# Yet another paradefinite logic: The role of conflation

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**Abstract**

In this study, we introduce a paradefinite logic (PL) given by a Gentzen-type sequent calculus as a modified extension of Arieli, Avron, and Zamansky's ideal four-valued paradefinite logic known as 4CC. Some combined implication-conflation logical inference rules in PL are formalized based on a characteristic axiom scheme for connexive logic. Our sequent calculus PL has the characteristic properties of quasi-paraconsistency and quasi-paracompleteness that represent the interaction between conflation and paraconsistent negation. We prove several theorems for syntactically and semantically embedding PL into LK, a Gentzen-type sequent calculus for classical logic, and vice versa. We obtain cut-elimination and completeness theorems for PL via these embedding theorems. Moreover, we introduce an extended paradefinite logic (EPL) that is theorem-equivalent to a Gentzen-type sequent calculus for 4CC. Our sequent calculus EPL has the novel characteristic property of negative symmetry that represents a type of symmetry between conflation and paraconsistent negation.

[1] The results of this paper include the results presented in the 47th IEEE International Symposium on Multiple-Valued Logic [16].

**Key words**: Paradefinite logic, paraconsistent logic, sequent calculus, cut-elimination, completeness.

# 1 Introduction

In this study, we introduce a paradefinite logic (PL) that is given by a Gentzen-type sequent calculus as a modified extension of *Arieli, Avron, and Zamansky's ideal four-valued paradefinite logic* known as 4CC [5, 6, 7]. 4CC is also regarded as a variant of the *logic of logical bilattices* [3, 4]. Here, the calculus PL has a paraconsistent negation connective $\sim$ and a dual conflation connective $-$ [11]. Some combined implication-conflation logical inference rules in PL are formalized on the basis of a characteristic axiom scheme for *connexive logic* [2, 20, 27, 28]. For more comprehensive information on connexive logic, see, for example, [29].

*Paradefinite logic* [5, 6], is a special kind of *paraconsistent logic* [22] that has multiple names; it is also called *non-alethic logic* by da Costa and *paranormal logic* by Béziau [9]. Regardless of its name, paradefinite logic incorporates the properties of both *paraconsistency*, which rejects *the principle* $(\alpha \wedge {\sim}\alpha) \to \beta$ *of explosion*, and *paracompleteness*, which rejects *the law* $\alpha \vee {\sim}\alpha$ *of excluded middle*. Thus, it is both paraconsistent and paracomplete. Paradefinite logic is known to be appropriate for handling inconsistent and incomplete information [5].

Our approach for handling and combining conflation and paraconsistent negation is also similar to the approach presented, for example, in [24, 17, 18] for some paraconsistent logics, including *multilattice logics* [24]. Similar to the multilattice logics and 4CC, the classical negation connective $\neg$ in classical logic can be defined in PL by the combination of $-$ and $\sim$ as $\neg\alpha := -{\sim}\alpha$. Another approach for defining classical negation in paraconsistent logics was proposed by Carnielli et al. [10] developing *the logics of formal inconsistency*, wherein $\neg$ is defined using a consistency operator.

In this study, we prove several theorems for syntactically and semantically embedding PL into LK, a Gentzen-type sequent calculus for classical logic, and vice versa. We then obtain cut-elimination and completeness theorems for PL using these embedding theorems. Such an embedding-based method of proof has also been studied, for example, in [14, 15, 17, 18] to prove cut-elimination and completeness theorems for some paraconsistent logics. Using the cut-elimination theorem for PL, we obtain the properties of both *quasi-paraconsistency*, which rejects *the principle* $(-\alpha \wedge {\sim}\alpha) \to \beta$ *of quasi-explosion*, and *quasi-paracompleteness*, which rejects *the law* $-\alpha \vee {\sim}\alpha$ *of quasi-excluded middle*.

Moreover, in this study, we introduce an extended paradefinite logic (EPL), which is obtained from PL by adding the initial sequents of the forms $(-p, {\sim}p \Rightarrow)$ and $(\Rightarrow -p, {\sim}p)$, and present cut-elimination and completeness theorems for EPL. Our calculus EPL is also shown to be theorem-equivalent to the Gentzen-type sequent calculus $\mathrm{G}_{4\mathrm{CC}}$ [5] for 4CC. Our calculus EPL (and also as a consequence of

$G_{4CC}$) has the following novel characteristic property of *negative symmetry*. For any formulas $\alpha$ and $\beta$, EPL $\vdash {\sim}\alpha \Rightarrow {\sim}\beta$ holds if and only if EPL $\vdash -\beta \Rightarrow -\alpha$ holds. This property implies that $\sim$ and $-$ are, in a sense, symmetric.

We also use this property to demonstrate that EPL is theorem-equivalent to $\text{EPL}^{-}$, that is a more simplified alternative cut-free Gentzen-type sequent calculus. Given this, some combined implication-conflation logical inference rules in EPL, which correspond to some connexive logic axiom schemes, play an important role in demonstrating the negative symmetry property. Note here that the embedding theorems for EPL and $\text{EPL}^{-}$ have not yet been obtained; however, we achieve cut-elimination and completeness using another method.

Our motivation for developing PL and EPL is to clarify the role of the conflation connective $-$ in paradefinite logics. To clarify the role of $-$, we must obtain some characteristic properties concerning the pair of $\sim$ and $-$. The negative symmetry property is regarded as such a property. To clarify the role of $-$, it is useful to see the differences between PL and EPL: (1) EPL has the negative symmetry property, which represents a type of symmetry between $-$ and $\sim$, but PL has no such a property; (2) PL has the quasi-paraconsistency and quasi-paracompleteness properties, which represent the interaction between $-$ and $\sim$, but EPL has no such properties.

Another motivation for developing PL is to obtain a good paradefinite logic that can simulate classical logic. Such a logic is required in application areas that use both paraconsistent (or inconsistency-tolerant) and classical negations. Some paraconsistent logics that can simulate classical negation via paraconsistent double negation have recently been studied in [17, 18], where it was shown that some *bidirectional* embeddings (i.e., embeddings from the underlying paraconsistent logic into classical logic and vice versa) characterize such logics. We believe that the existence of such bidirectional embeddings is important in formalizing the paradefinite logics that can simulate classical logic.

In addition to this introductory section, we structure our paper as follows. In Section 2, we introduce the Gentzen-type sequent calculi PL, EPL, $\text{EPL}^{-}$, and LK. Moreover, we show the negative symmetry property of EPL. Next, in Section 3, we prove several theorems for syntactically embedding PL into LK and vice versa. We also obtain the cut-elimination theorem for PL using the syntactical embedding theorem of PL into LK, and show the cut-elimination theorems for EPL and $\text{EPL}^{-}$ using the standard method of Gentzen. Using the cut-elimination theorem for PL, we also obtain the quasi-paraconsistency and quasi-paracompleteness properties for PL. In Section 4, we prove several theorems for semantically embedding PL into LK and vice versa. Moreover, we also obtain the completeness theorem with respect to a valuation semantics for PL using both the syntactical and semantical embedding theorems into LK. Then, we show the completeness theorem with respect to a valuation semantics for EPL via the method presented by Lahav and Avron in [19]. Finally, in Section 5, we conclude our paper and address some remarks.

## 2 Sequent calculi

*Formulas* of *paradefinite logic* are constructed from countably many propositional variables by the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\sim$ (paraconsistent negation) and $-$ (conflation). In the following we use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to represent finite (possibly empty) sets of formulas. We use expressions ${\sim}\Gamma$ and $-\Gamma$ to denote the sets $\{{\sim}\gamma \mid \gamma \in \Gamma\}$ and $\{-\gamma \mid \gamma \in \Gamma\}$, respectively. We use the symbol $\equiv$ to denote the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. We use an expression $\alpha \Leftrightarrow \beta$ to represent the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. An expression $L \vdash S$ means that a sequent $S$ is provable in a sequent calculus $L$. The *height* of a proof is defined as usual. Our proofs about the various sequent calculi use induction on this measure. If $L$ of $L \vdash S$ is clear from the context, we omit $L$ in it. We say that two sequent calculi $L_1$ and $L_2$ are *theorem-equivalent* if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. A rule $R$ of inference is said to be *admissible* in a sequent calculus $L$ if the following condition is satisfied: For any instance

$$\frac{S_1 \cdots S_n}{S}$$

of $R$, if $L \vdash S_i$ for all $i$, then $L \vdash S$. Moreover, $R$ is said to be *derivable* in $L$ if there is a derivation from $S_1, \cdots, S_n$ to $S$ in $L$. Note that a rule $R$ of inference is admissible in a sequent calculus $L$ if and only if two sequent calculi $L$ and $L + R$ are theorem-equivalent.

A Gentzen-type sequent calculus PL for a paradefinite logic is defined as follows.

