phi)$  as long as  $\phi \wedge \#(x_1, \dots, x_m)$  (for fresh variables  $x_i$ ) is not  $\mathcal{T}$ -satisfiable. It is formally defined in Definition 1, and its sharp combination theorem is proven in Corollary 2.

**Iteration 3:** The set obtained by the end of iteration 2 is already closed under intersection (see Propositions 1 and 4). Thus, we close it under  $G$ . From Table 1,  $G(\mathfrak{T}) = \mathfrak{T}_{\text{shiny}}$  and  $G(\mathfrak{T}_{\text{CFS}}) = \mathfrak{T}_{\text{gen}}$ . By Corollary 2,  $G(\mathfrak{T}_{\text{CBFS}}) = \mathfrak{T}_{\text{SI+gen}}$ . Thus, we only need to add  $G(\mathfrak{T}_{\text{SI+CS}})$ , which requires yet another new combination theorem, based on a new property, which we call *computable bounded spectra*, denoted  $\mathfrak{T}_{\text{CBS}}$ : fortunately, it can be defined as being infinitely decidable and having computable bounded finite spectra. Its formal definition is in Definition 2, and a sharp combination theorem for it is proven in Corollary 2.

**Iteration 4:** In addition to the inclusions specified in Iteration 2, we also have  $\mathfrak{T}_{\text{CFS}} \subseteq \mathfrak{T}_{\text{CBFS}} \subseteq \mathfrak{T}$  and  $\mathfrak{T}_{\text{SI}} \subseteq \mathfrak{T}_{\text{CBS}} \subseteq \mathfrak{T}_{\text{CBFS}}$  from Proposition 4, and the

intersection of every two properties in each resulting inclusion path is the smaller of the two. The only missing intersection is that of  $\mathfrak{T}_{\text{CFS}}$  and  $\mathfrak{T}_{\text{CBS}}$ , which we prove in Proposition 4 to be equal to  $\mathfrak{T}_{\text{CS}}$ , which we now add.

**There are no other properties:** Denote by  $S_0$  the set of 10 properties listed in Figure 1b. This set is  $S_\cap$  after line 3 of the fourth iteration of Algorithm 1. From Corollary 2 and table 1, we have that after line 4 of the fourth iteration, we get  $S_G = S_0$  as well, that is, that  $G(S_0) = S_0$ . Going to the fifth iteration of the loop, we have already seen that  $S_0$  is closed under intersection. Thus, in the fifth iteration, after lines 3 and 4 are executed, we have  $S_\cap = S_G = S_0 = S$ , and so in line 6 we return  $S_0$ .  $\square$

### 3.1 Other resulting sets

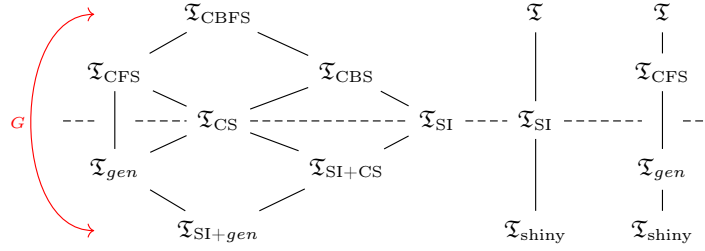


Fig. 3: The Hasse diagrams for the resulting sets given different inputs

We have generated a lattice from the three properties of stable infiniteness, gentleness and shininess. What would happen if we were to restrict ourselves to only two among these properties instead?

Stable infiniteness and gentleness are enough to generate most of Figure 1b: a reasoning similar to the one found at the proof of Theorem 3 shows we only miss  $\mathfrak{T}$  and  $\mathfrak{T}_{\text{shiny}}$ . For shininess and stable infiniteness, we only get  $\mathfrak{T}$ ,  $\mathfrak{T}_{\text{SI}}$  and  $\mathfrak{T}_{\text{shiny}}$ . For gentleness and shininess the situation is similar, except that we must add  $G(\mathfrak{T}_{\text{gen}}) = \mathfrak{T}_{\text{CFS}}$ . The resulting lattices are shown in Figure 3. Formally:

#### Theorem 4.

1.  $\text{GENERATE}(\{\mathfrak{T}_{\text{SI}}, \mathfrak{T}_{\text{gen}}\})$  terminates, and its output consists of the 8 properties that appear in the left diagram of Figure 3.
2.  $\text{GENERATE}(\{\mathfrak{T}_{\text{SI}}, \mathfrak{T}_{\text{shiny}}\})$  terminates, and its output consists of the 3 properties that appear in the middle diagram of Figure 3.
3.  $\text{GENERATE}(\{\mathfrak{T}_{\text{gen}}, \mathfrak{T}_{\text{shiny}}\})$  terminates, and its output consists of the 4 properties that appear in the right diagram of Figure 3.

