

The nonexistence of unicorns and many-sorted Löwenheim–Skolem theorems

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Abstract. Stable infiniteness, strong finite witnessability, and smoothness are model-theoretic properties relevant to theory combination in satisfiability modulo theories. Theories that are strongly finitely witnessable and smooth are called *strongly polite* and can be effectively combined with other theories. Toledo, Zohar, and Barrett conjectured that stably infinite and strongly finitely witnessable theories are smooth and therefore strongly polite. They called counterexamples to this conjecture *unicorn theories*, as their existence seemed unlikely. We prove that, indeed, unicorns do not exist. We also prove versions of the Löwenheim–Skolem theorem and the Łoś–Vaught test for many-sorted logic.

1 Introduction

Given decision procedures for theories \mathcal{T}_1 and \mathcal{T}_2 with disjoint signatures, is there a decision procedure for $\mathcal{T}_1 \cup \mathcal{T}_2$? In general, the answer is “not necessarily”, but a central question in Satisfiability Modulo Theories (SMT) [3] is: what assumptions on \mathcal{T}_1 and \mathcal{T}_2 suffice for theory combination? This line of research began with Nelson and Oppen’s theory combination procedure [16], which applies when \mathcal{T}_1 and \mathcal{T}_2 are stably infinite, roughly meaning that every \mathcal{T}_i -satisfiable quantifier-free formula is satisfied by an infinite \mathcal{T}_i -interpretation for $i \in \{1, 2\}$.

The Nelson–Oppen procedure is quite useful, but requires *both* theories to be stably infinite, which is not always the case (e.g., the theories of bit-vectors and finite datatypes are not stably infinite). Thus, sufficient properties of only one of the theories were identified, such as gentleness [8], shininess [20], and flexibility [10]. The most relevant property for our purposes is strong politeness [18,9,5,19]. It is essential to the functioning of the SMT solver cvc5 [1], which is called billions of times per day in industrial production code. A theory is *strongly polite* if it is smooth and strongly finitely witnessable, which are model-theoretic properties we will define later. These properties are more involved than stable infiniteness, so proving a theory to be strongly polite is more difficult. But the advantage of strongly polite theories is that they can be combined with any other decidable theory, including theories that are not stably infinite.

Given the abundance of model-theoretic properties relevant to theory combination, some of which interact in subtle ways, it behooves us to understand the logical relations between them. Recent papers [21,23] have sought to understand the relations between seven model-theoretic properties—including stable infiniteness, smoothness, and strong finite witnessability—by determining which combinations of properties are possible in various signatures. In most cases, a theory with the desired combination of properties was constructed, or it was proved that none exists. The sole exception was theories that are stably infinite and strongly finitely witnessable but not smooth, dubbed *unicorn theories* and conjectured not to exist. Our main result, Theorem 2, confirms this conjecture.

Besides completing the taxonomy of properties from [21,23], our result has practical consequences. The nonexistence of unicorns implies that strongly polite theories can be equivalently defined as those that are stably infinite and strongly finitely witnessable. Since it is easier to prove that a theory is stably infinite than to prove that it is smooth, this streamlines the process of proving that a theory is strongly polite. Thus, each time a new theory is introduced, proving that it can be combined with other theories becomes easier.³ Similarly, our results give a new characterization of shiny theories, which makes it easier to prove that a theory is amenable to the shiny combination procedure (see Corollary 2).

We also believe that our result is of theoretical interest. Theorem 3, which is the main ingredient in the proof of Theorem 2, can be seen as a variant of the upward Löwenheim–Skolem theorem for many-sorted logic, since proving that a theory is smooth amounts to proving that cardinalities of sorts can be increased arbitrarily, including to uncountable cardinals. This result may be of independent interest to logicians studying the model theory of many-sorted logic, and we hope the proof techniques are useful to them as well.

Speaking of proof techniques, our proof is curious in that it uses Ramsey’s theorem from finite combinatorics. This is not the first time Ramsey’s theorem has been used in logic. Ramsey proved his theorem in the course of solving a special case of the decision problem for first-order logic [17]. Ramsey’s theorem also shows up in the Ehrenfeucht–Mostowski construction in model theory [6]. Our proof actually requires a generalization of Ramsey’s theorem, which we prove using the standard version of Ramsey’s theorem.

A major component of the proof of Theorem 2 amounts to proving a many-sorted version of the Löwenheim–Skolem theorem. On the course to proving this, we realized that a proper understanding of this theorem for many-sorted logic appears to be missing from the literature, despite the fact that the SMT-LIB standard [2] is based on many-sorted logic. To fill this gap, we prove generalizations of the Löwenheim–Skolem theorem for many-sorted logic, and use them to prove a many-sorted Łoś–Vaught test, useful for proving theory completeness.

The remainder of this paper is structured as follows. Section 2 provides background and definitions on many-sorted logic and SMT. Section 3 proves the

³ [21] already proved that stably infinite and strongly finitely witnessable theories can be combined with other theories. Our result gives a new proof (see Corollary 1), and shows that their procedure is not more general than polite combination.

main result of this paper, namely the nonexistence of unicorn theories. Section 4 proves new many-sorted variants of the Löwenheim–Skolem theorem. Section 5 concludes and presents directions for future work.⁴

2 Preliminaries

2.1 Many-sorted first-order logic

We work in many-sorted first-order logic [15]. A *signature* Σ consists of a non-empty set \mathcal{S}_Σ of sorts, a set \mathcal{F}_Σ of function symbols, and a set \mathcal{P}_Σ of predicate symbols containing an equality symbol $=_\sigma$ for every sort $\sigma \in \mathcal{S}_\Sigma$.⁵ Every function symbol has an arity $(\sigma_1, \dots, \sigma_n, \sigma)$ and every predicate symbol an arity $(\sigma_1, \dots, \sigma_n)$, where $\sigma_1, \dots, \sigma_n, \sigma \in \mathcal{S}_\Sigma$ and $n \geq 0$. Every equality symbol $=_\sigma$ has arity (σ, σ) . To quantify a variable x of sort σ , we write $\forall x : \sigma.$ and $\exists x : \sigma.$ for the universal and existential quantifiers respectively. Let $|\Sigma| = |\mathcal{S}_\Sigma| + |\mathcal{F}_\Sigma| + |\mathcal{P}_\Sigma|$. If a signature contains only sorts and equalities, we say it is *empty*. Two signatures are said to be *disjoint* if they share at most sorts and equality symbols.

We define Σ -terms and Σ -formulas as usual. The set of free variables of sort σ in φ is denoted $\text{vars}_\sigma(\varphi)$. For $S \subseteq \mathcal{S}_\Sigma$, let $\text{vars}_S(\varphi) = \bigcup_{\sigma \in S} \text{vars}_\sigma(\varphi)$. We also let $\text{vars}(\varphi) = \text{vars}_{\mathcal{S}_\Sigma}(\varphi)$. A Σ -sentence is a Σ -formula with no free variables.

A Σ -*structure* \mathbb{A} interprets each sort $\sigma \in \mathcal{S}_\Sigma$ as a nonempty set $\sigma^\mathbb{A}$, each function symbol $f \in \mathcal{F}_\Sigma$ as a function $f^\mathbb{A}$ with the appropriate domain and codomain, and each predicate symbol $P \in \mathcal{P}_\Sigma$ as a relation $P^\mathbb{A}$ over the appropriate set, such that $=_\sigma^\mathbb{A}$ is the identity on $\sigma^\mathbb{A}$. A Σ -*interpretation* \mathcal{A} is a pair (\mathbb{A}, ν) , where \mathbb{A} is a Σ -structure and ν is a function, called an *assignment*, mapping each variable x of sort σ to an element $\nu(x) \in \sigma^\mathbb{A}$, denoted $x^\mathcal{A}$. We write $t^\mathcal{A}$ for the interpretation of the Σ -term t under \mathcal{A} , which is defined in the usual way. The entailment relation, denoted \models , is defined as usual.

Two structures are *elementarily equivalent* if they satisfy the same sentences. We say that \mathbb{A} is an *elementary substructure* of \mathbb{B} if \mathbb{A} is a substructure of \mathbb{B} and, for all formulas φ and all assignments ν on \mathbb{A} , we have $(\mathbb{A}, \nu) \models \varphi$ if and only if $(\mathbb{B}, \nu) \models \varphi$. Note that if \mathbb{A} is an elementary substructure of \mathbb{B} , then they are elementarily equivalent. \mathcal{A} is an elementary subinterpretation of \mathcal{B} if \mathbb{A} is an elementary substructure of \mathbb{B} and \mathcal{A} 's assignment is the same as \mathcal{B} 's assignment.

Given a Σ -structure \mathbb{A} , let $\mathcal{S}_{\geq \aleph_0}^\mathbb{A} = \{\sigma \in \mathcal{S}_\Sigma : |\sigma^\mathbb{A}| \geq \aleph_0\}$ and $\mathcal{S}_{< \aleph_0}^\mathbb{A} = \mathcal{S}_\Sigma \setminus \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$. We similarly define $\mathcal{S}_{\geq \aleph_0}^\mathcal{A}$ and $\mathcal{S}_{< \aleph_0}^\mathcal{A}$ for a Σ -interpretation \mathcal{A} .

A Σ -*theory* \mathcal{T} is a set of Σ -sentences, called the *axioms* of \mathcal{T} . We write $\vdash_{\mathcal{T}} \varphi$ instead of $\mathcal{T} \models \varphi$. Structures satisfying \mathcal{T} are called \mathcal{T} -*models*, and interpretations satisfying \mathcal{T} are called \mathcal{T} -*interpretations*. We say a Σ -formula is \mathcal{T} -*satisfiable* if it is satisfied by some \mathcal{T} -interpretation, and we say two Σ -formulas are \mathcal{T} -*equivalent* if every \mathcal{T} -interpretation satisfies one if and only if it satisfies the other. \mathcal{T} is *complete* if for every sentence φ , we have $\vdash_{\mathcal{T}} \varphi$ or $\vdash_{\mathcal{T}} \neg\varphi$. \mathcal{T} is *consistent* if there

⁴ Some proofs are omitted. They can be found in the appendix.

⁵ When specifying a signature, we often omit the equality symbols, and include them implicitly. We also omit σ from $=_\sigma$ when it does not cause confusion.

is no formula φ such that $\vdash_{\mathcal{T}} \varphi$ and $\vdash_{\mathcal{T}} \neg\varphi$. If Σ_1 and Σ_2 are disjoint, let $\Sigma_1 \cup \Sigma_2$ be the signature with the union of their sorts, function symbols, and predicate symbols. Given a Σ_1 -theory \mathcal{T}_1 and a Σ_2 -theory \mathcal{T}_2 , the $(\Sigma_1 \cup \Sigma_2)$ -theory $\mathcal{T}_1 \cup \mathcal{T}_2$ is the theory whose axioms are the union of the axioms of \mathcal{T}_1 and \mathcal{T}_2 .

The following theorem, proved in [15], is a many-sorted variant of the first-order compactness theorem.