**Definition 2.1 (PL)** *The initial sequents of* PL *are of the following form, for any propositional variable $p$,*

$$p \Rightarrow p \qquad {\sim}p \Rightarrow {\sim}p \qquad -p \Rightarrow -p.$$

*The structural inference rules of* PL *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right).}$$

*The non-negated logical inference rules of* PL *are of the form:*

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \text{ ($\wedge$left)} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \text{ ($\wedge$right)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \text{ ($\vee$left)} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \text{ ($\vee$right)}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\to}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\to}\mathrm{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\to}\beta}\ ({\to}\mathrm{right}).$$

*The* $\sim$*-combined logical inference rules of* PL *are of the form:*

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta \quad {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\wedge}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta)}\ ({\sim}{\wedge}\mathrm{right})$$

$$\frac{{\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\vee}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta)}\ ({\sim}{\vee}\mathrm{right})$$

$$\frac{\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha{\to}\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\to}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\alpha}\ ({\sim}{\sim}\mathrm{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{{\sim}{-}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{-}\mathrm{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}{-}\alpha}\ ({\sim}{-}\mathrm{right}).$$

*The* $-$*-combined logical inference rules of* PL *are of the form:*

$$\frac{-\alpha, -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ ({-}{\wedge}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)}\ ({-}{\wedge}\mathrm{right})$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta \quad -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({-}{\vee}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, -\alpha, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \vee \beta)}\ ({-}{\vee}\mathrm{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha{\to}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({-}{\to}\mathrm{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha{\to}\beta)}\ ({-}{\to}\mathrm{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{--\alpha, \Gamma \Rightarrow \Delta}\ ({-}{-}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, --\alpha}\ ({-}{-}\mathrm{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{-{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({-}{\sim}\mathrm{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -{\sim}\alpha}\ ({-}{\sim}\mathrm{right}).$$

In order to compare PL and Arieli-Avron's $\mathrm{G}_{4\mathrm{CC}}$ [5, 6], we introduce a Gentzen-type sequent calculus EPL for an extended paradefinite logic, which is an extension of PL with variants of excluded middle and explosion.

**Definition 2.2 (EPL)** EPL *is obtained from* PL *by adding the initial sequents of the following form, for any propositional variable* $p$,

$${\sim}p, -p \Rightarrow \qquad\qquad \Rightarrow {\sim}p, -p.$$

**Remark 2.3**

1. $({-}{\to}\mathrm{left})$ *and* $({-}{\to}\mathrm{right})$ *correspond to the Hilbert-style axiom scheme* $-(\alpha{\to}\beta) \leftrightarrow \alpha{\to}{-}\beta$*, which is a characteristic axiom scheme for* connexive logics [2, 20, 27, 28] *when* $-$ *is replaced with* $\sim$*. See* [29] *for a comprehensive information on connexive logics.*

2. $(-\!\to\!\text{left})$ *and* $(-\!\to\!\text{right})$ *play an important role for showing the* negative symmetry property *for* EPL. *See Theorem 2.11.*

3. *We can consider the following logical inference rules instead of* $(-\!\to\!\text{left})$ *and* $(-\!\to\!\text{right})$*:*

$$\frac{\Gamma \Rightarrow \Delta, -\alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha\to\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\!\to\!\text{left}') \qquad \frac{-\alpha, \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha\to\beta)}\ (-\!\to\!\text{right}')$$

   *which correspond to the Hilbert-style axiom scheme* $-(\alpha\to\beta) \leftrightarrow -\alpha\to-\beta$.

4. *We cannot show Theorem 2.11 for the logic which is obtained from* EPL *by replacing* $(-\!\to\!\text{left})$ *and* $(-\!\to\!\text{right})$ *with* $(-\!\to\!\text{left}')$ *and* $(-\!\to\!\text{right}')$.

5. *We can similarly show the cut-elimination and completeness theorems for the logic (called here* $\text{PL}^*$*) which is obtained from* PL *by replacing* $(-\!\to\!\text{left})$ *and* $(-\!\to\!\text{right})$ *with* $(-\!\to\!\text{left}')$ *and* $(-\!\to\!\text{right}')$. *The logic* $\text{PL}^*$ *is thus also regarded as a natural paradefinite logic.*

6. *We can define the classical negation connective* $\neg$ *in* PL, $\text{PL}^*$ *and* EPL *as* $\neg\alpha := {\sim}{-}\alpha$ *or* $\neg\alpha := -{\sim}\alpha$, *since the logical inference rules* $({\sim}{-}\text{left})$, $({\sim}{-}\text{right})$, $(-{\sim}\text{left})$ *and* $(-{\sim}\text{right})$ *are the same forms of the logical inference rules for* $\neg$ *in a sequent calculus* LK *for classical logic.*

7. *As shown in Proposition 2.9, the following rules are admissible in cut-free* EPL*:*

$$\frac{\Gamma \Rightarrow \Delta, -\alpha}{{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\text{left}) \qquad \frac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ ({\sim}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{-\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}) \qquad \frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\alpha}\ (-\text{right}).$$

8. *As shown in Definition 2.16,* $\text{G}_{4\text{CC}}$ [5, 6] *is obtained from* EPL *by replacing the following initial sequents and rules:*

   $(p \Rightarrow p)$, $({\sim}p \Rightarrow {\sim}p)$, $(-p \Rightarrow -p)$, $({\sim}p, -p \Rightarrow)$, $(\Rightarrow {\sim}p, -p)$, $(-\wedge\text{left})$, $(-\wedge\text{right})$, $(-\vee\text{left})$, $(-\vee\text{right})$, $(-\!\to\!\text{left})$, $(-\!\to\!\text{right})$, $(--\text{left})$, $(--\text{right})$, $(-{\sim}\text{left})$, $(-{\sim}\text{right})$

   *with the following initial sequents and rules:*

   $(\alpha \Rightarrow \alpha)$, $(-\text{left})$, $(-\text{right})$.

9. *As shown in Theorem 2.15,* EPL *is theorem-equivalent to the system which is obtained from* $\text{G}_{4\text{CC}}$ *by adding* $({\sim}\alpha, -\alpha \Rightarrow)$, $(\Rightarrow {\sim}\alpha, -\alpha)$, $({\sim}\text{left})$ *and* $({\sim}\text{right})$.

10. *As shown in Theorem* 2.17, EPL *is also theorem-equivalent to* $\text{G}_{4CC}$.

Next, we show some basic propositions for PL and EPL.

**Proposition 2.4** *Let $L$ be* PL *or* EPL*. Sequents of the form $\alpha \Rightarrow \alpha$ for any formula $\alpha$ are provable in cut-free $L$.*

**Proof.** By induction on $\alpha$. ∎

**Proposition 2.5** *Sequents of the form ${\sim}\alpha, -\alpha \Rightarrow$ and $\Rightarrow {\sim}\alpha, -\alpha$ for any formula $\alpha$ are provable in cut-free* EPL*.*

**Proof.** By induction on $\alpha$. We show only the following cases.

1. Case $\alpha \equiv {\sim}\beta$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{Prop. 2.4}\\ \beta \Rightarrow \beta\end{array}}{\beta, -{\sim}\beta \Rightarrow}\ (-{\sim}\text{left})}{{\sim}{\sim}\beta, -{\sim}\beta \Rightarrow}\ ({\sim}{\sim}\text{left}) \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\ \text{Prop. 2.4}\\ \beta \Rightarrow \beta\end{array}}{\Rightarrow \beta, -{\sim}\beta}\ (-{\sim}\text{right})}{\Rightarrow {\sim}{\sim}\beta, -{\sim}\beta}\ ({\sim}{\sim}\text{right}).$$

2. Case $\alpha \equiv \beta{\to}\gamma$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{Prop. 2.4}\\ \beta \Rightarrow \beta\end{array} \qquad \begin{array}{c}\vdots\ \text{Ind.hyp.}\\ {\sim}\gamma, -\gamma \Rightarrow\end{array}}{\beta, {\sim}\gamma, -(\beta{\to}\gamma) \Rightarrow}\ (-{\to}\text{left})}{{\sim}(\beta{\to}\gamma), -(\beta{\to}\gamma) \Rightarrow}\ ({\sim}{\to}\text{left}).$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{Prop. 2.4}\\ \beta \Rightarrow \beta\end{array}}{\beta \Rightarrow -\gamma, \beta}\ (\text{we-right}) \qquad \dfrac{\begin{array}{c}\vdots\ \text{Ind.hyp}\\ \Rightarrow {\sim}\gamma, -\gamma\end{array}}{\beta \Rightarrow {\sim}\gamma, -\gamma}\ (\text{we-left})}{\beta \Rightarrow {\sim}(\beta{\to}\gamma), -\gamma}\ ({\sim}{\to}\text{right})}{\Rightarrow {\sim}(\beta{\to}\gamma), -(\beta{\to}\gamma)}\ (-{\to}\text{right}).$$

∎

**Proposition 2.6** *The following sequents are provable in cut-free* PL*:*

1. ${\sim}{\sim}\alpha \Leftrightarrow \alpha$,
2. ${\sim}-\alpha \Leftrightarrow -{\sim}\alpha$,
3. ${\sim}(\alpha \wedge \beta) \Leftrightarrow {\sim}\alpha \vee {\sim}\beta$,
4. ${\sim}(\alpha \vee \beta) \Leftrightarrow {\sim}\alpha \wedge {\sim}\beta$,
5. ${\sim}(\alpha{\to}\beta) \Leftrightarrow \alpha \wedge {\sim}\beta$,
6. $--\alpha \Leftrightarrow \alpha$,
7. $-(\alpha \wedge \beta) \Leftrightarrow -\alpha \wedge -\beta$,

*8.* $-(\alpha \vee \beta) \Leftrightarrow -\alpha \vee -\beta$*,*

*9.* $-(\alpha {\rightarrow} \beta) \Leftrightarrow \alpha {\rightarrow} -\beta$*.*

**Proof.** We show only the cases (2) and (9) below.

1. Case (2):

$$\dfrac{\dfrac{\overset{\vdots\ \text{Prop. 2.4}}{\alpha \Rightarrow \alpha}}{\Rightarrow -{\sim}\alpha, \alpha}\ (-{\sim}\text{right})}{{\sim}{-}\alpha \Rightarrow -{\sim}\alpha}\ ({\sim}{-}\text{left})$$

$$\dfrac{\dfrac{\overset{\vdots\ \text{Prop. 2.4}}{\alpha \Rightarrow \alpha}}{\Rightarrow {\sim}{-}\alpha, \alpha}\ ({\sim}{-}\text{right})}{-{\sim}\alpha \Rightarrow {\sim}{-}\alpha}\ (-{\sim}\text{left}).$$