Table 5: Signatures

| Sig.                                      | Functions   | Predicates                            |
|-------------------------------------------|-------------|---------------------------------------|
| $\Sigma_1$                                | $\emptyset$ | $\emptyset$                           |
| $\Sigma_P$                                | $\emptyset$ | $\{P\}$                               |
| $\Sigma_P^{\mathbb{N}}$                   | $\emptyset$ | $\{P_n : n \in \mathbb{N}^*\}$        |
| $\Sigma_P^{\mathbb{N} \times \mathbb{N}}$ | $\emptyset$ | $\{P_{m,n} : m, n \in \mathbb{N}^*\}$ |

Table 6: Theories defined over signatures from Table 5. L.S. stands for the least set the theory belongs to.  $P_{\#} = \{P_i \rightarrow \neg P_j : i \neq j\}$ ,  $P_{\infty} = \{P_1 \rightarrow \psi_{\geq k} : k \in \mathbb{N}\}$ , and  $P_U(n) = \{P_n \rightarrow \neg \psi_{=k} : k \notin U\}$ .

| Sig.                    | Name                       | Axiomatization                                                                                                             | Source | L.S.                    |
|-------------------------|----------------------------|----------------------------------------------------------------------------------------------------------------------------|--------|-------------------------|
| $\Sigma_P^{\mathbb{N}}$ | $\mathcal{T}_{\{si\}}$     | $\{P_n \rightarrow \psi_{\geq m} : n \in U, m \in \mathbb{N}^*\} \cup P_{\#}$                                              | [10]   | $\mathfrak{T}_{SI}$     |
| $\Sigma_P$              | $\mathcal{T}_{\{cs\}}$     | $\{P \rightarrow \psi_{=1}\} \cup \{\neg P \rightarrow \psi_{\geq m} : m \in \mathbb{N}\}$                                 | [10]   | $\mathfrak{T}_{CS}$     |
| $\Sigma_1$              | $\mathcal{T}_{\leq n}$     | $\{\psi_{\leq n}\}$                                                                                                        | [13]   | $\mathfrak{T}_{gen}$    |
| $\Sigma_1$              | $\mathcal{T}_{\infty}$     | $\{\psi_{\geq k} : k \in \mathbb{N}^*\}$                                                                                   | [13]   | $\mathfrak{T}_{SI+CS}$  |
| $\Sigma_1$              | $\mathcal{T}_{\geq n}$     | $\{\psi_{\geq n}\}$                                                                                                        | [13]   | $\mathfrak{T}_{shiny}$  |
| $\Sigma_1$              | $\mathcal{T}_{\geq n+2}^n$ | $\{\psi_{=n} \vee \psi_{\geq n+2}\}$                                                                                       | New    | $\mathfrak{T}_{SI+gen}$ |
| $\Sigma_P^{\mathbb{N}}$ | $\mathcal{T}(m, n)$        | $\{\psi_{=m} \vee \psi_{=n}\} \cup \{P_k \rightarrow \psi_{=m} : k \in U\} \cup \{P_1 \rightarrow \psi_{=n}\} \cup P_{\#}$ | New    | $\mathfrak{T}$          |
| $\Sigma_P^{\mathbb{N}}$ | $\mathcal{T}_{\{cbfs\}}$   | $P_{\infty} \cup P_U(2) \cup \{P_k \rightarrow \psi_{\leq F(k)} : k > 2, F(k) \in \mathbb{N}\} \cup P_{\#}$                | New    | $\mathfrak{T}_{CBFS}$   |
| $\Sigma_P^{\mathbb{N}}$ | $\mathcal{T}_{\{cfs\}}$    | $P_{\infty} \cup \{P_k \rightarrow \psi_{\leq F(k)} : k > 1, F(k) \in \mathbb{N}\} \cup P_{\#}$                            | New    | $\mathfrak{T}_{CFS}$    |
| $\Sigma_P^{\mathbb{N}}$ | $\mathcal{T}_{\{cbs\}}$    | $P_U(1) \cup \{P_k \rightarrow \psi_{=k} : k > 1\} \cup P_{\#}$                                                            | New    | $\mathfrak{T}_{CBS}$    |