Theorem 1 (Compactness Theorem [15]). *A set of Σ -formulas Γ is satisfiable if and only if every finite subset of Γ is satisfiable.*

We say that a Σ -theory \mathcal{T} has *built-in Skolem functions* if for all formulas $\psi(\vec{x}, y)$, there is $f \in \mathcal{F}_{\Sigma}$ such that $\vdash_{\mathcal{T}} \forall \vec{x}. (\exists y. (\psi(\vec{x}, y)) \rightarrow \psi(\vec{x}, f(\vec{x})))$.⁶ The following is a many-sorted variant of Lemma 2.3.6 of [13]. The proof is almost identical to that of the single-sorted case from [13].

Lemma 1. *If \mathcal{T} is a Σ -theory for a countable Σ , then there is a countable signature $\Sigma^* \supseteq \Sigma$ and Σ^* -theory $\mathcal{T}^* \supseteq \mathcal{T}$ with built-in Skolem functions.*

We state a many-sorted generalization of the Tarski–Vaught test, whose proof is also similar to the single-sorted case [13, Proposition 2.3.5].

Lemma 2 (The Tarski–Vaught test). *Suppose \mathbb{A} is a substructure of \mathbb{B} . Then, \mathbb{A} is an elementary substructure of \mathbb{B} if and only if $(\mathbb{B}, \nu) \models \exists v. \varphi(\vec{x}, v)$ implies $(\mathbb{A}, \nu) \models \exists v. \varphi(\vec{x}, v)$ for every formula $\varphi(\vec{x}, v)$ and assignment ν over \mathbb{A} .*

2.2 Model-theoretic properties

Definition 1. *Let Σ be a many-sorted signature, $S \subseteq \mathcal{S}_{\Sigma}$, and \mathcal{T} a Σ -theory.*

- \mathcal{T} is *stably infinite with respect to S* if for every \mathcal{T} -satisfiable quantifier-free formula φ , there is a \mathcal{T} -interpretation \mathcal{A} satisfying φ with $|\sigma^{\mathcal{A}}| \geq \aleph_0$ for every $\sigma \in S$.
- \mathcal{T} is *stably finite with respect to S* if for every quantifier-free Σ -formula φ and \mathcal{T} -interpretation \mathcal{A} satisfying φ , there is a \mathcal{T} -interpretation \mathcal{B} satisfying φ such that $|\sigma^{\mathcal{B}}| \leq |\sigma^{\mathcal{A}}|$ and $|\sigma^{\mathcal{B}}| < \aleph_0$ for every $\sigma \in S$.
- \mathcal{T} is *smooth with respect to S* if for every quantifier-free formula φ , \mathcal{T} -interpretation \mathcal{A} satisfying φ , and function κ from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$ for every $\sigma \in S$, there is a \mathcal{T} -interpretation \mathcal{B} satisfying φ with $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$ for every $\sigma \in S$.

Next, we define *arrangements*. Given a set of sorts $S \subseteq \mathcal{S}_{\Sigma}$, finite sets of variables V_{σ} of sort σ for each $\sigma \in S$, and equivalence relations E_{σ} on V_{σ} , the *arrangement* δ_V on $V = \bigcup_{\sigma \in S} V_{\sigma}$ induced by $E = \bigcup_{\sigma \in S} E_{\sigma}$ is

$$\bigwedge_{\sigma \in S} \left[\bigwedge_{x E_{\sigma} y} (x = y) \wedge \bigwedge_{x \overline{E}_{\sigma} y} \neg(x = y) \right],$$

where \overline{E}_{σ} is the complement of E_{σ} .

⁶ Intuitively: \mathcal{T} has enough function symbols to witness all existential formulas.

Definition 2. Let Σ be a many-sorted signature, $S \subseteq \mathcal{S}_\Sigma$ a finite set, and \mathcal{T} a Σ -theory. Then \mathcal{T} is strongly finitely witnessable with respect to S if there is a computable function wit from the quantifier-free formulas into themselves such that for every quantifier-free formula φ :

- (i) φ and $\exists \vec{w}. \text{wit}(\varphi)$ are \mathcal{T} -equivalent, where $\vec{w} = \text{vars}(\text{wit}(\varphi)) \setminus \text{vars}(\varphi)$; and
- (ii) given a finite set of variables V and an arrangement δ_V on V , if $\text{wit}(\varphi) \wedge \delta_V$ is \mathcal{T} -satisfiable, then there is a \mathcal{T} -interpretation \mathcal{A} satisfying $\text{wit}(\varphi) \wedge \delta_V$ such that $\sigma^{\mathcal{A}} = \text{vars}_\sigma(\text{wit}(\varphi) \wedge \delta_V)^{\mathcal{A}}$ for every $\sigma \in S$.

2.3 Notation

\mathbb{N} denotes the set of non-negative integers. Given $m, n \in \mathbb{N}$, let $[m, n] := \{\ell \in \mathbb{N} : m \leq \ell \leq n\}$ and $[n] := [1, n]$. Given a set X , let $P_n(X) := \{Y \subseteq X : |Y| = n\}$, $X^n := \{(x_1, \dots, x_n) : x_i \in X \text{ for all } i \in [n]\}$, and $X^* := \bigcup_{n \in \mathbb{N}} X^n$. For any x , we denote (x, \dots, x) by $(x)^{\oplus n}$. Given a tuple of tuples $(\vec{x}_1, \dots, \vec{x}_n)$, where $\vec{x}_i \in X^*$ for all i , we will often treat it as an element of X^* by flattening the tuple.

3 The nonexistence of unicorns

We now state our main theorem, which implies that unicorn theories do not exist. Note that since we are motivated by applications to SMT, we hereafter assume all signatures are countable.⁷

Theorem 2. Assume that \mathcal{T} is a Σ -theory, where Σ is countable. If \mathcal{T} is stably infinite and strongly finitely witnessable, both with respect to $S \subseteq \mathcal{S}_\Sigma$, then \mathcal{T} is smooth with respect to S .

For our proof, we define a weaker variant of smoothness, that focuses the requirements only for finite cardinals.

Definition 3. A Σ -theory \mathcal{T} is finitely smooth with respect to $S \subseteq \mathcal{S}_\Sigma$ if for every quantifier-free formula φ , \mathcal{T} -interpretation \mathcal{A} with $\mathcal{A} \models \varphi$, and function κ from $\mathcal{S}_{<\aleph_0}^{\mathcal{A}} \cap S$ to the class of cardinals with $|\sigma^{\mathcal{A}}| \leq \kappa(\sigma) < \aleph_0$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^{\mathcal{A}} \cap S$, there is a \mathcal{T} -interpretation \mathcal{B} with $\mathcal{B} \models \varphi$ with $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^{\mathcal{A}} \cap S$.

We make use of the following two lemmas.

Lemma 3. If \mathcal{T} is stably infinite and strongly finitely witnessable, both with respect to some set of sorts $S \subseteq \mathcal{S}_\Sigma$, then \mathcal{T} is finitely smooth with respect to S .

Lemma 4 ([23, Theorem 3]). If \mathcal{T} is strongly finitely witnessable with respect to some set of sorts $S \subseteq \mathcal{S}_\Sigma$, then \mathcal{T} is stably finite with respect to S .

In light of the above two lemmas, the following theorem implies Theorem 2.

⁷ The paper that introduced unicorn theories [21] also made this assumption.

Theorem 3. *Assume that \mathcal{T} is a Σ -theory, where Σ is countable. If \mathcal{T} is stably finite and finitely smooth, both with respect to some set of sorts $S \subseteq \mathcal{S}_\Sigma$, then \mathcal{T} is smooth with respect to S .*

The remainder of this section is thus dedicated to the proof of Theorem 3.

3.1 Motivating the proof

In this section, we illustrate the proof technique with a simple example. The goal is to motivate the proof of Theorem 3 before delving into the details.

Suppose \mathcal{T} is a Σ -theory, where $\mathcal{S}_\Sigma = \{\sigma_1, \sigma_2\}$, $\mathcal{F}_\Sigma = \{f\}$, f has arity (σ_2, σ_1) , and the only predicate symbols are equalities. Suppose that \mathcal{T} is also stably finite and finitely smooth, both with respect to $S = \mathcal{S}_\Sigma$. Let φ be a \mathcal{T} -satisfiable quantifier-free formula and \mathcal{A} a \mathcal{T} -interpretation satisfying φ . Let κ be a function from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^\mathcal{A}|$ for both $\sigma \in S$. For concreteness, suppose $|\sigma_1^\mathcal{A}| = |\sigma_2^\mathcal{A}| = 10$, $\kappa(\sigma_1) = \aleph_0$, and $\kappa(\sigma_2) = \aleph_1$. Our goal is to show that there is a \mathcal{T} -interpretation \mathcal{B}^- satisfying φ with $|\sigma_1^{\mathcal{B}^-}| = \aleph_0$ and $|\sigma_2^{\mathcal{B}^-}| = \aleph_1$.⁸

A natural thought is to apply some variant of the upward Löwenheim–Skolem theorem, but this doesn't quite work. As will be seen in Section 4, generalizations of the Löwenheim–Skolem theorem to many-sorted logic do not let us control the cardinalities of σ_1 and σ_2 independently. Nevertheless, let us emulate the standard proof technique for the upward Löwenheim–Skolem theorem.

Here is the most natural way of generalizing the proof of the upward Löwenheim–Skolem theorem to our setting. For simplicity, assume that \mathcal{T} already has built-in Skolem functions. We introduce \aleph_0 new constants $\{c_{1,\alpha}\}_{\alpha < \omega}$ and \aleph_1 new constants $\{c_{2,\alpha}\}_{\alpha < \omega_1}$. We define a set of formulas $\Gamma = \{\varphi\} \cup \Gamma_1$, where

$$\Gamma_1 = \{\neg(c_{i,\alpha} = c_{i,\beta}) : i \in \{1, 2\}; \alpha, \beta < \kappa(\sigma_i); \alpha \neq \beta\}.$$

By Theorem 1 and finite smoothness, there is a \mathcal{T} -interpretation \mathcal{B} satisfying Γ : indeed, were that not true, Theorem 1 would guarantee that some finite subset of Γ is unsatisfiable; yet such a set would only demand the existence of finitely many new elements, which can be achieved by making use of finite smoothness. Since $\mathcal{B} \models \Gamma_1$, we have $|\sigma_1^\mathcal{B}| \geq \aleph_0$ and $|\sigma_2^\mathcal{B}| \geq \aleph_1$.

Since \mathcal{B} may be too large, we construct a subinterpretation \mathcal{B}^- with

$$\begin{aligned} \sigma_1^{\mathcal{B}^-} &= \{c_{1,\alpha}\}_{\alpha < \omega} \cup \{f^\mathcal{B}(c_{2,\alpha})\}_{\alpha < \omega_1} \\ \sigma_2^{\mathcal{B}^-} &= \{c_{2,\alpha}\}_{\alpha < \omega_1}. \end{aligned}$$

And using the assumption that \mathcal{T} has built-in Skolem functions, we can prove that \mathcal{B}^- is an elementary subinterpretation of \mathcal{B} , so $\mathcal{B}^- \models \Gamma$; we can then prove that $|\sigma_2^{\mathcal{B}^-}| = \aleph_1$, but we unfortunately cannot guarantee that $|\sigma_1^{\mathcal{B}^-}| = \aleph_0$. This is because \mathcal{B}^- has not only the \aleph_1 elements $\{c_{2,\alpha}\}_{\alpha < \omega_1}$ of sort σ_2 , but also

⁸ The reason for the $-$ superscript in \mathcal{B}^- will be clear presently.

the elements $\{f^{\mathcal{B}}(c_{2,\alpha}^{\mathcal{B}})\}_{\alpha < \omega_1}$ of sort σ_1 . The function symbol f has created a “spillover” of elements from σ_2 to σ_1 .