2. Case (9):

$$\dfrac{\dfrac{\overset{\vdots\ \text{Prop. 2.4}}{\alpha \Rightarrow \alpha} \qquad \overset{\vdots\ \text{Prop. 2.4}}{-\beta \Rightarrow -\beta}}{\alpha, -(\alpha{\rightarrow}\beta) \Rightarrow -\beta}\ (-{\rightarrow}\text{left})}{-(\alpha{\rightarrow}\beta) \Rightarrow \alpha{\rightarrow}-\beta}\ ({\rightarrow}\text{right})$$

$$\dfrac{\dfrac{\overset{\vdots\ \text{Prop. 2.4}}{\alpha \Rightarrow \alpha} \qquad \overset{\vdots\ \text{Prop. 2.4}}{-\beta \Rightarrow -\beta}}{\alpha, \alpha{\rightarrow}-\beta \Rightarrow -\beta}\ ({\rightarrow}\text{left})}{\alpha{\rightarrow}-\beta \Rightarrow -(\alpha{\rightarrow}\beta)}\ (-{\rightarrow}\text{right}).$$

$\blacksquare$

**Proposition 2.7** *The following sequents are provable in cut-free* EPL*:*

*1.* $(-\alpha \wedge {\sim}\alpha) \Rightarrow \beta$ *(the principle of quasi-explosion),*

*2.* $\Rightarrow -\alpha \vee {\sim}\alpha$ *(the law of quasi-excluded middle).*

**Proof.**

1. Follows from Proposition 2.5 using (we-right) and ($\wedge$left).

2. Follows from Proposition 2.5 using ($\vee$right). $\blacksquare$

**Remark 2.8** *Proposition 2.7 does not hold for* PL*. This fact can be obtained by using the cut-elimination theorem for* PL *(see Theorem 3.9).*

**Proposition 2.9** *The following rules are admissible in cut-free* EPL*:*

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha}{{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\text{left}) \qquad \dfrac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ ({\sim}\text{right})$$

$$\dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{-\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}) \qquad \dfrac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\alpha}\ (-\text{right}).$$

**Proof.** This proposition is proved by (simultaneous) induction on the cut-free proofs $P$ of the upper sequents of the rules ($\sim$left), ($\sim$right), ($-$left), and ($-$right). Here, we show only the case of ($\sim$left) by induction on the proofs $P$ of the upper sequent $\Gamma \Rightarrow \Delta, -\alpha$ of ($\sim$left) in cut-free EPL. We show some cases.

1. Case ($-p \Rightarrow -p$): The last inference of $P$ is of the form: $-p \Rightarrow -p$ for a propositional variable $p$. In this case, we have $\vdash\ {\sim}p, -p \Rightarrow$, since it is an initial sequent.

2. Case ($\Rightarrow {\sim}p, -p$): The last inference of $P$ is of the form: $\Rightarrow {\sim}p, -p$ for a propositional variable $p$. In this case, we have $\vdash {\sim}p \Rightarrow {\sim}p$, since it is an initial sequent.

3. Case ($--$right): The last inference of $P$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, --\beta}\ (--\text{right}).$$

   We then obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, \beta\end{matrix}}{{\sim}-\beta, \Gamma \Rightarrow \Delta}\ ({\sim}-\text{left}).$$

4. Case ($-{\sim}$right): The last inference of $P$ is of the form:

$$\frac{\beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -{\sim}\beta}\ (-{\sim}\text{right}).$$

   We then obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ \beta, \Gamma \Rightarrow \Delta\end{matrix}}{{\sim}{\sim}\beta, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{left}).$$

5. Case ($-{\rightarrow}$right): The last inference of $P$ is of the form:

$$\frac{\alpha_1, \Gamma \Rightarrow \Delta, -\alpha_2}{\Gamma \Rightarrow \Delta, -(\alpha_1{\rightarrow}\alpha_2)}\ (-{\rightarrow}\text{right}).$$

   By induction hypothesis, we have $\vdash \alpha_1, {\sim}\alpha_2, \Gamma \Rightarrow \Delta$. We then obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ \alpha_1, {\sim}\alpha_2, \Gamma \Rightarrow \Delta\end{matrix}}{{\sim}(\alpha_1{\rightarrow}\alpha_2), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left}).$$

∎

**Remark 2.10** *Proposition 2.9 can also be obtained by using the cut-elimination theorem for* EPL*, since the rules* ($\sim$left), ($\sim$right), ($-$left), *and* ($-$right) *are derivable in* EPL *by using* (cut)*.*

We then obtain the following characteristic property of EPL.

**Theorem 2.11 (Negative symmetry for EPL)** *For any formulas* $\alpha$ *and* $\beta$*,*

$$\mathrm{EPL} - (\mathrm{cut}) \vdash {\sim}\alpha \Rightarrow {\sim}\beta \ \ \textit{iff} \ \ \mathrm{EPL} - (\mathrm{cut}) \vdash -\beta \Rightarrow -\alpha.$$

**Proof.** By Proposition 2.9. ∎

**Remark 2.12**

1. *Theorem 2.11 does not hold for* PL*. A counterexample is presented as follows. For any propositional variable* $p$*, we have:* $\mathrm{PL} \vdash {\sim}{\sim}{\sim}p \Rightarrow {\sim}p$ *and* $\mathrm{PL} \nvdash -p \Rightarrow -{\sim}{\sim}p$*. The unprovability of* $-p \Rightarrow -{\sim}{\sim}p$ *in* PL *can be obtained by the cut-elimination theorem for* PL*, which will be presented as Theorem 3.4.*

2. *The following general fact also holds: For any sets* $\Gamma$ *and* $\Delta$ *of formulas,* $\mathrm{EPL} - (\mathrm{cut}) \vdash {\sim}\Gamma \Rightarrow {\sim}\Delta$ *iff* $\mathrm{EPL} - (\mathrm{cut}) \vdash -\Delta \Rightarrow -\Gamma$*. Based on this fact, we can define an alternative simple system* $\mathrm{EPL}^-$*, which corresponds to an extension of* $\mathrm{G_{4CC}}$ *by adding* $({\sim}\alpha, -\alpha \Rightarrow)$, $(\Rightarrow {\sim}\alpha, -\alpha)$, ($\sim$left) *and* ($\sim$right)*.*

**Definition 2.13 ($\mathbf{EPL}^-$)** $\mathrm{EPL}^-$ *is obtained from* EPL *by replacing the following initial sequents and rules:*

$(-p \Rightarrow -p)$, $({\sim}p, -p \Rightarrow)$, $(\Rightarrow {\sim}p, -p)$, ($-\wedge$left), ($-\wedge$right), ($-\vee$left), ($-\vee$right), ($-{\to}$left), ($-{\to}$right), ($--$left), ($--$right), ($-{\sim}$left), ($-{\sim}$right)

*with the following rules:*

($\sim$left), ($\sim$right), ($-$left), ($-$right).

**Remark 2.14** *Using* (cut)*, it can be shown that there is no need to add all four rules* {($\sim$left), ($\sim$right), ($-$left), ($-$right)}*. Instead, it suffices to choose one of the right rules and one of the left rules.*

**Theorem 2.15 (Cut-free equivalence between EPL and $\mathbf{EPL}^-$)** $\mathrm{EPL} - (\mathrm{cut})$ *and* $\mathrm{EPL}^- - (\mathrm{cut})$ *are theorem-equivalent.*

**Proof.** ($\Longrightarrow$): We show that for any sequent $S$, $\mathrm{EPL} - (\mathrm{cut}) \vdash S$ implies $\mathrm{EPL}^- - (\mathrm{cut}) \vdash S$. This is shown by induction on the proofs $Q$ of $S$ in cut-free EPL. We show some cases.

1. Case ($\sim p, -p \Rightarrow$) or ($\Rightarrow \sim p, -p$): By using ($\sim$left), ($\sim$right), ($-$left) and ($-$right).

2. Case ($--$left): The last inference of $Q$ is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{--\alpha, \Gamma \Rightarrow \Delta}\ (--\text{left}).$$

We obtain the required fact:

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ \alpha, \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta, \sim-\alpha}\ (\sim-\text{right})}{--\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}).$$

3. Case ($-\sim$left): The last inference of $Q$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{-\sim\alpha, \Gamma \Rightarrow \Delta}\ (-\sim\text{left}).$$

We obtain the required fact:

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{matrix}}{\Gamma \Rightarrow \Delta, \sim\sim\alpha}\ (\sim\sim\text{right})}{-\sim\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}).$$

4. Case ($-\!\to$left): The last inference of $Q$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha\to\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\!\to\text{left}).$$

We obtain the required fact:

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\\ \vdots\ (\text{we-left}), (\text{we-right})\\ \Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha\end{matrix} \qquad \begin{matrix}\cfrac{\begin{matrix}\vdots\\ -\beta, \Sigma \Rightarrow \Pi\end{matrix}}{\Sigma \Rightarrow \Pi, \sim\beta}\ (\sim\text{right})\\ \vdots\ (\text{we-left}), (\text{we-right})\\ \Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim\beta\end{matrix}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim(\alpha\to\beta)}\ (\sim\!\to\text{right})}{-(\alpha\to\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\text{left})$$

($\Longleftarrow$): We show that for any sequent $S$, $\text{EPL}^- - (\text{cut}) \vdash S$ implies $\text{EPL} - (\text{cut}) \vdash S$. This is shown by induction on the proofs $P$ of $S$ in cut-free $\text{EPL}^-$. The cases when the last inference of $P$ is ($\sim$left), ($\sim$right), ($-$left) or ($-$right) are obtained from Proposition 2.9. ∎

In order to show the equivalence between EPL and $\text{G}_{4\text{CC}}$, we give a precise definition of $\text{G}_{4\text{CC}}$ as follows.