### 3.2 The relationships between the properties

To conclude this section, we prove that the properties are correctly situated in Figure 1b. That is, if there is a line between two properties, the lower one is stronger than the upper one. Most of these relationships are already captured by Proposition 1. Relationships that involve the new properties,  $\mathfrak{T}_{CBS}$  and  $\mathfrak{T}_{CBFS}$ , are proven in Proposition 4 below. We just need two more connections:

**Proposition 2.**  $\mathfrak{T}_{SI} \cap \mathfrak{T}_{CFS} = \mathfrak{T}_{SI+CS}$  and  $\mathfrak{T}_{gen} \cap \mathfrak{T}_{SI+CS} = \mathfrak{T}_{SI+gen}$ .

*Proof.* By Proposition 1, stable infiniteness implies being infinitely decidable (for decidable theories), and by appealing to the definition of having computable spectra we get  $\mathfrak{T}_{SI} \cap \mathfrak{T}_{CFS} = \mathfrak{T}_{SI+CS}$ .

By Proposition 1, gentleness implies having computable spectra, so the conjunction of gentleness, stable infiniteness and computable spectra is the same as that of gentleness and stable infiniteness, that is,  $\mathfrak{T}_{gen} \cap \mathfrak{T}_{SI+CS} = \mathfrak{T}_{SI+gen}$ .  $\square$

We also prove that all inclusions in Figure 1b are strict. For this, we present theories that separate between the different sets. The signatures for these theories can be found in Table 5.  $\Sigma_1$  is empty,  $\Sigma_P$  includes a single nullary predicate

symbol  $P$ , and  $\Sigma_P^{\mathbb{N}}$  includes a nullary predicate symbol  $P_n$  for every  $n \in \mathbb{N}^*$ . Table 5 includes one more signature, with predicate symbols for each pair of natural numbers, to be used in Section 4 below.

The axiomatizations of the separating theories are found in Table 6. In it,  $F : \mathbb{N}^* \rightarrow \mathbb{N}^* \cup \{\aleph_0\}$  is a function such that: (i)  $\{(m, n) \in \mathbb{N}^* \times \mathbb{N}^* : F(m) \geq n\}$  is decidable; and (ii)  $\{n : F(n) = \aleph_0\}$  is undecidable.<sup>4</sup> Also,  $U \subset \mathbb{N}^*$  is an undecidable set such that  $1 \notin U$ .

The upper part of Table 6 includes theories that were defined in previous works (whose axiomatizations we present for completeness' sake). Briefly, a  $\mathcal{T}_{\langle si \rangle}$ -interpretation that satisfies a  $P_n$ , for  $n \in U$ , must be infinite; a  $\mathcal{T}_{\langle cs \rangle}$ -interpretation either satisfies  $P$  and has a single element, or  $\neg P$  and is infinite;  $\mathcal{T}_\infty$  has only infinite interpretations,  $\mathcal{T}_{\leq n}$  only interpretations of size at most  $n$ , and  $\mathcal{T}_{\geq n}$  interpretations of size at least  $n$ .

The last five theories will also be used in Section 5 below and are brand new.  $\mathcal{T}_{\geq n+2}^n$  is easy to understand: it has models of cardinality exactly  $n$ , or strictly greater than  $n+1$ .  $\mathcal{T}(m, n)$  has models of cardinality  $m$  or  $n$ ; if  $P_k$  is true, for some  $k \in U$ , then the cardinality must be  $m$ ; but if  $P_1$  is true (and we assumed  $1 \notin U$ ), then the cardinality must be  $n$ . In  $\mathcal{T}_{\langle cbfs \rangle}$  the truth of:  $P_1$  implies a model is infinite;  $P_2$  implies a model has cardinality in  $U$ ; and  $P_k$ , for  $k > 2$ , implies a model has cardinality at most  $F(k)$ , whenever  $F(k)$  is finite.  $\mathcal{T}_{\langle cfs \rangle}$  is similar to  $\mathcal{T}_{\langle cbfs \rangle}$ , except that the truth of  $P_2$  now implies a model has cardinality at most  $F(2)$ , if this value is finite. And, finally, in  $\mathcal{T}_{\langle cbs \rangle}$  the truth of  $P_1$  implies a model has cardinality in  $U$ , while the truth of  $P_k$  for  $k > 1$  implies a model has cardinality precisely  $k$ .