To fix this, we need to ensure that $|\{f^{\mathcal{B}}(c_{2,\alpha}^{\mathcal{B}})\}_{\alpha < \omega_1}| \leq \aleph_0$. To that end, define Γ to instead be $\{\varphi\} \cup \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_2 = \{f(b) = f(d) : b, d \in \{c_{2,\alpha}\}_{\alpha < \omega_1}\}.$$

Then, if there is a model \mathcal{B} satisfying Γ , we have $|\{f^{\mathcal{B}}(c_{2,\alpha}^{\mathcal{B}})\}_{\alpha < \omega_1}| = 1 \leq \aleph_0$. To show Γ is \mathcal{T} -satisfiable, it suffices by the compactness theorem to show that $\mathcal{T} \cup \Gamma'$ is satisfiable for every finite subset $\Gamma' \subseteq \Gamma$. So let $\Gamma'_1 \subseteq \Gamma_1$ and $\Gamma'_2 \subseteq \Gamma_2$ be finite subsets. We will construct a \mathcal{T} -interpretation \mathcal{B}' such that $\mathcal{B}' \models \{\varphi\} \cup \Gamma'_1 \cup \Gamma'_2$. For concreteness, suppose that $\{c_{1,0}, c_{1,1}, \dots, c_{1,99}\}$ and $\{c_{2,0}, c_{2,1}, \dots, c_{2,9}\}$ are the new constants that appear in $\Gamma'_1 \cup \Gamma'_2$. By finite smoothness, there is a \mathcal{T} -interpretation \mathcal{B}' satisfying φ such that $|\sigma_1^{\mathcal{B}'}| = 100$ and $|\sigma_2^{\mathcal{B}'}| = 901$. By the pigeonhole principle, there is a subset $Y \subseteq \sigma_2^{\mathcal{B}'}$ with $|Y| \geq 10$ such that $f^{\mathcal{B}'}$ is constant on Y ; if 901 pigeons are put in 100 holes, then some hole has at least 10 pigeons (although this is not true for 900 pigeons). Then, \mathcal{B}' can interpret the constants $\{c_{1,0}, c_{1,1}, \dots, c_{1,99}\}$ as distinct elements of $\sigma_1^{\mathcal{B}'}$ and the constants $\{c_{2,0}, c_{2,1}, \dots, c_{2,9}\}$ as distinct elements of Y . This proves that Γ is \mathcal{T} -satisfiable.

We illustrate the top level structure of the proof idea in Figure 1, applied to the working example. The x axis represents cardinalities of interpretations of σ_1 , and the y axis does the same for σ_2 . Starting from the interpretation \mathcal{A} with $|\sigma_1^{\mathcal{A}}| = |\sigma_2^{\mathcal{A}}| = 10$, we construct some interpretation \mathcal{B} , represented by the array of red dots as there is some degree of uncertainty regarding the precise cardinalities of its domains, with $|\sigma_1^{\mathcal{B}}| \geq \aleph_0$ and $|\sigma_2^{\mathcal{B}}| \geq \aleph_1$. From \mathcal{B} we hope to construct \mathcal{B}^- , which has $|\sigma_1^{\mathcal{B}^-}| = \aleph_0$ and $|\sigma_2^{\mathcal{B}^-}| = \aleph_1$: the latter can be achieved using techniques similar to the many-sorted Löwenheim–Skolem theorems (see Section 4 below), while the former requires the aforementioned pigeonhole principle arguments.

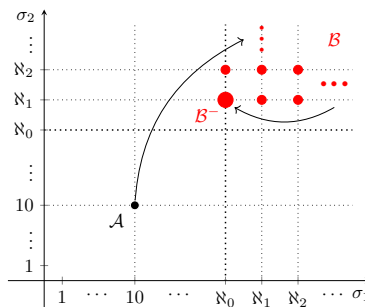


Fig. 1: How we move from interpretation to interpretation

The above proof sketch illustrates the main ideas behind the proof of Theorem 3. The generalization to more sorts and function symbols requires some extra bookkeeping. More interestingly, the generalization to functions of arity greater than one requires a version of Ramsey’s theorem, which is a generalization of the pigeonhole principle.

3.2 Ramsey’s theorem and generalizations

In this section, we state Ramsey’s theorem and a generalization of it.

Ramsey's theorem is sometimes stated in terms of coloring the edges of hypergraphs, but for our purposes it is more convenient to state it as follows. In the following lemma, the notations $P_n(X)$ and $[k]$ are defined as in Section 2.3.

Lemma 5 (Ramsey's theorem [17, Theorem B]). *For any $k, n, m \in \mathbb{N}$, there is an $R(k, n, m) \in \mathbb{N}$ such that for any set X with $|X| \geq R(k, n, m)$ and function $f : P_n(X) \rightarrow [k]$, there is a subset $Y \subseteq X$ with $|Y| \geq m$ such that f is constant on $P_n(Y)$.*

Note that in Ramsey's theorem, the set $[k]$ can be replaced by any set of cardinality k .

We want to generalize Ramsey's theorem to functions $f : X^n \rightarrow [k]$. The most natural generalization would state that there is a large subset $Y \subseteq X$ such that f is constant on Y^n . But this generalization is false, as the following example shows.

Example 1. Let $X = \mathbb{Z}$, and let $f : X^2 \rightarrow [2]$ be given by

$$f(m, n) = \begin{cases} 1 & \text{if } m < n \\ 2 & \text{otherwise.} \end{cases}$$

Then, $f(m, n) \neq f(n, m)$ for all $m, n \in X$ with $m \neq n$. Thus, there is no subset $Y \subseteq X$ with $|Y| \geq 2$ such that f is constant on Y^2 .

To avoid counterexamples like this, our generalization needs to consider the order of the arguments of f . This motivates the following definition.

Definition 4. *Let $(X, <)$ be a totally ordered set, and let $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ be elements of X^n . We write $\vec{x} \sim \vec{y}$ if for every $1 \leq i < j \leq n$ we have*

$$\begin{aligned} x_i < x_j &\iff y_i < y_j & \text{and} \\ x_i = x_j &\iff y_i = y_j. \end{aligned}$$

Observe that \sim is an equivalence relation on X^n with finitely many equivalence classes.⁹

Now we can state our first generalization of Ramsey's theorem.

Lemma 6. *For any $k, n, m \in \mathbb{N}$, there is an $R^*(k, n, m) \in \mathbb{N}$ such that for any totally ordered set $(X, <)$ with $|X| \geq R^*(k, n, m)$ and function $f : X^n \rightarrow [k]$, there is a subset $Y \subseteq X$ with $|Y| \geq m$ such that f is constant on each \sim -equivalence class of Y^n .*

Next, we further generalize Ramsey's theorem to multiple functions f_1, \dots, f_r .

Lemma 7. *For any $k, m \in \mathbb{N}$ and $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, there is a number $R^{**}(k, \vec{n}, m) \in \mathbb{N}$, such that for any totally ordered set $(X, <)$ with $|X| \geq R^{**}(k, \vec{n}, m)$ and functions $f_i : X^{n_i} \rightarrow [k]$ for $i \in [r]$, there is a subset $Y \subseteq X$ with $|Y| \geq m$, such that f_i is constant on each \sim -equivalence class of Y^{n_i} for all $i \in [r]$.*

⁹ The number of equivalence classes is given by the ordered Bell numbers (<https://oeis.org/A000670>).

3.3 The proof of Theorem 3

Fix a Σ -theory \mathcal{T} and a set of sorts $S \subseteq \mathcal{S}_\Sigma$. Assume that Σ is countable. Suppose that \mathcal{T} is stably finite and finitely smooth, both with respect to S . Let φ be a \mathcal{T} -satisfiable quantifier-free formula and \mathcal{A} a \mathcal{T} -interpretation satisfying φ . Let κ be a function from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$ for every $\sigma \in S$.

Write $S = \{\sigma_1, \sigma_2, \dots\}$ and, without loss of generality, assume $\kappa(\sigma_1) \leq \kappa(\sigma_2) \leq \dots$. For notational convenience, we write all Σ -terms in the form $t(\vec{x}_1, \vec{x}_2, \dots)$,¹⁰ where \vec{x}_i is a tuple of variables of sort σ_i . If $\kappa(\sigma_i) < \aleph_0$ for all i , then we are done by the fact \mathcal{T} is finitely smooth. Otherwise, let ℓ be the largest natural number such that $\kappa(\sigma_\ell) < \aleph_0$ if there is such a number, and let $\ell = 0$ otherwise.

The proof of Theorem 3 proceeds in two steps. First, we construct a set of formulas Γ such that $\varphi \in \Gamma$ and prove that there is a \mathcal{T} -interpretation \mathcal{B} satisfying Γ . Second, we prove that \mathcal{B} has an elementary subinterpretation \mathcal{B}^- such that $|\sigma_i^{\mathcal{B}^-}| = \kappa(\sigma_i)$ for all i . Since $\varphi \in \Gamma$, it will follow that \mathcal{T} is smooth.

The assumption that \mathcal{T} is stably finite and finitely smooth is used to construct \mathcal{T} -interpretations of the following form, which will be useful for a compactness argument.

Lemma 8. *There is a \mathcal{T} -interpretation \mathcal{B} satisfying φ such that $|\sigma_i^{\mathcal{B}}| = \kappa(\sigma_i)$ for all $i \leq \ell$, and $|\sigma_i^{\mathcal{B}}|$ is arbitrarily large but finite for all $i > \ell$.*

Proof. First, apply stable finiteness to get a \mathcal{T} -interpretation \mathcal{A}' satisfying φ such that $|\sigma_i^{\mathcal{A}'}| \leq |\sigma_i^{\mathcal{A}}|$ and $|\sigma_i^{\mathcal{A}'}| < \aleph_0$ for all i . Then, apply finite smoothness to \mathcal{A}' with κ' given by $\kappa'(\sigma_i) = \kappa(\sigma_i)$ for all $i \leq \ell$ and $\kappa'(\sigma_i)$ arbitrarily large but finite for all $i > \ell$. \square

It will be convenient to work with a theory with built-in Skolem functions, so we use Lemma 1 to get a Σ^* -theory $\mathcal{T}^* \supseteq \mathcal{T}$, where $\Sigma^* \supseteq \Sigma$ and Σ^* is countable. To construct our set of formulas Γ , we introduce $\kappa(\sigma_i)$ new constants $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$ of sort σ_i for each i . We consider these constants to be part of an even larger signature $\Sigma' \supseteq \Sigma^*$. In what follows, we construct sentences and interpretations over Σ' . Impose an arbitrary total order on each $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$ to be used for the \sim relation. For the definition below, recall that given a set X , we define $X^* = \bigcup_{n \in \mathbb{N}} X^n$.