**Definition 2.16 ($\mathrm{G}_{4\mathrm{CC}}$ [5, 6])** $\mathrm{G}_{4\mathrm{CC}}$ *is obtained from* EPL *by replacing the following initial sequents and rules:*

$(p \Rightarrow p)$, $({\sim}p \Rightarrow {\sim}p)$, $(-p \Rightarrow -p)$, $({\sim}p, -p \Rightarrow)$, $(\Rightarrow {\sim}p, -p)$, $(-\wedge\text{left})$, $(-\wedge\text{right})$, $(-\vee\text{left})$, $(-\vee\text{right})$, $(-{\to}\text{left})$, $(-{\to}\text{right})$, $(--\text{left})$, $(--\text{right})$, $(-{\sim}\text{left})$, $(-{\sim}\text{right})$

*with the following initial sequents and rules:*

$(\alpha \Rightarrow \alpha)$, $(-\text{left})$, $(-\text{right})$.

**Theorem 2.17 (Equivalence between EPL and $\mathrm{G}_{4\mathrm{CC}}$)** EPL *is theorem-equivalent to* $\mathrm{G}_{4\mathrm{CC}}$.

**Proof.** As presented in Remark 2.3 (item 8) and Definition 2.13, $\mathrm{G}_{4\mathrm{CC}}$ [5] is also obtained from $\mathrm{EPL}^-$ by adding $\alpha \Rightarrow \alpha$, (${\sim}$left) and (${\sim}$right). By Theorem 2.15, EPL is theorem-equivalent to $\mathrm{EPL}^-$. Thus it suffices to show that $\mathrm{EPL}^-$ is theorem-equivalent to $\mathrm{G}_{4\mathrm{CC}}$. To show that every sequent that is provable in $\mathrm{G}_{4\mathrm{CC}}$ is also provable in $\mathrm{EPL}^-$, it suffices to prove that $\alpha \Rightarrow \alpha$ is derivable in $\mathrm{EPL}^-$ for every formula $\alpha$, which was proven in Proposition 2.4 for PL, and thus also holds for $\mathrm{EPL}^-$. For the converse, we show that the rules (${\sim}$left) and (${\sim}$right) are derivable in $\mathrm{G}_{4\mathrm{CC}}$. For the first, consider the following proof in $\mathrm{G}_{4\mathrm{CC}}$:

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha \qquad \dfrac{\dfrac{{\sim}\alpha \Rightarrow {\sim}\alpha}{{\sim}\alpha, \Gamma \Rightarrow \Delta, {\sim}\alpha}\ \vdots\,(\text{we-left}), (\text{we-right})}{{\sim}\alpha, -\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left})}{{\sim}\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut})$$

For the second, consider the following proof in $\mathrm{G}_{4\mathrm{CC}}$:

$$\dfrac{-\alpha, \Gamma \Rightarrow \Delta \qquad \dfrac{\dfrac{{\sim}\alpha \Rightarrow {\sim}\alpha}{{\sim}\alpha, \Gamma \Rightarrow \Delta, {\sim}\alpha}\ \vdots\,(\text{we-left}), (\text{we-right})}{\Gamma \Rightarrow \Delta, {\sim}\alpha, -\alpha}\ (-\text{right})}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ (\text{cut})$$

■

In order to show some syntactical embedding theorems, we introduce a Gentzen-type sequent calculus LK for classical logic, which is a variant of the propositional fragment of Gentzen's original calculus for this logic [12]. Formulas of LK are constructed from countably many propositional variables by logical connectives $\wedge$, $\vee$, $\to$ and $\neg$ (classical negation).

**Definition 2.18 (LK)** LK *is obtained from the* $\{{\sim}, -\}$*-free fragment of* PL *by adding the classical negation inference rules of the form:*

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \dfrac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}).$$

As well-known, the cut-elimination theorem holds for LK (see e.g., [12, 25]).

## 3 Syntactical embedding and cut-elimination

We introduce an LK-translation function for formulas of PL, and by using this translation, we show several theorems for embedding PL into LK.

**Definition 3.1** *We fix a set* $\Phi$ *of propositional variables, and define the sets* $\Phi^n := \{p^n \mid p \in \Phi\}$ *and* $\Phi^c := \{p^c \mid p \in \Phi\}$ *of propositional variables. The language* $\mathcal{L}_{\mathrm{PL}}$ *of* PL *is defined using* $\Phi$, $\wedge, \vee, \rightarrow, \sim$ *and* $-$. *The language* $\mathcal{L}_{\mathrm{LK}}$ *of* LK *is defined using* $\Phi$, $\Phi^n$, $\Phi^c$, $\wedge$, $\vee$, $\rightarrow$ *and* $\neg$. *A mapping* $f$ *from* $\mathcal{L}_{\mathrm{PL}}$ *to* $\mathcal{L}_{\mathrm{LK}}$ *is defined inductively by:*

1. *For any* $p \in \Phi$, $f(p) := p$, $f({\sim}p) := p^n \in \Phi^n$ *and* $f(-p) := p^c \in \Phi^c$,
2. $f(\alpha \wedge \beta) := f(\alpha) \wedge f(\beta)$,
3. $f(\alpha \vee \beta) := f(\alpha) \vee f(\beta)$,
4. $f(\alpha{\rightarrow}\beta) := f(\alpha){\rightarrow}f(\beta)$,
5. $f({\sim}(\alpha \wedge \beta)) := f({\sim}\alpha) \vee f({\sim}\beta)$,
6. $f({\sim}(\alpha \vee \beta)) := f({\sim}\alpha) \wedge f({\sim}\beta)$,
7. $f({\sim}(\alpha{\rightarrow}\beta)) := f(\alpha) \wedge f({\sim}\beta)$,
8. $f({\sim}{\sim}\alpha) := f(\alpha)$,
9. $f({\sim}{-}\alpha) := \neg f(\alpha)$,
10. $f(-(\alpha \wedge \beta)) := f(-\alpha) \wedge f(-\beta)$,
11. $f(-(\alpha \vee \beta)) := f(-\alpha) \vee f(-\beta)$,
12. $f(-(\alpha{\rightarrow}\beta)) := f(\alpha){\rightarrow}f(-\beta)$,
13. $f(--\alpha) := f(\alpha)$,
14. $f(-{\sim}\alpha) := \neg f(\alpha)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula $\alpha$ in $\Gamma$ by an occurrence of $f(\alpha)$. Analogous notation is used for the other mapping $g$ discussed later.

**Remark 3.2** *A similar translation as defined in Definition 3.1 has been used by Gurevich* [13], *Rautenberg* [23] *and Vorob'ev* [26] *to embed Nelson's constructive logic* [1, 21] *into intuitionistic logic. Some similar translations have also recently been used, for example, in* [14, 15, 17, 18] *to embed some paraconsistent logics into classical logic.*

We now show a weak theorem for syntactically embedding PL into LK.

**Theorem 3.3 (Weak syntactical embedding from PL into LK)** *Let* $\Gamma$, $\Delta$ *be sets of formulas in* $\mathcal{L}_{\mathrm{PL}}$*, and* $f$ *be the mapping defined in Definition 3.1.*

1. *If* $\mathrm{PL} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

2. *If* $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$*, then* $\mathrm{PL} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

**Proof.** • (1): By induction on the proofs $P$ of $\Gamma \Rightarrow \Delta$ in PL. We distinguish the cases according to the last inference of $P$, and show some cases.

1. Case $\sim p \Rightarrow \sim p$: The last inference of $P$ is of the form: $\sim p \Rightarrow \sim p$ for any $p \in \Phi$. In this case, we obtain $\mathrm{LK} \vdash f(\sim p) \Rightarrow f(\sim p)$, i.e., $\mathrm{LK} \vdash p^n \Rightarrow p^n$ $(p^n \in \Phi^n)$, by the definition of $f$.

2. Case ($\sim\wedge$left): The last inference of $P$ is of the form:

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\sim\wedge\mathrm{left}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\sim\alpha), f(\Gamma) \Rightarrow f(\Delta)$ and $\mathrm{LK} \vdash f(\sim\beta), f(\Gamma) \Rightarrow f(\Delta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\sim\alpha), f(\Gamma) \Rightarrow f(\Delta)\end{array} \quad \begin{array}{c}\vdots\\ f(\sim\beta), f(\Gamma) \Rightarrow f(\Delta)\end{array}}{f(\sim\alpha) \vee f(\sim\beta), f(\Gamma) \Rightarrow f(\Delta)}\ (\vee\mathrm{left})$$

where $f(\sim\alpha) \vee f(\sim\beta)$ coincides with $f(\sim(\alpha \wedge \beta))$ by the definition of $f$.

3. Case ($\sim-$left): The last inference of $P$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim-\alpha, \Gamma \Rightarrow \Delta}\ (\sim-\mathrm{left}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array}}{\neg f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\mathrm{left})$$

where $\neg f(\alpha)$ coincides with $f(\sim-\alpha)$ by the definition of $f$.

4. Case ($-\!\rightarrow$left): The last inference of $P$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\!\rightarrow\mathrm{left}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$ and $\mathrm{LK} \vdash f(-\beta), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\dfrac{\begin{matrix}\vdots \\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{matrix} \quad \begin{matrix}\vdots \\ f(-\beta), f(\Sigma) \Rightarrow f(\Pi)\end{matrix}}{f(\alpha)\to f(-\beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\to\text{left})$$

where $f(\alpha)\to f(-\beta)$ coincides with $f(-(\alpha\to\beta))$ by the definition of $f$.