Indeed, the theories in Table 6 separate the properties of Figure 1b, meaning that each has a property without having any of the properties below it. This is represented in Table 6 by column L.S., which stands for the “least set” of Figure 1b the theory belongs to.

**Proposition 3.** *The last column of Table 6 correctly identifies the least set (L.S.) the theories belong to.*

*Example 1.* Theory  $\mathcal{T}_{\langle si \rangle}$  is stably infinite but does not have computable spectra. Thus, it shows that the inclusion  $\mathfrak{T}_{SI+CS} \subseteq \mathfrak{T}_{SI}$  observed in Figure 1b is strict.

## 4 Two new combination methods

In this section, we introduce two new combination methods. While they were tailored precisely for “running” Algorithm 1 in search of the smallest lattice that contains Figure 1a, they are interesting in their own right, and allow combinations of theories that other methods do not support (see Example 2 below).

The new methods are inspired by gentleness and, in particular, Fontaine’s lemma (Lemma 1) plays a crucial role in their correctness proofs.

<sup>4</sup> Uses and explicit examples of such functions may be found in [2,14].

---

**Algorithm 2** Determining non-emptiness of  $\text{Spec}_{\mathcal{T}_1}(\phi_1) \cap \text{Spec}_{\mathcal{T}_2}(\phi_2)$ , assuming  $\mathcal{T}_1 \in \mathfrak{T}_{\text{CBFS}}$  and  $\mathcal{T}_2 \in \mathfrak{T}_{\text{SI+gen}}$ .

---

```

1: function CBFS+(SI+GEN)( $\phi_1, \phi_2$ )
2:   if  $\phi_2$  is not  $\mathcal{T}_2$ -satisfiable then return false
3:    $m \leftarrow \max(\mathbb{N}_\omega \setminus \text{Spec}_{\mathcal{T}_2}(\phi_2))$ .
4:   if  $\phi_1 \wedge \#(x_1, \dots, x_{m+1})$  is  $\mathcal{T}_1$ -satisfiable then  $\triangleright x_1, \dots, x_{m+1}$  are fresh
5:     return true
6:    $X \leftarrow \text{Spec}_{\mathcal{T}_1}(\phi_1) \setminus (\mathbb{N}_\omega \setminus \text{Spec}_{\mathcal{T}_2}(\phi_2))$ 
7:   for  $n \leftarrow [1, m-1]$  do
8:     if  $n \in X$  then
9:       return true
10:  return false

```

---



---

**Algorithm 3** Determining non-emptiness of  $\text{Spec}_{\mathcal{T}_1}(\phi_1) \cap \text{Spec}_{\mathcal{T}_2}(\phi_2)$ , assuming  $\mathcal{T}_1 \in \mathfrak{T}_{\text{SI+CS}}$  and  $\mathcal{T}_2 \in \mathfrak{T}_{\text{CBS}}$ .

---

```

1: function (SI+CS)+CBS( $\phi_1, \phi_2$ )
2:   if  $\phi_1$  is  $\mathcal{T}_1$ -satisfiable and  $\aleph_0 \in \text{Spec}_{\mathcal{T}_2}(\phi_2)$  then
3:     return true
4:    $m \leftarrow 1$ 
5:   while  $\phi_2 \wedge \#(x_1, \dots, x_m)$  is  $\mathcal{T}_2$ -satisfiable do  $\triangleright x_1, \dots, x_m$  are fresh
6:      $m \leftarrow m + 1$ 
7:   for  $n \leftarrow [1, m-1]$  do
8:     if  $n \in \text{Spec}_{\mathcal{T}_1}(\phi_1) \cap \text{Spec}_{\mathcal{T}_2}(\phi_2)$  then
9:       return true
10:  return false

```

---

Recall that having computable finite spectra amounts to the existence of an algorithm that decides whether a given natural number is in the spectrum of a given formula. We here weaken this requirement by allowing the algorithm to return arbitrary values in case the spectra is unbounded: if it is bounded, however, it must answer correctly whether an element  $n$  is or not in the spectra, given an adequate upper bound  $m$ .