Definition 5. *We define a set of formulas $\Gamma = \{\varphi\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where*

$$\begin{aligned} \Gamma_1 &= \{ \neg(c_{i,\alpha} = c_{i,\beta}) : 1 \leq i \leq |S|; \alpha, \beta < \kappa(\sigma_i); \alpha \neq \beta \} \\ \Gamma_2 &= \left\{ t \left(\vec{c}_1, \dots, \vec{c}_i, \vec{b}_{i+1}, \vec{b}_{i+2}, \dots \right) = t \left(\vec{c}_1, \dots, \vec{c}_i, \vec{d}_{i+1}, \vec{d}_{i+2}, \dots \right) : \right. \\ &\quad \left. t \text{ is a } \Sigma^* \text{-term of sort } \sigma_i; i > \ell; \vec{c}_k, \vec{b}_k, \vec{d}_k \in (\{c_{k,\alpha}\}_{\alpha < \kappa(\sigma_k)})^* \right. \\ &\quad \left. \text{for all } k; \vec{b}_j \sim \vec{d}_j \text{ for all } j > i \right\} \end{aligned}$$

¹⁰ Even if S is infinite, the denoted term is still finite since each term only has a finite number of variables occurring in it.

$$\Gamma_3 = \left\{ \forall x : \sigma_i. \bigvee_{\alpha < \kappa(\sigma_i)} x = c_{i,\alpha} : i \leq \ell \right\}.$$

Note that the disjunctions in Γ_3 are finite given the condition $i \leq \ell$.

Lemma 9. *There is a \mathcal{T}^* -interpretation \mathcal{B} such that $\mathcal{B} \models \Gamma$.*

This lemma, whose proof is in the appendix, forms the core of the argument. By the compactness theorem, it suffices to prove that for any finite subset $\Gamma' \subseteq \Gamma$, there is a \mathcal{T}^* -interpretation \mathcal{B}' such that $\mathcal{B}' \models \Gamma'$. The tricky part is making \mathcal{B}' satisfy $\Gamma' \cap \Gamma_2$. The strategy is to use Lemma 8 to construct a model \mathcal{B}' in which $|\sigma_{i+1}^{\mathcal{B}'}|$ is very large in terms of $|\sigma_i^{\mathcal{B}'}|$ for each $i > \ell$. Lemma 7 will ensure that there is some way of interpreting the constants $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$ so that $\mathcal{B}' \models \Gamma' \cap \Gamma_2$.

We are now ready to prove Theorem 3.

Proof (Theorem 3). By Lemma 9, there is a \mathcal{T}^* -interpretation \mathcal{B} such that $\mathcal{B} \models \Gamma$. Let

$$B = \{t^{\mathcal{B}}((\vec{c}_1)^{\mathcal{B}}, (\vec{c}_2)^{\mathcal{B}}, \dots) : t \text{ is a } \Sigma^* \text{-term; } \vec{c}_i \in (\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)})^* \text{ for all } i\}.$$

For every $f \in \mathcal{F}_\Sigma$, the set B is closed under $f^{\mathcal{B}}$. Thus, we can define \mathcal{B}^- to be the subinterpretation of \mathcal{B} obtained by restricting the sorts, functions, and predicates to B .¹¹ Since the Σ^* -theory \mathcal{T}^* has built-in Skolem functions, \mathcal{B}^- is an elementary subinterpretation of \mathcal{B} by Lemma 2. We claim $|\sigma_i^{\mathcal{B}^-}| = \kappa(\sigma_i)$ for all i .

First, $\{c_{i,\alpha}^{\mathcal{B}^-}\}_{\alpha < \kappa(\sigma_i)}$ is a set of $\kappa(\sigma_i)$ distinct elements in $\sigma_i^{\mathcal{B}^-}$, because $\mathcal{B}^- \models \Gamma_1$. Thus, $|\sigma_i^{\mathcal{B}^-}| \geq \kappa(\sigma_i)$ for all i .

Second, $|\sigma_i^{\mathcal{B}^-}| \leq |\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}| = \kappa(\sigma_i)$ for all $i \in [\ell]$, as $\mathcal{B}^- \models \Gamma_3$.

Finally, it remains to show that $|\sigma_i^{\mathcal{B}^-}| \leq \kappa(\sigma_i)$ for all $i > \ell$. Inductively suppose that $|\sigma_j^{\mathcal{B}^-}| \leq \kappa(\sigma_j)$ for all $j < i$. Now, every element of $\sigma_i^{\mathcal{B}^-}$ is of the form

$$t^{\mathcal{B}}((\vec{c}_1)^{\mathcal{B}}, \dots, (\vec{c}_i)^{\mathcal{B}}, (\vec{c}_{i+1})^{\mathcal{B}}, (\vec{c}_{i+2})^{\mathcal{B}}, \dots),$$

where t is a Σ^* -term of sort σ_i . Since Σ^* is countable, there are at most \aleph_0 choices for t . We have at most $\kappa(\sigma_i)$ choices for $(\vec{c}_1)^{\mathcal{B}}, \dots, (\vec{c}_i)^{\mathcal{B}}$. Finally, we have finitely many choices for $(\vec{c}_{i+1})^{\mathcal{B}}, (\vec{c}_{i+2})^{\mathcal{B}}, \dots$ up to \sim -equivalence. Since $\mathcal{B}^- \models \Gamma_2$, it follows that there are at most $\kappa(\sigma_i)$ elements of $\sigma_i^{\mathcal{B}^-}$. Therefore, \mathcal{B}^- is a \mathcal{T}^* -interpretation satisfying φ with $|\sigma_i^{\mathcal{B}^-}| = \kappa(\sigma_i)$ for all i . Taking the reduct of \mathcal{B}^- to Σ gives the desired \mathcal{T} -interpretation. \square

3.4 Applications to theory combination

Since Theorem 2 implies that stably infinite and strongly finitely witnessable theories are strongly polite, we can restate the theorem on strongly polite theory combination with weaker hypotheses. This was already proved in [21] via a different method, but is now obtained as an immediate corollary of Theorem 2.

¹¹ In other words, \mathcal{B}^- is the Skolem hull of $\bigcup_i \{c_{i,\alpha}^{\mathcal{B}}\}_{\alpha < \kappa(\sigma_i)}$ in \mathcal{B} [13, p. 180].

Corollary 1. *Let Σ_1 and Σ_2 be disjoint countable signatures. Let \mathcal{T}_1 and \mathcal{T}_2 be Σ_1 - and Σ_2 -theories respectively, and let φ_1 and φ_2 be quantifier-free Σ_1 - and Σ_2 -formulas respectively. Suppose \mathcal{T}_1 is stably infinite and strongly finitely witnessable, both with respect to $\mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}$, and let $V = \text{vars}_{\mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}}(\text{wit}(\varphi_1))$. Then, $\varphi_1 \wedge \varphi_2$ is $(\mathcal{T}_1 \cup \mathcal{T}_2)$ -satisfiable if and only if there is an arrangement δ_V on V such that $\text{wit}(\varphi_1) \wedge \delta_V$ is \mathcal{T}_1 -satisfiable and $\varphi_2 \wedge \delta_V$ is \mathcal{T}_2 -satisfiable.*

We can also use our results to give a new characterization of shiny theories, which allows us to restate shiny combination theorem with weaker hypotheses.

To define shininess, we first need a few other notions. Let Σ be a signature with \mathcal{S}_Σ finite, and let $S \subseteq \mathcal{S}_\Sigma$. Write $S = \{\sigma_1, \dots, \sigma_n\}$. Then, the S -size of a Σ -interpretation \mathcal{A} is given by the tuple $(|\sigma_1^{\mathcal{A}}|, \dots, |\sigma_n^{\mathcal{A}}|)$. Such n -tuples are partially ordered by the product order: $(x_1, \dots, x_n) \preceq (y_1, \dots, y_n)$ if and only if $x_i \leq y_i$ for all $i \in [n]$. Given a quantifier-free formula φ , let $\text{minmods}_{\mathcal{T}, S}(\varphi)$ be the set of minimal S -sizes of \mathcal{T} -interpretations satisfying φ . It follows from results in [11] that $\text{minmods}_{\mathcal{T}, S}(\varphi)$ is a finite set of tuples.¹²

Then, we say a Σ -theory \mathcal{T} is *shiny* with respect to some subset of sorts $S \subseteq \mathcal{S}_\Sigma$ if \mathcal{S}_Σ is finite, \mathcal{T} is stably finite and smooth, both with respect to S , and $\text{minmods}_{\mathcal{T}, S}$ is computable. Theorem 3 implies that we can replace smoothness by finite smoothness, which may make it easier to prove that some theories are shiny. We can therefore improve the shiny theory combination theorem from [5, Theorem 2] as an immediate corollary of Theorem 3.

Corollary 2. *Let Σ_1 and Σ_2 be disjoint countable signatures, where \mathcal{S}_{Σ_1} and \mathcal{S}_{Σ_2} are finite. Let \mathcal{T}_1 and \mathcal{T}_2 be Σ_1 - and Σ_2 -theories respectively, and assume the satisfiability problems for quantifier-free formulas of both \mathcal{T}_1 and \mathcal{T}_2 are decidable. Suppose \mathcal{T}_1 is stably finite and finitely smooth, both with respect to $\mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}$, and $\text{minmods}_{\mathcal{T}_1, \mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}}$ is computable. Then, the satisfiability problem for quantifier-free formulas of $\mathcal{T}_1 \cup \mathcal{T}_2$ is decidable.*

4 Many-sorted Löwenheim–Skolem theorems

In this section, we state many-sorted generalizations of the Löwenheim–Skolem theorem. Our first results, in Section 4.2, hold with no assumptions on the signature. Later, in Section 4.3, we state stronger results for restricted signatures, which we then use for a many-sorted variant of the Łoś–Vaught test in Section 4.4. But first, in Section 4.1, we explain the limitations of relying solely on translations to single-sorted first-order logic.