• (2): By induction on the proofs $Q$ of $f(\Gamma) \Rightarrow f(\Delta)$ in LK $-$ (cut). We distinguish the cases according to the last inference of $Q$.

We explicitly consider some of the cases. The rest are handled similarly.

1. Case ($\neg$left): The last inference of $Q$ is ($\neg$left).

   (a) Subcase (1): The last inference of $Q$ is of the form:

   $$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha)}{f({\sim}{-}\alpha), f(\Gamma) \Rightarrow f(\Delta)} \ (\neg\text{left})$$

   where $f({\sim}{-}\alpha)$ coincides with $\neg f(\alpha)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

   $$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \alpha\end{matrix}}{{\sim}{-}\alpha, \Gamma \Rightarrow \Delta} \ ({\sim}{-}\text{left}).$$

   (b) Subcase (2): The last inference of $Q$ is of the form:

   $$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha)}{f({-}{\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)} \ (\neg\text{left})$$

   where $f({-}{\sim}\alpha)$ coincides with $\neg f(\alpha)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

   $$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \alpha\end{matrix}}{{-}{\sim}\alpha, \Gamma \Rightarrow \Delta} \ ({-}{\sim}\text{left}).$$

2. Case ($\wedge$right): The last inference of $Q$ is ($\wedge$right).

   (a) Subcase (1): The last inference of $Q$ is of the form:

   $$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(\alpha \wedge \beta)} \ (\wedge\text{right})$$

   where $f(\alpha \wedge \beta)$ coincides with $f(\alpha) \wedge f(\beta)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and PL $-$ (cut) $\vdash$ $\Gamma \Rightarrow \Delta, \beta$. We thus obtain the required fact:

   $$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \beta\end{matrix}}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \ (\wedge\text{right}).$$

(b) Subcase (2): The last inference of $Q$ is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\sim\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(\sim(\alpha \vee \beta))} \ (\wedge\text{right})$$

where $f(\sim(\alpha \vee \beta))$ coincides with $f(\sim\alpha) \wedge f(\sim\beta)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim\alpha$ and PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim\beta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \sim\alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \sim\beta\end{matrix}}{\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)} \ (\sim\vee\text{right}).$$

(c) Subcase (3): The last inference of $Q$ is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(-\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(-\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(-(\alpha \wedge \beta))} \ (\wedge\text{right})$$

where $f(-(\alpha \wedge \beta))$ coincides with $f(-\alpha) \wedge f(-\beta)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, -\alpha$ and PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, -\beta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, -\alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, -\beta\end{matrix}}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)} \ (-\wedge\text{right}).$$

(d) Subcase (4): The last inference of $Q$ is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(\sim(\alpha\to\beta))} \ (\wedge\text{right})$$

where $f(\sim(\alpha\to\beta))$ coincides with $f(\alpha) \wedge f(\sim\beta)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim\beta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta, \sim\beta\end{matrix}}{\Gamma \Rightarrow \Delta, \sim(\alpha\to\beta)} \ (\sim\to\text{right}).$$

3. Case ($\to$left): The last inference of $Q$ is ($\to$left).

(a) Subcase (1): The last inference of $Q$ is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\alpha\to\beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\to\text{left})$$

where $f(\alpha{\rightarrow}\beta)$ coincides with $f(\alpha){\rightarrow}f(\beta)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and PL $-$ (cut) $\vdash \beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ \beta, \Sigma \Rightarrow \Pi\end{array}}{\alpha{\rightarrow}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}).$$

(b) Subcase (2): The last inference of $Q$ is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(-\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(-(\alpha{\rightarrow}\beta)), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ ({\rightarrow}\text{left})$$

where $f(-(\alpha{\rightarrow}\beta))$ coincides with $f(\alpha){\rightarrow}f(-\beta)$ by the definition of $f$. By induction hypothesis, we have PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and PL $-$ (cut) $\vdash -\beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ -\beta, \Sigma \Rightarrow \Pi\end{array}}{-(\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({-}{\rightarrow}\text{left}).$$

∎

Using Theorem 3.3 and the cut-elimination theorem for LK, we obtain the following cut-elimination theorem for PL.

**Theorem 3.4 (Cut-elimination for PL)** *The rule* (cut) *is admissible in cut-free* PL*.*

**Proof.** Suppose PL $\vdash \Gamma \Rightarrow \Delta$. Then, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.3 (1), and hence LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. By Theorem 3.3 (2), we obtain PL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. ∎

**Remark 3.5**

1. *The counterexample which was presented in Remark 2.12 for the negative symmetry properties for* PL *can be obtained using Theorem 3.4. Indeed, as mentioned in Remark 2.12, the unprovability of* $-p \Rightarrow -{\sim}{\sim}p$ *in* PL *is guaranteed by Theorem 3.4.*

2. *The cut-elimination theorem for* PL *can also be proved directly (as shown by Gentzen for* LK [12]*) without using Theorem 3.3.*

3. *As shown above, the cut-elimination theorem for* PL *is easily obtained from the embedding theorem. That is why we give here the embedding-based proof of the cut-elimination theorem.*

4. *A variant of the subformula property that allows negated and conflated subformulas of the form* $\sim\alpha$ *or* $-\alpha$ *can be obtained by using Theorem 3.4.*

Using Theorem 3.3 and the cut-elimination theorem for LK, we obtain a strong theorem for syntactically embedding PL into LK.

**Theorem 3.6 (Syntactical embedding from PL into LK)** *Let* $\Gamma$, $\Delta$ *be sets of formulas in* $\mathcal{L}_{\mathrm{PL}}$, *and* $f$ *be the mapping defined in Definition 3.1.*

1. $\mathrm{PL} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

2. $\mathrm{PL} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

**Proof.** • (1): ($\Longrightarrow$): By Theorem 3.3 (1). ($\Longleftarrow$): Suppose $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$. Then we have $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. We thus obtain $\mathrm{PL} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ by Theorem 3.3 (2). Therefore we have $\mathrm{PL} \vdash \Gamma \Rightarrow \Delta$.

• (2): ($\Longrightarrow$): Suppose $\mathrm{PL} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. Then we have $\mathrm{PL} \vdash \Gamma \Rightarrow \Delta$. We then obtain $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.3 (1). Therefore we obtain $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. ($\Longleftarrow$): By Theorem 3.3 (2). ▮

**Theorem 3.7 (Decidability for PL)** PL *is decidable.*

**Proof.** By decidability of LK, for each $\alpha$, it is possible to decide if $f(\alpha)$ is provable in LK. Then, by Theorem 3.6, PL is also decidable. ▮

By using Theorem 3.4, we show the paraconsistency, quasi-paraconsistency, paracompleteness, and quasi-paracompleteness properties for PL. The quasi-paraconsistency and quasi-paracompleteness properties for PL do not hold for EPL (see Proposition 2.7).

**Definition 3.8** *We define the following notions.*

1. *A sequent system L is called* explosive *with respect to a negation-like connective* $\sharp$ *if* $L \vdash \alpha, \sharp\alpha \Rightarrow \beta$ *for any formulas* $\alpha$ *and* $\beta$. *A sequent system L is called* paraconsistent *with respect to* $\sharp$ *if L is not explosive with respect to* $\sharp$.

2. *A sequent system L is called* quasi-explosive *with respect to the combination of two different negation-like connectives* $\sharp$ *and* $\natural$ *if* $L \vdash \sharp\alpha, \natural\alpha \Rightarrow \beta$ *for any formulas* $\alpha$ *and* $\beta$. *A sequent system L is called* quasi-paraconsistent *with respect to the combination of* $\sharp$ *and* $\natural$ *if L is not quasi-explosive with respect to the combination of* $\sharp$ *and* $\natural$.

3. *A sequent system L is called* exclusive *with respect to a negation-like connective* $\sharp$ *if* $L \vdash\ \Rightarrow \alpha, \sharp\alpha$ *for any formula* $\alpha$. *A sequent system L is called* paracomplete *with respect to* $\sharp$ *if L is not exclusive with respect to* $\sharp$.

4. *A sequent system $L$ is called* quasi-exclusive *with respect to the combination of two different negation-like connectives $\sharp$ and $\natural$ if $L \vdash \Rightarrow \sharp\alpha, \natural\alpha$ for any formula $\alpha$. A sequent system $L$ is called* quasi-paracomplete *with respect to the combination of $\sharp$ and $\natural$ if $L$ is not quasi-exclusive with respect to the combination of $\sharp$ and $\natural$.*

**Theorem 3.9 (Quasi-paraconsistency and quasi-paracompleteness for PL)** *We have:*

1. *(Paraconsistency):* PL *is paraconsistent with respect to $\sim$ and $-$.*
2. *(Quasi-paraconsistency):* PL *is quasi-paraconsistent with respect to the combination of $\sim$ and $-$.*
3. *(Paracompleteness):* PL *is paracomplete with respect to $\sim$ and $-$.*
4. *(Quasi-paracompleteness):* PL *is quasi-paracomplete with respect to the combination of $\sim$ and $-$.*

**Proof.** We show only (1) below. Consider sequent $({\sim}p, -p \Rightarrow q)$ where $p$ and $q$ are distinct propositional variables. Then, the unprovability of this sequent is guaranteed by Theorem 3.4. ∎

Next, we introduce a PL-translation function for formulas of LK, and by using this translation, we show some theorems for embedding LK into PL.