**Definition 1.** A theory  $\mathcal{T}$  is said to have computable bounded finite spectra if there is an algorithm that takes a quantifier-free formula  $\phi$  and two  $m, n \in \mathbb{N}^*$ , and adheres to the following specification: if  $\phi$  is not  $\mathcal{T}$ -satisfiable by an interpretation with at least  $m$  elements, then the algorithm must return whether  $n \in \text{Spec}_{\mathcal{T}}(\phi)$ . Otherwise, the algorithm can return any Boolean value.

Just like computable spectra adds infinite decidability to computable finite spectra, a similar change to computable bounded finite spectra is pivotal:

**Definition 2.** A theory  $\mathcal{T}$  is said to have computable bounded spectra if it is infinitely decidable and has computable bounded finite spectra.

We now examine the relations between these properties and others we have met so far. The result is formalized in the next theorem, depicted in Figure 4.

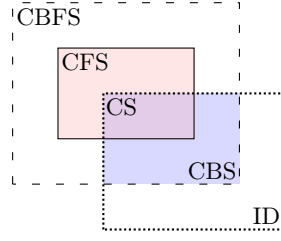


Fig. 4: Relationships between having computable bounded finite spectra (CBFS, dashed box), having computable finite spectra (CFS, red), having computable bounded spectra (CBS, blue), having computable spectra (CS, purple), and being infinitely decidable (ID, dotted box)

**Proposition 4.**  $\mathfrak{T}_{SI} \subseteq \mathfrak{T}_{CBS} \subseteq \mathfrak{T}_{CBFS}$ ,  $\mathfrak{T}_{CFS} \subseteq \mathfrak{T}_{CBFS}$ , and  $\mathfrak{T}_{CFS} \cap \mathfrak{T}_{CBS} = \mathfrak{T}_{CS}$ .

*Proof.* To see that  $\mathfrak{T}_{SI} \subseteq \mathfrak{T}_{CBS}$ , given a quantifier-free  $\phi$  and  $m, n \in \mathbb{N}^*$ , our algorithm may return an arbitrary boolean, as a stably infinite theory is infinitely decidable by [10] and  $\phi$  always has models of cardinality greater than  $m$ .

To see that  $\mathfrak{T}_{CFS} \subseteq \mathfrak{T}_{CBFS}$ , given a quantifier-free formula  $\phi$  and  $m, n \in \mathbb{N}^*$ , we can define the algorithm that proves  $\mathcal{T}$  has computable bounded finite spectra by simply ignoring  $m$  and feeding  $\phi$  and  $n$  to the algorithm that guarantees  $\mathcal{T}$  has computable finite spectra.

That  $\mathfrak{T}_{CBS} \subseteq \mathfrak{T}_{CBFS}$  follows from the definition of the latter. Finally, as a theory with computable bounded spectra is infinitely decidable by definition, and computable spectra is defined as being infinitely decidable and having computable finite spectra,  $\mathfrak{T}_{CBS} \cap \mathfrak{T}_{CFS} = \mathfrak{T}_{CS}$ .  $\square$

The next theorem provides a new combination method between theories with computable bounded finite spectra, and theories both stably infinite and gentle.

**Theorem 5.** If  $\mathcal{T}_1 \in \mathfrak{T}_{CBFS}$  and  $\mathcal{T}_2 \in \mathfrak{T}_{SI+gen}$ , then  $\mathcal{T}_1 \sqcup \mathcal{T}_2$  is decidable. That is,  $\mathfrak{T}_{CBFS} \subseteq G(\mathfrak{T}_{SI+gen})$  (and  $\mathfrak{T}_{SI+gen} \subseteq G(\mathfrak{T}_{CBFS})$ ).

*Proof (sketch).* By Lemma 1, it suffices to show a decision procedure that decides, given  $\phi_1$  and  $\phi_2$  whether  $Spec_{\mathcal{T}_1}(\phi_1) \cap Spec_{\mathcal{T}_2}(\phi_2)$  is empty. We present such a decision procedure in Algorithm 2. It relies on the properties of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  in order to determine whether the models of the two formulas share a cardinality, by iterating over all possible cardinalities. Termination is guaranteed by Corollary 1. Notice that in line 3,  $m \in \mathbb{N}^*$ : since  $\mathcal{T}_2$  is stably infinite we have  $\aleph_0 \in Spec_{\mathcal{T}_2}(\phi_2)$ , and since  $\mathcal{T}_2$  is also gentle the set  $Spec_{\mathcal{T}_2}(\phi_2)$  must be cofinite, making of  $\mathbb{N}_\omega^* \setminus Spec_{\mathcal{T}_2}(\phi_2)$  a finite subset of  $\mathbb{N}^*$ .  $\square$

A similar yet distinct combination theorem is obtained between theories that are stably infinite and have computable spectra, and theories with computable bounded spectra. It is proved in a similar fashion to Theorem 5. In particular, it relies on Lemma 1 and Algorithm 3.