4.1 Lost in translation

We may transform a many-sorted signature into a single-sorted signature by adding unary predicates signifying the sorts; of course, some restrictions are

¹² [5] proves this assuming that \mathcal{T} is stably finite, using Hilbert’s basis theorem. This assumption can be dropped by using the fact that if (X, \leq) is a well-quasi-order, then so is (X^n, \prec) , where \prec is the product order. Here X is the class of cardinals.

necessary, distinctness of sorts, etc. This procedure [25,7,14] is often used to lift results from single-sorted to many-sorted logic. As one example, standard versions of the downward Löwenheim–Skolem theorem for many-sorted logic, found in [15], are proven using this translation; we can, however, strengthen these results while still using only translations:

Theorem 4 (Downward). *Let Σ be a many-sorted signature with $|\mathcal{S}_\Sigma| < \aleph_0$. Suppose we have a Σ -structure \mathbb{A} with $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$, a cardinal κ satisfying $\max\{|\Sigma|, \aleph_0\} \leq \kappa \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}$, and sets $A_\sigma \subseteq \sigma^\mathbb{A}$ with $|A_\sigma| \leq \kappa$ for each $\sigma \in \mathcal{S}_\Sigma$. Then, there is an elementary substructure \mathbb{B} of \mathbb{A} such that $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$, $\aleph_0 \leq |\sigma^\mathbb{B}| \leq \kappa$ for all $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, $|\sigma^\mathbb{B}| = \kappa$ for some $\sigma \in \mathcal{S}_\Sigma$, and $A_\sigma \subseteq \sigma^\mathbb{B}$ for all $\sigma \in \mathcal{S}_\Sigma$.*

Theorem 5 (Upward). *Let Σ be a many-sorted signature with $|\mathcal{S}_\Sigma| < \aleph_0$. Suppose we have a Σ -structure \mathbb{A} with $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$ and a cardinal $\kappa \geq \max\{|\Sigma|, \max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\}\}$. Then, there is a Σ -structure \mathbb{B} containing \mathbb{A} as an elementary substructure such that $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for all $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$, $\aleph_0 \leq |\sigma^\mathbb{B}| \leq \kappa$ for all $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, and $|\sigma^\mathbb{B}| = \kappa$ for some sort $\sigma \in \mathcal{S}_\Sigma$.*

As convenient as translation arguments are, the above Löwenheim–Skolem theorems seem unsatisfactory, as they only allow us to choose a single cardinal, rather than one for each sort.

4.2 Downward, upward, and combined versions

The following are generalizations of the downward and upward Löwenheim–Skolem theorems to many-sorted logic, which are proved by adapting the proofs of the single-sorted case. Notice that we set all infinite domains to the same cardinality, while finite domains preserve their cardinalities.

Theorem 6 (Downward). *Fix a first-order many-sorted signature Σ . Suppose we have a Σ -structure \mathbb{A} , a cardinal κ such that $\max\{\aleph_0, |\Sigma|\} \leq \kappa \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}$, and sets $A_\sigma \subseteq \sigma^\mathbb{A}$ with $|A_\sigma| \leq \kappa$ for each $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$. Then, there is an elementary substructure \mathbb{B} of \mathbb{A} that satisfies $|\sigma^\mathbb{B}| = \kappa$ and $\sigma^\mathbb{B} \supseteq A_\sigma$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, and also $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

Theorem 7 (Upward). *Fix a first-order many-sorted signature Σ . Given a Σ -structure \mathbb{A} , pick a cardinal $\kappa \geq \max\{|\Sigma|, \aleph_0, \sup\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}\}$. Then, there is a Σ -structure \mathbb{B} containing \mathbb{A} as an elementary substructure that satisfies $|\sigma^\mathbb{B}| = \kappa$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, and also $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

Theorems 6 and 7 can be combined to yield yet another variant of the Löwenheim–Skolem theorem, which may be called the combined version.

Corollary 3 (Combined). *Fix a many-sorted signature Σ . Given a Σ -structure \mathbb{A} , pick a cardinal $\kappa \geq \max\{|\Sigma|, \aleph_0\}$. Then, there is a Σ -structure \mathbb{B} elementarily equivalent to \mathbb{A} with $|\sigma^\mathbb{B}| = \kappa$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, and $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

We illustrate Corollary 3 in Figure 2. In black, we represent the cardinalities of the resulting structure, and in red, those of the original one. When they coincide, we use marks split between the two colors. This representation shows a set of sorts in the horizontal axis, and the heights of the marks represent the cardinalities of the respective domains. We clearly separate cardinals larger and smaller than \aleph_0 with a rule. Assume, without

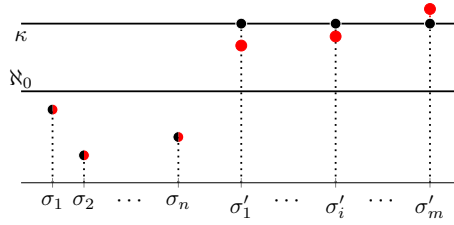


Fig. 2: Illustration of Corollary 3.

loss of generality, that initially $\sigma_1 \dots \sigma_n$ have finite cardinalities and σ'_1 has the least and σ'_m the greatest infinite cardinality.¹³ Corollary 3 allows us to pick an infinite cardinal κ in between the least and greatest infinite cardinalities, and set all infinite cardinalities in the interpretation to κ .

The above theorems require that the desired cardinalities of the infinite sorts are all equal. The following example shows that this limitation is necessary.

Example 2. Take the signature Σ with sorts $S = \{\sigma_1, \sigma_2\}$, no predicates, and only one function f of arity (σ_1, σ_2) . Take the Σ -structure \mathbb{A} with: $\sigma_1^{\mathbb{A}}$ and $\sigma_2^{\mathbb{A}}$ of cardinality \aleph_1 , and $f^{\mathbb{A}}$ a bijection. It is then true that $\mathbb{A} \models \varphi_{inj} \wedge \varphi_{sur}$, where $\varphi_{inj} = \forall x : \sigma_1. \forall y : \sigma_1. [[f(x) = f(y)] \rightarrow [x = y]]$ and $\varphi_{sur} = \forall u : \sigma_2. \exists x : \sigma_1. [f(x) = u]$, codifying that f is injective and surjective respectively. Notice then that, although $\max\{|\Sigma|, \aleph_0\} = \aleph_0$, there cannot be an elementary substructure \mathbb{B} of \mathbb{A} with $|\sigma_1^{\mathbb{B}}| = \aleph_0$ and $|\sigma_2^{\mathbb{B}}| = \aleph_1$: for if $\mathbb{B} \models \varphi_{inj} \wedge \varphi_{sur}$, $f^{\mathbb{B}}$ must be a bijection between $\sigma_1^{\mathbb{B}}$ and $\sigma_2^{\mathbb{B}}$. A similar argument shows that the corresponding generalization of the upwards theorem fails as well.

4.3 A stronger result for split signatures

Example 2 relies on “mixing sorts” by using a function symbol with arities spanning different sorts. We can state stronger versions of the many-sorted Löwenheim–Skolem theorems when such mixing of sorts is restricted.

Definition 6. A signature Σ is said to be split by A into a family of signatures $\{\Sigma_\lambda : \lambda \in A\}$ if A is a partition of \mathcal{S}_Σ , $\mathcal{S}_{\Sigma_\lambda} = \lambda$ for each $\lambda \in A$, $\mathcal{F}_\Sigma = \bigcup_{\lambda \in A} \mathcal{F}_{\Sigma_\lambda}$, and $\mathcal{P}_\Sigma = \bigcup_{\lambda \in A} \mathcal{P}_{\Sigma_\lambda}$. If Σ is split by A and each $\lambda \in A$ is a singleton, then we say that Σ is completely split by A .

If Σ is split by A , then the function/predicate symbols of Σ_λ must be disjoint from $\Sigma_{\lambda'}$ for $\lambda \neq \lambda'$. Given a partition A of \mathcal{S}_Σ and $\lambda \in A$, let $\mathcal{S}_{\geq \aleph_0}^{\mathbb{A}}(\lambda) = \mathcal{S}_{\geq \aleph_0}^{\mathbb{A}} \cap \lambda$. We state the downward, upward, and combined theorems for split signatures.

¹³ For greater clarity, the diagram only depicts the cases where there are finitely many sorts and the signature is countable.

Theorem 8 (Downward). *Fix a first-order many-sorted signature Σ split by Λ . Suppose we have a Σ -structure \mathbb{A} , a cardinal κ_λ such that $\max\{\aleph_0, |\Sigma_\lambda|\} \leq \kappa_\lambda \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)\}$ for each $\lambda \in \Lambda$, and sets $A_\sigma \subseteq \sigma^\mathbb{A}$ with $|A_\sigma| \leq \kappa_\lambda$ for each $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$. Then, there is an elementary substructure \mathbb{B} of \mathbb{A} that satisfies $|\sigma^\mathbb{B}| = \kappa_\lambda$ and $\sigma^\mathbb{B} \supseteq A_\sigma$ for $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, and $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

Theorem 9 (Upward). *Suppose Σ is split by Λ . Given a Σ -structure \mathbb{A} , pick a cardinal $\kappa_\lambda \geq \max\{|\Sigma_\lambda|, \aleph_0, \sup\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)\}\}$ for each $\lambda \in \Lambda$. Then, there is a Σ -structure \mathbb{B} containing \mathbb{A} as an elementary substructure that satisfies $|\sigma^\mathbb{B}| = \kappa_\lambda$ for $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, and $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

Corollary 4 (Combined). *Suppose Σ is split by Λ . Given a Σ -structure \mathbb{A} , pick a cardinal $\kappa_\lambda \geq \max\{|\Sigma_\lambda|, \aleph_0\}$ for each $\lambda \in \Lambda$. Then, there is a Σ -structure \mathbb{B} elementarily equivalent to \mathbb{A} with $|\sigma^\mathbb{B}| = \kappa_\lambda$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, and also $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

Corollary 4 is illustrated in Figure 3. We add sorts $S'' = \{\sigma''_1, \dots, \sigma''_m\}$, and assume our signature is split into Σ_{λ_1} and Σ_{λ_2} , where $\mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda_1) = \{\sigma'_1, \dots, \sigma'_m\}$ and $\mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda_2) = S''$ (the sorts with finite cardinalities can belong to either). Then, κ' is the cardinal associated with Σ_{λ_1} , and κ'' with Σ_{λ_2} . Thus, we are able to choose a cardinality for each class of sorts.

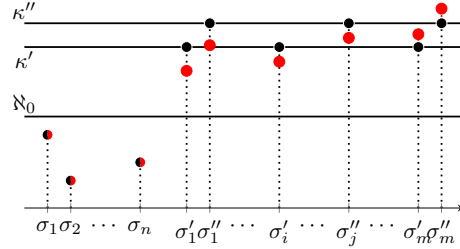


Fig. 3: Illustration of Corollary 4.

4.4 An application: the Łoś–Vaught test

We describe an application of our Löwenheim–Skolem theorems for theory-completeness: the Łoś–Vaught test. This is particularly relevant to SMT, as if a complete theory \mathcal{T} has a decidable set of axioms, then it is decidable whether $\vdash_{\mathcal{T}} \varphi$ [13, Lemma 2.2.8]. The single-sorted Łoś–Vaught is the following.