**Definition 3.10** *Let $\mathcal{L}_{\mathrm{PL}}$ and $\mathcal{L}_{\mathrm{LK}}$ be the languages defined in Definition 3.1. A mapping $g$ from $\mathcal{L}_{\mathrm{LK}}$ to $\mathcal{L}_{\mathrm{PL}}$ is defined inductively by:*

1. *For any $p \in \Phi$, any $p^n \in \Phi^n$ and any $p^c \in \Phi^c$, $g(p) := p$, $g(p^n) := {\sim}p$ and $g(p^c) := -p$,*
2. *$g(\alpha \wedge \beta) := g(\alpha) \wedge g(\beta)$,*
3. *$g(\alpha \vee \beta) := g(\alpha) \vee g(\beta)$,*
4. *$g(\alpha \rightarrow \beta) := g(\alpha) \rightarrow g(\beta)$,*
5. *$g(\neg\alpha) := {\sim}{-}g(\alpha)$.*

**Theorem 3.11 (Weak syntactical embedding from LK into PL)** *Let $\Gamma, \Delta$ be sets of formulas in $\mathcal{L}_{\mathrm{LK}}$, and $g$ be the mapping defined in Definition 3.10.*

1. *If* $\mathrm{LK} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{PL} \vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. *If* $\mathrm{PL} - (\mathrm{cut}) \vdash g(\Gamma) \Rightarrow g(\Delta)$*, then* $\mathrm{LK} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

**Proof.** • (1): By induction on the proofs $P$ of $\Gamma \Rightarrow \Delta$ in LK. We distinguish the cases according to the last inference of $P$, and show only the following cases.

1. Case $p^\star \Rightarrow p^\star$ with $\star \in \{n, c\}$: The last inference of $P$ is of the form: $p^\star \Rightarrow p^\star$ for any $p^\star \in \Phi^\star$ with $\star \in \{n, c\}$. In this case, we obtain PL $\vdash g(p^n) \Rightarrow g(p^n)$ and PL $\vdash g(p^c) \Rightarrow g(p^c)$ i.e., PL $\vdash {\sim}p \Rightarrow {\sim}p$ and PL $\vdash -p \Rightarrow -p$, by the definition of $g$.

2. Case ($\neg$left): The last inference of $P$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left})$$

By induction hypothesis, we have PL $\vdash g(\Gamma) \Rightarrow g(\Delta), g(\alpha)$. We then obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ g(\Gamma) \Rightarrow g(\Delta), g(\alpha)\end{array}}{{\sim}{-}g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ ({\sim}{-}\text{left})$$

where ${\sim}{-}g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of $g$.

• (2): By induction on the proofs $Q$ of $g(\Gamma) \Rightarrow g(\Delta)$ in PL $-$ (cut). We distinguish the cases according to the last inference of $Q$, and show only the following cases.

1. Case (${\sim}{-}$left): The last inference of $Q$ is of the form:

$$\frac{g(\Gamma) \Rightarrow g(\Delta), g(\alpha)}{{\sim}{-}g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ ({\sim}{-}\text{left})$$

where ${\sim}{-}g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of $g$. By induction hypothesis, we have LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array}}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}).$$

2. Case ($\wedge$right): The last inference of $Q$ is of the form:

$$\frac{g(\Gamma) \Rightarrow g(\Delta), g(\alpha) \quad g(\Gamma) \Rightarrow g(\Delta), g(\beta)}{g(\Gamma) \Rightarrow g(\Delta), g(\alpha) \wedge g(\beta)}\ (\wedge\text{right})$$

where $g(\alpha) \wedge g(\beta)$ coincides with $g(\alpha \wedge \beta)$ by the definition of $g$. By induction hypothesis, we have LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \beta$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \beta\end{array}}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right}).$$

∎

**Theorem 3.12 (Syntactical embedding from LK into PL)** *Let* $\Gamma$, $\Delta$ *be sets of formulas in* $\mathcal{L}_{\mathrm{LK}}$*, and* $g$ *be the mapping defined in Definition 3.10.*

1. $\mathrm{LK} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{PL} \vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. $\mathrm{LK} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{PL} - (\mathrm{cut}) \vdash g(\Gamma) \Rightarrow g(\Delta)$.

**Proof.** By using Theorems 3.11 and 3.4. Similar to Theorem 3.6. ∎

We have the following cut-elimination theorems for EPL and EPL$^-$, but these are not proved using an embedding-based method.

**Theorem 3.13 (Cut-elimination for EPL and EPL$^-$)** *Let* $L$ *be* EPL *or* EPL$^-$. *The rule* (cut) *is admissible in cut-free* $L$.

**Proof.** (Sketch). The theorem for EPL$^-$ is obtained from that for EPL and Theorem 2.15. Thus, we here consider the theorem for EPL. In this regard, it is sufficient to consider the cases of the initial sequents of the form $({\sim}p, -p \Rightarrow)$ and $(\Rightarrow {\sim}p, -p)$, since the cut-elimination theorem (Theorem 3.4) holds for the subsystem PL of EPL by deleting $({\sim}p, -p \Rightarrow)$ and $(\Rightarrow {\sim}p, -p)$. We thus demonstrate such cases below.

1. Case when the left upper sequent of the cut is derived from a single premise left logical inference rule $R$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta, {\sim}p\end{array}}{\Gamma^* \Rightarrow \Delta, {\sim}p}\,(R) \qquad {\sim}p, -p \Rightarrow}{-p, \Gamma^* \Rightarrow \Delta}\,(\mathrm{cut})$$

where $P$ is a cut-free proof. In this case, we can transform this proof into the following proof which can eliminate the cut by induction hypothesis:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta, {\sim}p\end{array} \qquad {\sim}p, -p \Rightarrow}{-p, \Gamma \Rightarrow \Delta}\,(\mathrm{cut})}{-p, \Gamma^* \Rightarrow \Delta}\,(R).$$

2. Case when the left upper sequent of the cut is derived from (we-right) where the principal formula of (we-right) is ${\sim}p$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta, {\sim}p}\,(\text{we-right}) \qquad {\sim}p, -p \Rightarrow}{-p, \Gamma \Rightarrow \Delta}\,(\mathrm{cut})$$

where $P$ is a cut-free proof. In this case, we can transform this proof into the following cut-free proof:

$$\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta\end{array}}{-p, \Gamma \Rightarrow \Delta}\ (\text{we-left}).$$

∎

**Theorem 3.14 (Paraconsistency and paracompleteness for EPL)** *We have:*

1. *(Paraconsistency):* EPL *(also* $\mathrm{EPL}^-$ *or* $\mathrm{G}_{4CC}$*) is paraconsistent with respect to* $\sim$ *and* $-$.

2. *(Paracompleteness):* EPL *(also* $\mathrm{EPL}^-$ *or* $\mathrm{G}_{4CC}$*) is paracomplete with respect to* $\sim$ *and* $-$.

**Proof.** By using Theorem 3.13. ∎

# 4 Semantical embedding and completeness

We now introduce a valuation semantics for PL by defining the valuation function on the two-element set of classical truth-values.

**Definition 4.1 (Semantics for PL)** *Let $\Phi$ be the set of all propositional variables, $\Phi^{\sim}$ be the set $\{\sim p \mid p \in \Phi\}$ and $\Phi^{-}$ be the set $\{-p \mid p \in \Phi\}$. A* paraconsistent valuation *$v^*$ is a mapping from $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to the set $\{t, f\}$ of truth values. The paraconsistent valuation $v^*$ is extended to the mapping from the set of all formulas to $\{t, f\}$ by the following clauses.*

1. $v^*(\alpha \wedge \beta) = t$ *iff* $v^*(\alpha) = t$ *and* $v^*(\beta) = t$,

2. $v^*(\alpha \vee \beta) = t$ *iff* $v^*(\alpha) = t$ *or* $v^*(\beta) = t$,

3. $v^*(\alpha \rightarrow \beta) = t$ *iff* $v^*(\alpha) = f$ *or* $v^*(\beta) = t$,

4. $v^*(\sim(\alpha \wedge \beta)) = t$ *iff* $v^*(\sim\alpha) = t$ *or* $v^*(\sim\beta) = t$,

5. $v^*(\sim(\alpha \vee \beta)) = t$ *iff* $v^*(\sim\alpha) = t$ *and* $v^*(\sim\beta) = t$,

6. $v^*(\sim(\alpha \rightarrow \beta)) = t$ *iff* $v^*(\alpha) = t$ *and* $v^*(\sim\beta) = t$,

7. $v^*(\sim\sim\alpha) = t$ *iff* $v^*(\alpha) = t$,

8. $v^*(\sim-\alpha) = t$ *iff* $v^*(\alpha) = f$,

9. $v^*(-(\alpha \wedge \beta)) = t$ *iff* $v^*(-\alpha) = t$ *and* $v^*(-\beta) = t$,

10. $v^*(-(\alpha \vee \beta)) = t$ *iff* $v^*(-\alpha) = t$ *or* $v^*(-\beta) = t$,

11. $v^*(-(\alpha{\rightarrow}\beta)) = t$ *iff* $v^*(\alpha) = f$ *or* $v^*(-\beta) = t$,

12. $v^*(--\alpha) = t$ *iff* $v^*(\alpha) = t$,

13. $v^*(-{\sim}\alpha) = t$ *iff* $v^*(\alpha) = f$.

*A formula* $\alpha$ *is called* PL-valid *iff* $v^*(\alpha) = t$ *holds for all paraconsistent valuation* $v^*$.

In order to show some semantical embedding theorems, we present the standard two-valued semantics for LK.