**Theorem 6.** *If  $\mathcal{T}_1 \in \mathfrak{T}_{SI+CS}$  and  $\mathcal{T}_2 \in \mathfrak{T}_{CBS}$ , then  $\mathcal{T}_1 \sqcup \mathcal{T}_2$  is decidable. That is,  $\mathfrak{T}_{CBS} \subseteq G(\mathfrak{T}_{SI+CS})$  (and  $\mathfrak{T}_{SI+CS} \subseteq G(\mathfrak{T}_{CBS})$ ).*

Examples of theories with computable bounded finite spectra include all of those in Table 6 but  $\mathcal{T}(m, n)$ , and examples of theories with computable bounded spectra include all of those in Table 6 but  $\mathcal{T}(m, n)$ ,  $\mathcal{T}_{\langle cbfs \rangle}$  and  $\mathcal{T}_{\langle cfs \rangle}$ . More importantly,  $\mathcal{T}_{\langle cbfs \rangle}$  is an example of a theory with computable bounded finite spectra but neither computable bounded spectra or computable finite spectra; and  $\mathcal{T}_{\langle cbs \rangle}$  is an example of a theory with computable bounded spectra but that neither is stably infinite nor has computable spectra. Such examples may seem artificial but finding a theory with a given property, and none of the stronger ones, can be non-trivial; if we relax these restrictions, any stably infinite theory, for example, has both forms of bounded spectra. The following example shows a combination made possible by Theorem 6 not known to be possible by existing methods, in particular those in Figure 1a.

*Example 2.* The combination of  $\mathcal{T}_{\langle cbs \rangle}$  and  $\mathcal{T}_\infty$  from Table 6 is decidable by Theorem 6, because the former has computable bounded spectra, and the latter is stably infinite and has computable spectra. We don't get this from any of the combination methods that correspond to Figure 1a: not from Nelson–Oppen (as  $\mathcal{T}_{\langle cbs \rangle}$  is not stably infinite), nor from the shiny combination method (as neither of the theories is shiny), nor from the gentle combination method (as  $\mathcal{T}_{\langle cbs \rangle}$  does not have computable finite spectra).

#### 4.1 Sharpness

We now prove that Theorems 5 and 6 are, in fact, sharp. For Theorem 5, this means that every decidable theory that can be combined with all decidable theories that are both stably infinite and gentle, must have computable bounded finite spectra, and vice-versa. A similar result holds for Theorem 6.

**Theorem 7.**  $G(\mathfrak{T}_{SI+gen}) \subseteq \mathfrak{T}_{CBFS}$  and  $G(\mathfrak{T}_{CBFS}) \subseteq \mathfrak{T}_{SI+gen}$ .

**Theorem 8.**  $G(\mathfrak{T}_{CBS}) \subseteq \mathfrak{T}_{SI+CS}$  and  $G(\mathfrak{T}_{SI+CS}) \subseteq \mathfrak{T}_{CBS}$ .

As a consequence, we obtain the following equalities, which are especially useful for the execution of Algorithm 1:

**Corollary 2.**  $G(\mathfrak{T}_{SI+gen}) = \mathfrak{T}_{CBFS}$ ,  $G(\mathfrak{T}_{CBFS}) = \mathfrak{T}_{SI+gen}$ ,  $G(\mathfrak{T}_{SI+CS}) = \mathfrak{T}_{CBS}$ , and  $G(\mathfrak{T}_{CBS}) = \mathfrak{T}_{SI+CS}$ .