Definition 7. *Let Σ be a signature and κ a function from \mathcal{S}_Σ to the class of cardinals. A Σ -theory \mathcal{T} is κ -categorical if it has exactly one model \mathbb{A} (up to isomorphism) with the property that $|\sigma^\mathbb{A}| = \kappa(\sigma)$ for every $\sigma \in \mathcal{S}_\Sigma$. If there is only one sort $\sigma \in \mathcal{S}_\Sigma$, we abuse notation by using κ to denote the cardinal $\kappa(\sigma)$.*

Theorem 10 ([12,24]). *Suppose Σ is single-sorted and \mathcal{T} is a Σ -theory with only infinite models. If \mathcal{T} is κ -categorical for some $\kappa \geq |\Sigma|$, then \mathcal{T} is complete.*

The Łoś–Vaught test is quite useful, e.g., for the completeness of dense linear orders without endpoints and algebraically closed fields. We generalize it to many sorts. Translating to one-sorted logic and using Theorem 10 gives us:

Corollary 5. *Let Σ be a signature with $|\mathcal{S}_\Sigma| < \aleph_0$. Suppose \mathcal{T} is a Σ -theory, all of whose models \mathbb{A} satisfy $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$. Suppose further that for some cardinal $\kappa \geq |\Sigma|$, \mathcal{T} has exactly one model \mathbb{A} (up to isomorphism) such that $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} = \kappa$. Then, \mathcal{T} is complete.*

This is not the result one would hope for, because it excludes some many-sorted κ -categorical theories, as the following example demonstrates.

Example 3. Suppose Σ has $S = \{\sigma_1, \sigma_2\}$, no predicate symbols, and function symbols $0, 1, +$, and \times , of the expected arities. Let $\mathcal{T} = \text{ACF}_0 \cup \{\psi_{\geq n}^{\sigma_2} : n \in \mathbb{N}\}$, where ACF_0 is the theory of algebraically closed fields of characteristic zero (with respect to σ_1) and $\psi_{\geq n}^\sigma = \exists x_1 : \sigma. \dots \exists x_n : \sigma. \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)$, which asserts that there are at least n elements of sort σ . \mathcal{T} is κ -categorical, where $\kappa(\sigma_1) = \aleph_1$ and $\kappa(\sigma_2) = \aleph_0$. But \mathcal{T} is also κ' -categorical, where $\kappa'(\sigma_1) = \kappa'(\sigma_2) = \aleph_1$. Thus, \mathcal{T} has multiple models \mathbb{A} satisfying $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} = \aleph_1$. Similar reasoning holds for other infinite cardinals, so Corollary 5 does not apply.

For completely split signatures, we prove a more natural Łoś–Vaught test:

Definition 8. *A Σ -structure \mathbb{A} is strongly infinite if $|\sigma^\mathbb{A}| \geq \aleph_0$ for all $\sigma \in \mathcal{S}_\Sigma$.*

Theorem 11. *Suppose Σ is completely split into $\{\Sigma_\sigma : \sigma \in \mathcal{S}_\Sigma\}$, \mathcal{T} is a Σ -theory all of whose models are strongly infinite, and \mathcal{T} is κ -categorical for some function κ such that $\kappa(\sigma) \geq |\Sigma_\sigma|$ for every $\sigma \in \mathcal{S}_\Sigma$. Then, \mathcal{T} is complete.*

The assumption that Σ is completely split is necessary for Theorem 11:

Example 4. Let Σ have sorts σ_1, σ_2 , and function symbol f of arity (σ_1, σ_2) . Let $\mathcal{T} = \{\psi_{\geq n}^{\sigma_1} : n \in \mathbb{N}\} \cup \{\psi_{\geq n}^{\sigma_2} : n \in \mathbb{N}\} \cup \{\varphi_{inj} \vee \forall x : \sigma_1. \forall y : \sigma_1. [f(x) = f(y)]\}$. In \mathcal{T} , σ_1, σ_2 are infinite, and f is injective or constant. \mathcal{T} is κ -categorical for $\kappa(\sigma_1) = \aleph_1, \kappa(\sigma_2) = \aleph_0$, but not complete, due to the sentence $\forall x, y : \sigma_1. f(x) = f(y)$. This does not contradict Theorem 11, as Σ is not completely split.

5 Conclusion

We closed the problem of the existence of unicorn theories and discussed applications to SMT. This included a result similar to the Löwenheim–Skolem theorem, which inspired us to investigate the adaptation of this theorem to many-sorted logic. We also obtained a many-sorted version of the Łoś–Vaught test.

In future work, we plan to investigate whether Theorem 3 can be extended to uncountable signatures. More broadly, we intend to continue studying the relationships among many-sorted model-theoretic properties related to SMT.

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Appendix

A Proof of Lemma 3

This lemma was used in [21], and is explicitly found, with a proof, in its extended technical report [22], as Lemma 73. For completeness, we include its proof in this appendix.

Lemma 3. *If \mathcal{T} is stably infinite and strongly finitely witnessable, both with respect to some set of sorts $S \subseteq \mathcal{S}_\Sigma$, then \mathcal{T} is finitely smooth with respect to S .*

Proof. Let \mathcal{T} be stably infinite and strongly finitely witnessable, both with respect to S . Let φ be a \mathcal{T} -satisfiable quantifier-free formula, \mathcal{A} a \mathcal{T} -interpretation satisfying φ , and κ a function from $\mathcal{S}_{<\aleph_0}^A \cap S$ to the class of cardinals such that $|\sigma^A| \leq \kappa(\sigma) < \aleph_0$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^A \cap S$. We have $\mathcal{A} \models \exists \vec{w}. \text{wit}(\varphi)$, where $\vec{w} = \text{vars}(\text{wit}(\varphi)) \setminus \text{vars}(\varphi)$. Hence, by modifying the interpretation of the variables in \vec{w} , we obtain a \mathcal{T} -interpretation \mathcal{A}' satisfying $\text{wit}(\varphi)$. Let $V = \text{vars}(\text{wit}(\varphi))$, and let δ_V be the arrangement on V induced by the equalities in \mathcal{A}' . Then, $\text{wit}(\varphi) \wedge \delta_V$ is \mathcal{T} -satisfiable, so there exists a \mathcal{T} -interpretation \mathcal{A}'' satisfying $\text{wit}(\varphi) \wedge \delta_V$ such that $\sigma^{\mathcal{A}''} = \text{vars}_\sigma(\text{wit}(\varphi) \wedge \delta_V)^{\mathcal{A}''}$ for every $\sigma \in S$. The map from $\sigma^{\mathcal{A}''}$ to $\sigma^{\mathcal{A}'}$ given by $x^{\mathcal{A}''} \mapsto x^{\mathcal{A}'}$, where $x \in \text{vars}_\sigma(\text{wit}(\varphi))$, is well-defined and injective, so we have $|\sigma^{\mathcal{A}''}| \leq |\sigma^{\mathcal{A}'}| = |\sigma^A|$ for every $\sigma \in S$. In particular, $\kappa(\sigma) - |\sigma^{\mathcal{A}''}| \geq \kappa(\sigma) - |\sigma^A| \geq 0$. For each $\sigma \in \mathcal{S}_{<\aleph_0}^A \cap S$, introduce $\kappa(\sigma) - |\sigma^{\mathcal{A}''}|$ fresh variables W_σ of sort σ . Let $W = \bigcup_{\sigma \in \mathcal{S}_{<\aleph_0}^A \cap S} W_\sigma$, and extend the arrangement δ_V to an arrangement $\delta_{V \cup W}$ by asserting that all variables in W are distinct from each other and other variables in V . Since \mathcal{T} is stably infinite, $\text{wit}(\varphi) \wedge \delta_{V \cup W}$ is \mathcal{T} -satisfiable, so there exists a \mathcal{T} -interpretation \mathcal{B} satisfying $\text{wit}(\varphi) \wedge \delta_{V \cup W}$ such that $\sigma^{\mathcal{B}} = \text{vars}_\sigma(\text{wit}(\varphi) \wedge \delta_{V \cup W})^{\mathcal{B}}$ for every $\sigma \in S$. Since \mathcal{B} satisfies $\text{wit}(\varphi)$, it also satisfies φ . We also have $|\sigma^{\mathcal{B}}| = |\sigma^{\mathcal{A}''}| + |W_\sigma| = \kappa(\sigma)$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^A \cap S$. Therefore, \mathcal{T} is finitely smooth with respect to S . \square

B Proof of Lemma 6

Lemma 6. *For any $k, n, m \in \mathbb{N}$, there is an $R^*(k, n, m) \in \mathbb{N}$ such that for any totally ordered set $(X, <)$ with $|X| \geq R^*(k, n, m)$ and function $f : X^n \rightarrow [k]$, there is a subset $Y \subseteq X$ with $|Y| \geq m$ such that f is constant on each \sim -equivalence class of Y^n .*

Proof. Let

$$R^*(k, n, m) = R(k^{n^n}, n, m + n - 1).$$

For any function $f : X^n \rightarrow [k]$, let $f_\rho(x_1, \dots, x_n) = f(x_{\rho(1)}, \dots, x_{\rho(n)})$, where $\rho : [n] \rightarrow [n]$ is an arbitrary function. Fix an ordering $\rho_1, \dots, \rho_{n^n}$ on the set of functions from $[n]$ to itself. Then, let $F : P_n(X) \rightarrow [k]^{n^n}$ be given by, for $x_1 < \dots < x_n$,

$$F(\{x_1, \dots, x_n\}) = (f_{\rho_1}(x_1, \dots, x_n), \dots, f_{\rho_{n^n}}(x_1, \dots, x_n)).$$

By Lemma 5, for any totally ordered set $(X, <)$ with $|X| \geq R^*(k, n, m)$, there is a subset $Y' \subseteq X$ with $|Y'| \geq m + n - 1$ such that F is constant on $P_n(Y')$. Let $Y \subseteq Y'$ be the initial m elements of Y' according to the order on Y' inherited from X . Let $\vec{x}, \vec{y} \in Y^n$ with $\vec{x} \sim \vec{y}$, and let the distinct elements of \vec{x} be $x_1 < \dots < x_\ell$ and let those of \vec{y} be $y_1 < \dots < y_\ell$ for some $\ell \in [n]$. Add additional elements from Y' to get $\{x_1, \dots, x_n\} \supseteq \{x_1, \dots, x_\ell\}$ and $\{y_1, \dots, y_n\} \supseteq \{y_1, \dots, y_\ell\}$, where $x_1 < \dots < x_n$ and $y_1 < \dots < y_n$. Then, $F(\{x_1, \dots, x_n\}) = F(\{y_1, \dots, y_n\})$, since $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\} \in P_n(Y')$. Hence,

$$f_{\rho_i}(x_1, \dots, x_n) = f_{\rho_i}(y_1, \dots, y_n)$$

for all $i \in [n^n]$. In particular, let ρ_i be such that

$$\begin{aligned} (x_{\rho_i(1)}, \dots, x_{\rho_i(n)}) &= \vec{x} \quad \text{and} \\ (y_{\rho_i(1)}, \dots, y_{\rho_i(n)}) &= \vec{y}. \end{aligned}$$

Then,

$$f(\vec{x}) = f_{\rho_i}(x_1, \dots, x_n) = f_{\rho_i}(y_1, \dots, y_n) = f(\vec{y}),$$

as desired. \square

C Proof of Lemma 7

Lemma 7. *For any $k, m \in \mathbb{N}$ and $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, there is a number $R^{**}(k, \vec{n}, m) \in \mathbb{N}$, such that for any totally ordered set $(X, <)$ with $|X| \geq R^{**}(k, \vec{n}, m)$ and functions $f_i : X^{n_i} \rightarrow [k]$ for $i \in [r]$, there is a subset $Y \subseteq X$ with $|Y| \geq m$, such that f_i is constant on each \sim -equivalence class of Y^{n_i} for all $i \in [r]$.*