**Definition 4.2 (Semantics for LK)** *A valuation* $v$ *is a mapping from the set of all propositional variables to the set* $\{t, f\}$ *of truth values. The valuation* $v$ *is extended to the mapping from the set of all formulas to* $\{t, f\}$ *by the following clauses.*

1. $v(\alpha \wedge \beta) = t$ *iff* $v(\alpha) = t$ *and* $v(\beta) = t$,

2. $v(\alpha \vee \beta) = t$ *iff* $v(\alpha) = t$ *or* $v(\beta) = t$,

3. $v(\alpha{\rightarrow}\beta) = t$ *iff* $v(\alpha) = f$ *or* $v(\beta) = t$,

4. $v(\neg\alpha) = t$ *iff* $v(\alpha) = f$.

*A formula* $\alpha$ *is called* LK-valid *iff* $v(\alpha) = t$ *holds for all valuation* $v$.

**Remark 4.3** *The following completeness theorem holds for* LK*: For any formula* $\alpha$*,* LK $\vdash\ \Rightarrow \alpha$ *iff* $\alpha$ *is LK-valid.*

Next, we show a theorem for semantically embedding PL into LK.

**Lemma 4.4** *Let* $f$ *be the mapping defined in Definition 3.1. For any paraconsistent valuation* $v^*$*, we can construct a valuation* $v$ *such that for any formula* $\alpha$*,*

$$v^*(\alpha) = t \text{ iff } v(f(\alpha)) = t.$$

**Proof.** Let $\Phi$ be a set of propositional variables, and for each $\star \in \{n, c\}$, let $\Phi^\star$ be the set $\{p^\star \mid p \in \Phi\}$ of propositional variables. Suppose that $v^*$ is a paraconsistent valuation. Suppose that $v$ is a mapping from $\Phi \cup \Phi^n \cup \Phi^c$ to $\{t, f\}$ such that

1. $v^*(p) = t$ iff $v(p) = t$,

2. $v^*({\sim}p) = t$ iff $v(p^n) = t$,

3. $v^*(-p) = t$ iff $v(p^c) = t$.

Then, the lemma is proved by induction on $\alpha$.

- Base step:

1. Case $\alpha \equiv p$ where $p$ is a propositional variable: $v^*(p) = t$ iff $v(p) = t$ (by the assumption) iff $v(f(p)) = t$ (by the definition of $f$).

2. Case $\alpha \equiv {\sim}p$ where $p$ is a propositional variable: $v^*({\sim}p) = t$ iff $v(p^n) = t$ (by the assumption) iff $v(f({\sim}p)) = t$ (by the definition of $f$).

3. Case $\alpha \equiv -p$ where $p$ is a propositional variable: Similar to the above case.

- Induction step: We show some cases.

1. Case $\alpha \equiv \beta \wedge \gamma$: $v^*(\beta \wedge \gamma) = t$ iff $v^*(\beta) = t$ and $v^*(\gamma) = t$ iff $v(f(\beta)) = t$ and $v(f(\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta) \wedge f(\gamma)) = t$ iff $v(f(\beta \wedge \gamma)) = t$ (by the definition of $f$).

2. Case $\alpha \equiv {\sim}{\sim}\beta$: $v^*({\sim}{\sim}\beta) = t$ iff $v^*(\beta) = t$ iff $v(f(\beta)) = t$ (by induction hypothesis) iff $v(f({\sim}{\sim}\beta)) = t$ (by the definition of $f$).

3. Case $\alpha \equiv {\sim}(\beta \wedge \gamma)$: $v^*({\sim}(\beta \wedge \gamma)) = t$ iff $v^*({\sim}\beta) = t$ or $v^*({\sim}\gamma) = t$ iff $v(f({\sim}\beta)) = t$ or $v(f({\sim}\gamma)) = t$ (by induction hypothesis) iff $v(f({\sim}\beta) \vee f({\sim}\gamma)) = t$ iff $v(f({\sim}(\beta \wedge \gamma))) = t$ (by the definition of $f$).

4. Case $\alpha \equiv {\sim}{-}\beta$: $v^*({\sim}{-}\beta) = t$ iff $v^*(\beta) = f$ iff $v(f(\beta)) = f$ (by induction hypothesis) iff $v(\neg f(\beta)) = t$ iff $v(f({\sim}{-}\beta)) = t$ (by the definition of $f$).

5. Case $\alpha \equiv -(\beta \wedge \gamma)$: $v^*(-(\beta \wedge \gamma)) = t$ iff $v^*(-\beta) = t$ and $v^*(-\gamma) = t$ iff $v(f(-\beta)) = t$ and $v(f(-\gamma)) = t$ (by induction hypothesis) iff $v(f(-\beta) \wedge f(-\gamma)) = t$ iff $v(f(-(\beta \wedge \gamma))) = t$ (by the definition of $f$).

6. Case $\alpha \equiv -(\beta \rightarrow \gamma)$: $v^*(-(\beta \rightarrow \gamma)) = t$ iff $v^*(\beta) = f$ or $v^*(-\gamma) = t$ iff $v(f(\beta)) = f$ or $v(f(-\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta) \rightarrow f(-\gamma)) = t$ iff $v(f(-(\beta \rightarrow \gamma))) = t$ (by the definition of $f$). ∎

**Lemma 4.5** *Let $f$ be the mapping defined in Definition 3.1. For any valuation $v$, we can construct a paraconsistent valuation $v^*$ such that for any formula $\alpha$,*

$v(f(\alpha)) = t$ *iff* $v^*(\alpha) = t$.

**Proof.** Similar to the proof of Lemma 4.4. ∎

**Theorem 4.6 (Semantical embedding from PL into LK)** *Let $f$ be the mapping defined in Definition 3.1. For any formula $\alpha$,*

*$\alpha$ is PL-valid iff $f(\alpha)$ is LK-valid.*

**Proof.** By Lemmas 4.4 and 4.5. ∎

**Theorem 4.7 (Completeness for PL)** *For any formula $\alpha$,*

$\mathrm{PL} \vdash \Rightarrow \alpha$ *iff* $\alpha$ *is PL-valid.*

**Proof.** We have:

$\mathrm{PL} \vdash \Rightarrow \alpha$

iff $\mathrm{LK} \vdash \Rightarrow f(\alpha)$ (by Theorem 3.6)

iff $f(\alpha)$ is LK-valid (by the completeness theorem for LK)

iff $\alpha$ is PL-valid (by Theorem 4.6). ∎

Next, we show a theorem for semantically embedding LK into PL.

**Lemma 4.8** *Let $g$ be the mapping defined in Definition 3.10. For any valuation $v$, we can construct a paraconsistent valuation $v^*$ such that for any formula $\alpha$,*

$v(\alpha) = t$ *iff* $v^*(g(\alpha)) = t$.

**Proof.** Let $\Phi$ be a set of propositional variables, and for each $\star \in \{n, c\}$, let $\Phi^\star$ be the set $\{p^\star \mid p \in \Phi\}$ of propositional variables. Suppose that $v^*$ is a paraconsistent valuation. Suppose that $v$ is a mapping from $\Phi \cup \Phi^n \cup \Phi^c$ to $\{t, f\}$ such that

1. $v^*(p) = t$ iff $v(p) = t$,
2. $v^*({\sim}p) = t$ iff $v(p^n) = t$,
3. $v^*(-p) = t$ iff $v(p^c) = t$.

Then, the lemma is proved by induction on $\alpha$.

- Base step:

1. Case $\alpha \equiv p$ where $p$ is a propositional variable: $v(p) = t$ iff $v^*(p) = t$ (by the assumption) iff $v^*(g(p)) = t$ (by the definition of $g$).
2. Case $\alpha \equiv p^n$ where $p$ is a propositional variable: $v(p^n) = t$ iff $v^*({\sim}p) = t$ (by the assumption) iff $v^*(g(p^n)) = t$ (by the definition of $g$).
3. Case $\alpha \equiv p^c$ where $p$ is a propositional variable: Similar to the above case.

- Induction step: We show some cases.

1. Case $\alpha \equiv \beta \wedge \gamma$: $v(\beta \wedge \gamma) = t$ iff $v(\beta) = t$ and $v(\gamma) = t$ iff $v^*(g(\beta)) = t$ and $v^*(g(\gamma)) = t$ (by induction hypothesis) iff $v^*(g(\beta) \wedge g(\gamma)) = t$ iff $v^*(g(\beta \wedge \gamma))$ $= t$ (by the definition of $g$).

2. Case $\alpha \equiv \neg\beta$: $v(\neg\beta) = t$ iff $v(\beta) = f$ iff $v^*(g(\beta)) = f$ (by induction hypothesis) iff $v^*({\sim}{-}g(\beta)) = t$ iff $v^*(g(\neg\beta)) = t$ (by the definition of $g$). ∎

**Lemma 4.9** *Let $g$ be the mapping defined in Definition 3.10. For any paraconsistent valuation $v^*$, we can construct a valuation $v$ such that for any formula $\alpha$,*

$$v^*(g(\alpha)) = t \text{ iff } v(\alpha) = t.$$

**Proof.** Similar to the proof of Lemma 4.8. ∎

**Theorem 4.10 (Semantical embedding from LK into PL)** *Let $g$ be the mapping defined in Definition 3.10. For any formula $\alpha$,*

*$\alpha$ is LK-valid iff $g(\alpha)$ is PL-valid.*

**Proof.** By Lemmas 4.8 and 4.9. ∎

For EPL, we also provide a valuation semantics. Completeness, however, is proven without going through the embedding theorems.