The proofs of Theorems 7 and 8 are carried out using the technique of *test theories*, introduced in [10]. The idea is, given a reference set, to define a family of *test theories* that is, in a sense, “maximally difficult” to combine: then, if a theory is combinable with these test theories, it must be combinable with the entire reference set. We mostly use the test theories introduced in [10]. For them, we have to prove that they can indeed serve as test theories for the two

$$\{P_{m,n} \rightarrow (\psi_{=n} \vee \psi_{\geq m+1}) : 1 \leq n \leq m\} \cup \{P_{m,n} \rightarrow \neg P_{p,q} : (m,n) \neq (p,q)\}$$

Fig. 5: Axiomatization of  $\mathcal{T}_{=}^>$ 

new properties that we introduce. We also introduce one new theory, denoted  $\mathcal{T}_{=}^>$ , which is axiomatized in Figure 5 over signature  $\Sigma_P^{\mathbb{N} \times \mathbb{N}}$  from Table 5. Essentially, in a  $\mathcal{T}_{=}^>$ -interpretation  $\mathcal{A}$ , if  $P_{m,n}$  holds true then either  $|\text{dom}(\mathcal{A})| = n$  or  $|\text{dom}(\mathcal{A})| > m$ . It has the following properties:

**Proposition 5.** *The theory  $\mathcal{T}_{=}^>$  is decidable, stably infinite, gentle, and for  $\aleph_0 \notin \text{Spec}_{\mathcal{T}}(\phi)$ ,  $m = \max \text{Spec}_{\mathcal{T}}(\phi)$  finite, and  $n \leq m$ , we have that  $\phi \wedge P_{m,n}$  is  $\mathcal{T} \sqcup \mathcal{T}_{=}^>$ -satisfiable if and only if  $n \in \text{Spec}_{\mathcal{T}}(\phi)$ .*

With this theory, and its properties as listed in Proposition 5, we can provide a sketch for the proof of Theorem 7. The proof of Theorem 8 is similar.

*Proof (of Theorem 7, sketch).* We show that  $G(\mathfrak{T}_{\text{SI}+\text{gen}}) \subseteq \mathfrak{T}_{\text{CBFS}}$ . The second part can be done in a similar fashion, but requires further test theories from [10].

Suppose that  $\mathcal{T}$  is decidable and can be combined with every decidable, gentle and stably infinite theory. We prove that  $\mathcal{T}$  has computable bounded finite spectra. Take a quantifier-free  $\phi$ , and  $m, n \in \mathbb{N}^*$ : our algorithm simply returns whether  $\phi \wedge P_{m,n}$  is  $\mathcal{T} \sqcup \mathcal{T}_{=}^>$ -satisfiable; in case  $\phi \wedge \#(x_1, \dots, x_m)$  is not  $\mathcal{T}$ -satisfiable this happens iff  $n \in \text{Spec}_{\mathcal{T}}(\phi)$ , for  $1 \leq n \leq m$ , so the algorithm has the required properties. We are then done as  $\mathcal{T} \sqcup \mathcal{T}_{=}^>$  is indeed decidable, given that  $\mathcal{T}_{=}^>$  is decidable, gentle and stably infinite, see Proposition 5.  $\square$

## 5 Theories that cannot be combined with themselves

In Figure 1b,  $G$  behaves as a reflection over the dashed line. For each set of theories  $\mathfrak{T}_X$  in the diagram, either  $\mathfrak{T}_X$  sits above  $G(\mathfrak{T}_X)$  (and so  $G(\mathfrak{T}_X) \subseteq \mathfrak{T}_X$ ), or  $G(\mathfrak{T}_X)$  sits above  $\mathfrak{T}_X$  (and so  $\mathfrak{T}_X \subseteq G(\mathfrak{T}_X)$ ). This is true for all combination properties we studied so far in this paper, and for all those found in [10], but is it true in general? In other words, must  $G$  look like a reflection? The answer, surprisingly, is no, as we proceed to show.

We first introduce the notion of self-combinable theories, and prove that all theories up to the dashed line of Figure 1b are self-combinable. This is an interesting property, and to our knowledge it was not studied so far. We will use it below to prove that  $G$  does not have to be a reflection.

**Definition 3.** *A decidable theory  $\mathcal{T}$  is self-combinable if  $\mathcal{T} \sqcup \mathcal{T}$  is decidable.*

Recall that  $\sqcup$  makes the signatures disjoint before joining the axiomatizations, and so this definition, while strange, is acceptable. We show below examples of both self-combinable and non self-combinable theories.