Proof. Let $n = n_1 + \dots + n_r$, and let $R^{**}(k, \vec{n}, m) = R^*(k^r, n, m + 1)$. Given functions $f_i : X^{n_i} \rightarrow [k]$, let $F : X^n \rightarrow [k]^r$ be given by

$$F(\vec{x}_1, \dots, \vec{x}_r) = (f_1(\vec{x}_1), \dots, f_r(\vec{x}_r)).$$

As proven in Lemma 6, for any totally ordered set $(X, <)$ with $|X| \geq R^{**}(k, \vec{n}, m)$, there is a subset $Y' \subseteq X$ with $|Y'| \geq m + 1$ such that F is constant on \sim -equivalence classes of Y'^n . Let $y' \in Y'$ be the maximum element according to the order on Y' inherited from X , and let $Y = Y' \setminus \{y'\}$. Given some $i \in [r]$, let $\vec{x}', \vec{y}' \in Y^{n_i}$ with $\vec{x}' \sim \vec{y}'$. Then,

$$\begin{aligned} & ((y')^{\oplus n_1}, \dots, (y')^{\oplus n_{i-1}}, \vec{x}', (y')^{\oplus n_{i+1}}, \dots, (y')^{\oplus n_r}) \sim \\ & ((y')^{\oplus n_1}, \dots, (y')^{\oplus n_{i-1}}, \vec{y}', (y')^{\oplus n_{i+1}}, \dots, (y')^{\oplus n_r}), \end{aligned}$$

since y' is strictly greater than every element in \vec{x}' and \vec{y}' . Therefore,

$$\begin{aligned} & F((y')^{\oplus n_1}, \dots, (y')^{\oplus n_{i-1}}, \vec{x}', (y')^{\oplus n_{i+1}}, \dots, (y')^{\oplus n_r}) = \\ & F((y')^{\oplus n_1}, \dots, (y')^{\oplus n_{i-1}}, \vec{y}', (y')^{\oplus n_{i+1}}, \dots, (y')^{\oplus n_r}), \end{aligned}$$

so $f_i(\vec{x}') = f_i(\vec{y}')$, as desired. \square

D Proof of Lemma 9

Lemma 9. *There is a \mathcal{T}^* -interpretation \mathcal{B} such that $\mathcal{B} \models \Gamma$.*

Proof. By the compactness theorem, it suffices to prove that $\mathcal{T}^* \cup \Gamma'$ is satisfiable for every finite subset $\Gamma' \subseteq \Gamma$. So let $\Gamma'_1 \subseteq \Gamma_1$, $\Gamma'_2 \subseteq \Gamma_2$, and $\Gamma'_3 \subseteq \Gamma_3$ be finite subsets. We will construct a \mathcal{T}^* -interpretation \mathcal{B}' such that $\mathcal{B}' \models \{\varphi\} \cup \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$. The tricky part is making \mathcal{B}' satisfy Γ'_2 . The strategy is to use Lemma 8 to construct a model \mathcal{B}' in which $|\sigma_{i+1}^{\mathcal{B}'}|$ is very large in terms of $|\sigma_i^{\mathcal{B}'}|$ for each $i > \ell$. Lemma 7 will ensure that there is some way of interpreting the constants $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$ so that $\mathcal{B}' \models \Gamma'_2$.

For each i , let $C_i = \{c_{i,\alpha} : \alpha < \kappa(\sigma_i); c_{i,\alpha} \text{ appears in } \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3\}$. Since $|\bigcup_i C_i| < \aleph_0$, there is a maximum natural number i such that $C_i \neq \emptyset$, which we denote s . For each $i > \ell$, let

$$T_i = \{t : t \text{ is a } \Sigma^* \text{-term of sort } \sigma_k \text{ appearing in } \Gamma'_2 \text{ for some } k < i\}.$$

Since $T_i \subseteq T_{i+1}$ for each i , and each T_i is finite, we can enumerate the terms of $\bigcup_i T_i$ so that for each i , there is an r_i such that t_1, \dots, t_{r_i} is an enumeration of T_i .

Let \mathcal{B}' be a \mathcal{T} -interpretation satisfying φ obtained according to Lemma 8, where $|\sigma_i^{\mathcal{B}'}|$ when $i > \ell$ is specified as follows. Suppose inductively that $|\sigma_k^{\mathcal{B}'}|$ has been determined for all $k < i$. Given a term of the form $t(\vec{x}_1, \dots, \vec{x}_s)$, let

$$m_{t,i} = \left| \sigma_1^{\mathcal{B}'} \right|^{|\vec{x}_1|} \times \dots \times \left| \sigma_{i-1}^{\mathcal{B}'} \right|^{|\vec{x}_{i-1}|} \times |C_{i+1}|^{|\vec{x}_{i+1}|} \times \dots \times |C_s|^{|\vec{x}_s|},$$

and let $n_{t,i} = |\vec{x}_i|$. Let $\vec{n}_i = ((n_{t_1,i})^{\oplus m_{t_1,i}}, \dots, (n_{t_{r_i},i})^{\oplus m_{t_{r_i},i}})$.¹⁴ Then, choose $|\sigma_i^{\mathcal{B}'}|$ so that

$$\left| \sigma_i^{\mathcal{B}'} \right| \geq R^{**} \left(\left| \sigma_{\ell+1}^{\mathcal{B}'} \right| + \dots + \left| \sigma_{i-1}^{\mathcal{B}'} \right|, \vec{n}_i, |C_i| \right),$$

where R^{**} is the function from Lemma 7.

Now, we specify how \mathcal{B}' interprets the constants C_i . Note that it does not matter how \mathcal{B}' interprets the constants in $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)} \setminus C_i$, since these constants do not appear in $\{\varphi\} \cup \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$. Impose an arbitrary total order on each $\sigma_i^{\mathcal{B}'}$ to be used for the \sim relation. If $i \leq \ell$, then interpret the elements of C_i as distinct elements of $\sigma_i^{\mathcal{B}'}$, which is possible because $|C_i| \leq |\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}| = \kappa(\sigma_i) = |\sigma_i^{\mathcal{B}'}|$. Otherwise, if $i > \ell$, we specify the interpretation of the constants C_i by induction on $s - i$. That is, we specify the interpretation of C_s , then that of C_{s-1} , and so on.

¹⁴ Recall that $(x)^{\oplus n}$ denotes the tuple consisting of x repeated n times.

Suppose that the interpretation of the constants C_j has been determined for all $j > i$. Given a term $t \in T_i$, define the following family of functions in $(\sigma_i^{\mathcal{B}'})^{n_{t,i}} \rightarrow \sigma_{\ell+1}^{\mathcal{B}'} \cup \dots \cup \sigma_{i-1}^{\mathcal{B}'}$:

$$\mathfrak{f}_{t,i} = \left\{ \vec{a} \mapsto t^{\mathcal{B}'} \left(\vec{a}_1, \dots, \vec{a}_{i-1}, \vec{a}, (\vec{c}_{i+1})^{\mathcal{B}'}, \dots, (\vec{c}_s)^{\mathcal{B}'} \right) : \right. \\ \left. \vec{a}_k \in \sigma_k^{\mathcal{B}'} \text{ for all } k < i; \vec{c}_j \in (C_j)^* \text{ for all } j > i \right\}.$$

Observe that $|\mathfrak{f}_{t,i}| = m_{t,i}$. By our choice of $|\sigma_i^{\mathcal{B}'}|$, we can apply Lemma 7 to the functions $\mathfrak{f}_i := \mathfrak{f}_{t_1,i} \cup \dots \cup \mathfrak{f}_{t_{r_i},i}$ to conclude that there is a subset $Y_i \subseteq \sigma_i^{\mathcal{B}'}$ with $|Y_i| \geq |C_i|$ such that each $f \in \mathfrak{f}_i$ is constant on \sim -equivalence classes of Y_i^n , where n is the arity of f . Then, interpret the constants C_i as distinct elements of Y_i in a way that is compatible with their respective total orders (i.e., $c_{i,\alpha} < c_{i,\beta}$ if and only if $c_{i,\alpha}^{\mathcal{B}'} < c_{i,\beta}^{\mathcal{B}'}$).

This completes the description of \mathcal{B}' . It remains to show that $\mathcal{B}' \models \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$.

First, we show that $\mathcal{B}' \models \Gamma'_1$. By construction, \mathcal{B}' interprets the constants C_i as distinct elements of $\sigma_i^{\mathcal{B}'}$ for all $i \in [s]$. Therefore, $\mathcal{B}' \models \Gamma'_1$.

Second, we show that $\mathcal{B}' \models \Gamma'_2$. For each $i > \ell$, let

$$\Gamma_2^i = \left\{ t \left(\vec{c}_1, \dots, \vec{c}_{i-1}, \vec{b}_i, \vec{c}_{i+1}, \dots, \vec{c}_s \right) = t \left(\vec{c}_1, \dots, \vec{c}_{i-1}, \vec{d}_i, \vec{c}_{i+1}, \dots, \vec{c}_s \right) : \right. \\ \left. t \in T_i; \vec{c}_k, \vec{b}_k, \vec{d}_k \in (C_k)^* \text{ for all } k; \vec{b}_i \sim \vec{d}_i \right\}.$$

Since $C_i^{\mathcal{B}'} \subseteq Y_i$ and each $f \in \mathfrak{f}_i$ is constant on \sim -equivalence classes of Y_i^n , where n is the arity of f , we have

$$t^{\mathcal{B}'} \left((\vec{c}_1)^{\mathcal{B}'}, \dots, (\vec{c}_{i-1})^{\mathcal{B}'}, (\vec{b}_i)^{\mathcal{B}'}, (\vec{c}_{i+1})^{\mathcal{B}'}, \dots, (\vec{c}_s)^{\mathcal{B}'} \right) = \\ t^{\mathcal{B}'} \left((\vec{c}_1)^{\mathcal{B}'}, \dots, (\vec{c}_{i-1})^{\mathcal{B}'}, (\vec{d}_i)^{\mathcal{B}'}, (\vec{c}_{i+1})^{\mathcal{B}'}, \dots, (\vec{c}_s)^{\mathcal{B}'} \right)$$

for each $t \in T_i$ whenever $\vec{b}_i \sim \vec{d}_i$. Thus, $\mathcal{B}' \models \Gamma_2^i$ for all $i > \ell$. Now, observe that $\bigcup_{i>\ell} \Gamma_2^i$ entails Γ'_2 . Therefore, $\mathcal{B}' \models \Gamma'_2$.