**Definition 4.11 (Semantics for EPL)** *Let $\Phi$ be the set of all propositional variables, and $\Phi^{\sim}$ be the set $\{{\sim}p \mid p \in \Phi\}$. A* paraconsistent EPL valuation *$v^*$ is a mapping from $\Phi \cup \Phi^{\sim}$ to the set $\{t, f\}$ of truth values. The paraconsistent EPL valuation $v^*$ is extended to the mapping from the set of all formulas to $\{t, f\}$ by the clauses 1–8 of Definition 4.1, with the addition of the following clause:*

$$v^*(-\alpha) = t \text{ iff } v^*({\sim}\alpha) = f.$$

*A formula $\alpha$ is called* EPL-valid *iff $v^*(\alpha) = t$ for every paraconsistent EPL valuation $v^*$.*

We then obtain the completeness theorem for ELP.

**Theorem 4.12 (Completeness for EPL)** *For any formula $\alpha$,*

EPL $\vdash \Rightarrow \alpha$ *iff $\alpha$ is EPL-valid.*

**Proof.** By Theorem 2.15, it suffices to prove completeness for EPL$^-$. Since EPL$^-$ extends PL, $\alpha \Rightarrow \alpha$ is provable in it for every formula $\alpha$ (see Proposition 2.4). Since it also includes (cut), (we-left) and (we-right), and sequents are taken to be pairs of finite sets of formulas, this system can be regarded as a *basic system*, in the terms of [19] (that are a generalization of the *structurally standard systems of sequents* [8]). [19] provides a method to obtain a semantics for such systems, that is based on a "semantic reading" of the derivation rules. For the case of EPL$^-$, it is easy to verify that this semantics is equivalent to the one defined in Definition 4.11. ∎

# 5 Concluding remarks

In this study, we have introduced PL, a Gentzen-type sequent calculus, as a modified extension of Arieli, Avron, and Zamansky's ideal four-valued paradefinite logic 4CC. Here, PL is formalized based on the idea of connexive logic. Indeed, some conflation and implication combined logical inference rules that correspond to characteristic axiom schemes for connexive logics are used in PL. Given this, we have proved the theorems for syntactically and semantically embedding PL into LK and vice versa. Next, we have obtained the cut-elimination and completeness theorems for PL via these embedding theorems. Using the cut-elimination theorem for PL, we have also obtained the characteristic properties of quasi-paraconsistency and quasi-paracompleteness that represent the interaction between conflation and paraconsistent negation. We have also introduced an extension of PL called EPL, which is theorem-equivalent to the Gentzen-type sequent calculus $\mathrm{G}_{4\mathrm{CC}}$ for 4CC. Our calculus EPL enjoys the novel characteristic property of negative symmetry that represents the symmetry between conflation and paraconsistent negation although EPL does not enjoy the quasi-paraconsistency and quasi-paracompleteness properties. We have then directly proved the cut-elimination and completeness theorems for EPL without using an embedding theorem. In conclusion, by investigating and comparing the properties of PL and EPL, we have shed light on the role of conflation in paradefinite logics.

We now describe ongoing and future work. Using the same embedding-based method proposed and used in [15, 18], we have been able to obtain a modified Craig interpolation theorem for PL, which is shown below. However, we have not yet obtained such a result for EPL, because we have not yet obtained several theorems for embedding EPL into LK.

An expression $V(\alpha)$ denotes the set of all propositional variables occurring in $\alpha$.

**Theorem 5.1 (Modified Craig interpolation for PL)** *Suppose* $\mathrm{PL} \vdash \alpha \Rightarrow \beta$ *for any formulas* $\alpha$ *and* $\beta$*. If* $V(\alpha) \cap V(\beta) \neq \emptyset$*, then there exists a formula* $\gamma$ *such that:*

1. $\mathrm{PL} \vdash \alpha \Rightarrow \gamma$ *and* $\mathrm{PL} \vdash \gamma \Rightarrow \beta$*;*
2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

*If* $V(\alpha) \cap V(\beta) = \emptyset$*, then:*

3. $\mathrm{PL} \vdash\ \Rightarrow {\sim}{-}\alpha$ *or* $\mathrm{PL} \vdash\ \Rightarrow \beta$.

Using this theorem, we can derive the following Maksimova principle of variable separation.

**Theorem 5.2 (Maksimova principle for PL)** *Suppose* $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ *for any formulas* $\alpha_1, \alpha_2, \beta_1$ *and* $\beta_2$*. If* $\mathrm{PL} \vdash \alpha_1 \wedge \beta_1 \Rightarrow \alpha_2 \vee \beta_2$*, then either* $\mathrm{PL} \vdash \alpha_1 \Rightarrow \alpha_2$ *or* $\mathrm{PL} \vdash \beta_1 \Rightarrow \beta_2$.

We are also able to introduce a first-order extension of PL here called FPL as well as its valuation semantics in a natural way. Thus, we can show several theorems for syntactically and semantically embedding FPL into FLK, which is a Gentzen-type sequent calculus for first-order classical logic. Using these embedding theorems, we can obtain the cut-elimination, completeness, and modified Craig interpolation theorems for FPL. Part of our future work will therefore be focused on obtaining results for a first-order extension of EPL.

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# References

[1] A. Almukdad and D. Nelson, Constructible falsity and inexact predicates, Journal of Symbolic Logic 49 (1), pp. 231-233, 1984.

[2] R. Angell, A propositional logics with subjunctive conditionals, Journal of Symbolic Logic 27, pp. 327-343, 1962.

[3] O. Arieli and A. Avron, Reasoning with logical bilattices, Journal of Logic, Language and Information 5, pp. 25-63, 1996.

[4] O. Arieli and A. Avron, The value of the four values, Artificial Intelligence 102 (1), pp. 97-141, 1998.

[5] O. Arieli and A. Avron, Minimal paradefinite logics for reasoning with incompleteness and inconsistency, Proceedings of the 1st International Conference on Formal Structures for Computation and Deduction (FSCD), Leibniz International Proceedings in Informatics (LIPIcs) 52, pp. 7:1-7:15, 2016.

[6] O. Arieli and A. Avron, Four-valued paradefinitie logics, Studia Logica 105 (6), pp. 1087-1122, 2017.

[7] O. Arieli, A. Avron and A. Zamansky, Ideal paraconsistent logics, Studia Logica 99 (1-3), pp. 31-60, 2011.

[8] J.-Y. Béziau, Sequents and bivaluations, Logique et Analyse 44 (176), pp. 373-394, 2001

[9] J.-Y. Béziau, Bivalent semantics for De Morgan logic (The uselessness of four-valuedness), In W.A. Carnieli, M.E. Coniglio and I.M. D'Ottaviano, editors, The many sides of logic, pp. 391-402, College Publications, 2009.

[10] W. Carnielli, M.E. Coniglio and J. Marcos, Logics of formal inconsistency, D.M. Gabbay and F. Guenthner (eds.), Handbook of Philosophical Logic, 2nd Edition, Volume 14, pp. 1-93, Springer, 2007.

[11] M. Fitting, Bilattices and the semantics of logic programming, The Journal of Logic Programming 11 (2), pp. 91-116, 1991.

[12] G. Gentzen, Collected papers of Gerhard Gentzen, M.E. Szabo, ed., Studies in logic and the foundations of mathematics, North-Holland (English translation), 1969.

[13] Y. Gurevich, Intuitionistic logic with strong negation, Studia Logica 36, pp. 49-59, 1977.

[14] N. Kamide, A hierarchy of weak double negations, Studia Logica 101 (6), pp. 1277-1297, 2013.

[15] N. Kamide, Trilattice logic: an embedding-based approach, Journal of Logic and Computation 25 (3), pp. 581-611, 2015.

[16] N. Kamide, Extending ideal paraconsistent four-valued logic, Proceedings of the 47th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2017), pp. 49-54, 2017.

[17] N. Kamide, Paraconsistent double negations as classical and intuitionistic negations, Studia Logica 105 (6), pp. 1167-1191, 2017.

[18] N. Kamide and Y. Shramko, Embedding from multilattice logic into classical logic and vice versa, Journal of Logic and Computation 27 (5), pp. 1549-1575, 2017.

[19] O. Lahav and A. Avron, A unified semantic framework for fully structural propositional sequent systems, ACM Transaction on Computational Logic 14 (4), pp. 27:1-27:33, 2013.

[20] S. McCall, Connexive implication, Journal of Symbolic Logic 31, pp. 415-433, 1966.

[21] D. Nelson, Constructible falsity, Journal of Symbolic Logic 14, pp. 16-26, 1949.

[22] G. Priest, Paraconsistent logic, Handbook of Philosophical Logic (Second Edition), Vol. 6, D. Gabbay and F. Guenthner (eds.), Kluwer Academic Publishers, Dordrecht, pp. 287-393, 2002.

[23] W. Rautenberg, Klassische und nicht-klassische Aussagenlogik, Vieweg, Braunschweig, 1979.

[24] Y. Shramko, Truth, falsehood, information and beyond: the American plan generalized, In: K. Bimbo (ed.), J. Michael Dunn on Information Based Logics, Outstanding Contributions to Logic 8, pp. 191-212, Springer, 2016.

[25] G. Takeuti, Proof theory (second edition), Dover Publications, Inc. Mineola, New York, 2013.

[26] N.N. Vorob'ev, A constructive propositional calculus with strong negation (in Russian), Doklady Akademii Nauk SSSR 85, pp. 465-468, 1952.

[27] H. Wansing, Connexive modal logic, Advances in Modal Logic 5, pp. 367-385, 2005.

[28] H. Wansing, Constructive negation, implication, and co-implication, Journal of Applied Non-Classical Logics 18 (2-3), pp. 341-364, 2008.

[29] H. Wansing, Connexive logic, Stanford Encyclopedia of Philosophy, 2014: http://plato.stanford.edu/entries/logic-connexive/