**Proposition 6.** *If  $\mathcal{T} \in \mathfrak{T}_X$  for  $\mathfrak{T}_X \in \{\mathfrak{T}_{\text{shiny}}, \mathfrak{T}_{\text{SI}+\text{gen}}, \mathfrak{T}_{\text{gen}}, \mathfrak{T}_{\text{SI}+\text{CS}}, \mathfrak{T}_{\text{CS}}, \mathfrak{T}_{\text{SI}}\}$  then it is self-combinable.*

*Proof.*  $\mathfrak{T}_X \subseteq G(\mathfrak{T}_X)$ , and so  $\mathcal{T} \in \mathfrak{T}_X \cap G(\mathfrak{T}_X)$ , hence  $\mathcal{T} \sqcup \mathcal{T}$  is decidable.  $\square$

*Example 3.* We show the theory  $\mathcal{T}(m, n)$  from Table 6, with  $m > n$ , is not self-combinable. For simplicity, let  $\Sigma_{P'}^{\mathbb{N}}$  be a disjoint copy of  $\Sigma_P^{\mathbb{N}}$ , with predicates  $P'_k$  instead of  $P_k$ . Consider the quantifier-free formulas  $P_k \wedge P'_1$ : they are  $\mathcal{T}(m, n) \sqcup \mathcal{T}(m, n)$ -satisfiable if and only if  $k \notin U$ . Indeed, if  $k \in U$  then  $P_k$  implies  $\psi_{=m}$ , while  $P'_1$  implies  $\psi_{=n}$ ; and if  $k \notin U$ , define a  $\mathcal{T}(m, n) \sqcup \mathcal{T}(m, n)$ -interpretation  $\mathcal{A}_k$  with  $n$  elements, where  $P_k$  is true, all other  $P_i$  are false,  $P'_1$  is true, and all other  $P'_i$  are false; of course it satisfies  $P_k \wedge P'_1$ . Thus, if  $\mathcal{T}(m, n) \sqcup \mathcal{T}(m, n)$  is decidable, we get a decision procedure for the undecidable set  $U$ , a contradiction.

Now,  $\mathcal{T}(m, n)$  is decidable, but has none of the other properties. We show that for each property strictly above the dashed line of Figure 1b there is a theory that exhibits it but is not self-combinable. We actually already defined these theories in the bottom part of Table 6:

**Proposition 7.**  $\mathcal{T}_{\langle cbs \rangle}$ ,  $\mathcal{T}_{\langle cfs \rangle}$ , and  $\mathcal{T}_{\langle cbs \rangle}$  from Table 6 are not self-combinable.

Returning to the reflection principle, we obtain the following negative result:

**Proposition 8.** There is a closed set  $\mathfrak{T}_X$  with  $\mathfrak{T}_X \notin G(\mathfrak{T}_X)$  and  $G(\mathfrak{T}_X) \notin \mathfrak{T}_X$ .

*Proof.* Take  $\mathfrak{T}_X$  to be  $G(\{\mathcal{T}(m, n)\})$  for some  $m > n$ : it is not in Figure 1b due to Proposition 6. It is closed because every output of  $G$  is closed, thus  $G(G(\mathfrak{T}_X)) = \mathfrak{T}_X$ . Notice that for any  $p > q$  with  $\{m, n\} \cap \{p, q\} = \emptyset$ ,  $\mathcal{T}(m, n)$  is combinable with  $\mathcal{T}(p, q)$ , as the models of one do not share cardinalities with models of the other.

By Example 3,  $\mathcal{T}(m, n)$  is not in  $\mathfrak{T}_X$  but  $\mathcal{T}(p, q)$  is, and because of that  $\mathcal{T}(m, n)$  is in  $G(\mathfrak{T}_X)$  but  $\mathcal{T}(p, q)$  is not. Therefore  $\mathfrak{T}_X \notin G(\mathfrak{T}_X)$  and  $G(\mathfrak{T}_X) \notin \mathfrak{T}_X$ .  $\square$

## 6 Conclusion

We have computed the minimal lattice of theory combination properties that contains the classical properties of stable infiniteness, shininess, and gentleness. This process generated two new sharp combination methods, based on the novel notions of theories with computable bounded (finite) spectra.

Going forward, we notice that [10] showed the feasibility of proving sharpness of combination theorems. We here adhere to this raised standard: the two new combination theorems that we present are proven sharp. However, the sharpness proofs both here and in [10] require tailored test theories. We plan to streamline this process, looking for more general sufficient conditions for sharpness, and study the completeness of test theories, as defined in [10], for the new properties shown here.

## Acknowledgments

This research was supported in part by Defense Advanced Research Projects Agency (DARPA) contract FA875024-2-1001, Israel Science Foundation (ISF) grant number 209/25, and the Binational Science Foundation (BSF) grant number 2024049. Przybocki was supported by the NSF Graduate Research Fellowship Program under Grant No. DGE-2140739.

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