Finally, we show that $\mathcal{B}' \models \Gamma'_3$. Suppose Γ'_3 contains a sentence of the form

$$\forall x \in \sigma_i. \bigvee_{\alpha < \kappa(\sigma_i)} x = c_{i,\alpha},$$

where $i \leq \ell$. Then, $C_i = \{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$, so $|C_i| = \kappa(\sigma_i) = |\sigma_i^{\mathcal{B}'}|$. Since \mathcal{B}' interprets the constants C_i distinctly, $C_i^{\mathcal{B}'} = \sigma_i^{\mathcal{B}'}$. Thus, the sentence above is equivalent to the fact that every element of $\sigma_i^{\mathcal{B}'}$ is denoted by some constant in C_i . Therefore, $\mathcal{B}' \models \Gamma'_3$. \square

E Proof of Theorem 4

Theorem 4 (Downward). *Let Σ be a many-sorted signature with $|\mathcal{S}_\Sigma| < \aleph_0$. Suppose we have a Σ -structure \mathbb{A} with $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$, a cardinal κ satisfying $\max\{|\Sigma|, \aleph_0\} \leq \kappa \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}$, and sets $A_\sigma \subseteq \sigma^\mathbb{A}$ with $|A_\sigma| \leq \kappa$ for each $\sigma \in \mathcal{S}_\Sigma$. Then, there is an elementary substructure \mathbb{B} of \mathbb{A} such that $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$, $\aleph_0 \leq |\sigma^\mathbb{B}| \leq \kappa$ for all $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, $|\sigma^\mathbb{B}| = \kappa$ for some $\sigma \in \mathcal{S}_\Sigma$, and $A_\sigma \subseteq \sigma^\mathbb{B}$ for all $\sigma \in \mathcal{S}_\Sigma$.*

Proof. Consider the Σ^\dagger -structure \mathbb{A}^\dagger with: domain $\bigcup_{\sigma \in \mathcal{S}_\Sigma} \sigma^\mathbb{A}$; for every function f of arity $(\sigma_1, \dots, \sigma_n, \sigma)$ (in Σ), $f^{\mathbb{A}^\dagger}$ equals $f^\mathbb{A}$ when restricted to $\sigma_1^\mathbb{A} \times \dots \times \sigma_n^\mathbb{A}$, and is arbitrary otherwise; for every predicate P of arity $(\sigma_1, \dots, \sigma_n)$ (in Σ), $P^{\mathbb{A}^\dagger}$ equals $P^\mathbb{A}$; and $a \in P_\sigma^{\mathbb{A}^\dagger}$ iff $a \in \sigma^\mathbb{A}$. Notice that, because \mathbb{A} is a Σ -structure, we get \mathbb{A}^\dagger satisfies the following additional formulas:

- I for every $\sigma \in \mathcal{S}_\Sigma$, $\exists x. P_\sigma(x)$;
- II for any two distinct $\sigma, \tau \in \mathcal{S}_\Sigma$, $\forall x. (P_\sigma(x) \rightarrow \neg P_\tau(x))$;
- III if $\mathcal{S}_\Sigma = \{\sigma_1, \dots, \sigma_n\}$, $\forall x. P_{\sigma_1}(x) \vee \dots \vee P_{\sigma_n}(x)$;
- IV for f of arity $(\sigma_1, \dots, \sigma_n, \sigma)$, $\forall x_1, \dots, x_n. \bigwedge_{i=1}^n P_{\sigma_i}(x_i) \rightarrow P_\sigma(f(x_1, \dots, x_n))$ (with some obvious care being necessary if $n = 0$).

Now, $|\Sigma| = |\Sigma^\dagger|$ since $\mathcal{F}_{\Sigma^\dagger}$ must be in bijection with \mathcal{F}_Σ , and $\mathcal{P}_{\Sigma^\dagger}$ with $\mathcal{S}_\Sigma \cup \mathcal{P}_\Sigma$; and because \mathbb{A} has an infinite domain and $\kappa \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\}$, \mathbb{A}^\dagger is infinite and has cardinality greater than κ . Taking $A = \bigcup_{\sigma \in \mathcal{S}_\Sigma} A_\sigma$, $|A| \leq \kappa$, and we can therefore apply the classical downward Löwenheim–Skolem to obtain an elementary substructure \mathbb{B}^\dagger of \mathbb{A}^\dagger with domain of cardinality κ , and containing A .

Finally, we define a Σ -structure \mathbb{B} by making: $\sigma^\mathbb{B} = P^{\mathbb{B}^\dagger}$ for every sort σ (these are nonempty and disjoint, given \mathbb{A}^\dagger satisfies the sets of formulas in I and II); for a function f of arity $(\sigma_1, \dots, \sigma_n, \sigma)$, $f^\mathbb{B}$ equals $f^{\mathbb{B}^\dagger}$ restricted to $\sigma_1^\mathbb{B} \times \dots \times \sigma_n^\mathbb{B}$ (which is well-defined because \mathbb{A}^\dagger satisfies the set of formulas in IV); and, for a predicate P of arity $(\sigma_1, \dots, \sigma_n)$, $P^\mathbb{B}$ equals the intersection of $P^{\mathbb{B}^\dagger}$ and $\sigma_1^\mathbb{B} \times \dots \times \sigma_n^\mathbb{B}$, making of \mathbb{B} a substructure of \mathbb{A} . It is easy to prove that \mathbb{B} is elementary equivalent to \mathbb{A} , and therefore $\sigma^\mathbb{B}$ has the same cardinality as $\sigma^\mathbb{A}$ if the latter is finite (and thus $\sigma^\mathbb{B} = \sigma^\mathbb{A}$), and is infinite if the latter is infinite.

We also get that, since A is contained in \mathbb{B}^\dagger , A_σ is contained in $\sigma^\mathbb{B}$. Finally, $\bigcup_{\sigma \in \mathcal{S}_\Sigma} \sigma^\mathbb{B}$ equals, given \mathbb{B}^\dagger satisfies the formula in III , the domain of \mathbb{B}^\dagger , meaning $\sum_{\sigma \in \mathcal{S}_\Sigma} |\sigma^\mathbb{B}| = \kappa$; given \mathcal{S}_Σ is finite, this means some domain of \mathbb{B} has cardinality κ , finishing the proof. \square

F Proof of Theorem 5

Theorem 5 (Upward). *Let Σ be a many-sorted signature with $|\mathcal{S}_\Sigma| < \aleph_0$. Suppose we have a Σ -structure \mathbb{A} with $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$ and a cardinal $\kappa \geq \max\{|\Sigma|, \max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\}\}$. Then, there is a Σ -structure \mathbb{B} containing \mathbb{A} as an elementary substructure such that $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for all $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$, $\aleph_0 \leq |\sigma^\mathbb{B}| \leq \kappa$ for all $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$, and $|\sigma^\mathbb{B}| = \kappa$ for some sort $\sigma \in \mathcal{S}_\Sigma$.*

Proof. Construct the Σ^\dagger -structure \mathbb{A}^\dagger as in the proof of Theorem 4, and since the cardinality of the domain of \mathbb{A}^\dagger is $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\}$ we can apply the classical upward Löwenheim–Skolem to obtain a Σ^\dagger -structure \mathbb{B}^\dagger with an elementary substructure isomorphic to \mathbb{A}^\dagger , and a domain of cardinality κ . Furthermore, since \mathcal{S}_Σ is finite (say it equals $\{\sigma_1, \dots, \sigma_n\}$), then \mathbb{A}^\dagger satisfies $\forall x. (P_{\sigma_1}(x) \vee \dots \vee P_{\sigma_n}(x))$, and so must \mathbb{B}^\dagger . Translating \mathbb{B}^\dagger back to a Σ -structure \mathbb{B} , again as done in the proof of Theorem 4, we obtain that \mathbb{B} has an elementary substructure isomorphic to \mathbb{A} (and so $|\sigma^\mathbb{B}| = |\sigma^\mathbb{A}|$ for every $\sigma \in \mathcal{S}_\Sigma$ such that $\sigma^\mathbb{A}$ is finite); and for some $\sigma \in \mathcal{S}_\Sigma$, \mathbb{B} has cardinality κ , because every element of \mathbb{B}^\dagger must be in some domain of \mathbb{B} . \square

G Proofs of Theorems 6 and 7 and Corollary 3

These results are obtained as corollaries from Theorems 8 and 9 and Corollary 4, whose proofs can be found in the following sections. The reason being that every signature can be trivially split to a singleton partition.

H Proof of Theorem 8

Definition 9. *Suppose Σ is split by Λ . A Σ -formula is said to be a generalized Λ -cube if it is a conjunction $\bigwedge_{i=1}^n \varphi_i$, where each φ_i is a Σ_{λ_i} -formula, where $\lambda_i \in \Lambda$ and for $i \neq j$, we have $\lambda_i \neq \lambda_j$; similarly, a Σ -formula is said to be a generalized Λ -clause if it is a disjunction $\bigvee_{i=1}^m \varphi_i$, where each φ_i is a Σ_{λ_i} -formula, where $\lambda_i \in \Lambda$ and for $i \neq j$, we have $\lambda_i \neq \lambda_j$.*

A Σ -formula that is a disjunction of generalized Λ -cubes (respectively, a conjunction of generalized Λ -clauses) is said to be in generalized disjunctive Λ -normal form, or Λ -GDNF (respectively, generalized conjunctive normal Λ -form, or Λ -GCNF).¹⁵

We start with some technical lemmas.

Lemma 10. *Suppose Σ is split into $\{\Sigma_\lambda : \lambda \in \Lambda\}$; then a formula that is equivalent to a formula in GDNF is also equivalent to a formula in GCNF, and vice-versa.*

Proof. We prove that if φ that is equivalent to a formula in GDNF is also equivalent to a formula in GCNF: the reciprocal has an analogous proof.

So, suppose that φ is equivalent to $\psi = \bigvee_{i=1}^m \bigwedge_{j=1}^{n_i} \varphi_j^i$, and define the number of generalized literals in ψ as $n = \sum_{i=1}^m n_i$: notice that any quantifiers in ψ must be inside one of the φ_j^i . We proceed by induction on n : if $n = 1$, ψ is already in GCNF as well, so there is nothing to prove. Suppose then that the result holds for some $n \geq 1$,

¹⁵ Whenever Λ is clear from context, we will omit it from the nomenclature.

and take a generalized cube $\bigwedge_{i=1}^{n_i} \varphi_j^i$ with $n_i > 1$ (if there are none, again ψ is already in GCNF): without loss of generality, assume that $i = m$, and that $m > 1$ (otherwise ψ is again in GCNF, and there is nothing to be done). Then we have that, denoting by $\theta \equiv \theta'$ the fact that θ and θ' are equivalent,

$$\psi = \bigvee_{i=1}^m \bigwedge_{j=1}^{n_i} \varphi_j^i = \left[\bigvee_{i=1}^{m-1} \bigwedge_{j=1}^{n_i} \varphi_j^i \right] \vee \left[\left(\bigwedge_{j=1}^{n_{m-1}} \varphi_j^m \right) \wedge \varphi_{n_m}^m \right] \equiv \left[\left(\bigvee_{i=1}^{m-1} \bigwedge_{j=1}^{n_i} \varphi_j^i \right) \vee \left(\bigwedge_{j=1}^{n_{m-1}} \varphi_j^m \right) \right] \wedge \left[\left(\bigvee_{i=1}^{m-1} \bigwedge_{j=1}^{n_i} \varphi_j^i \right) \vee \varphi_{n_m}^m \right],$$

by using the distributivity of disjunction over conjunction. Now, in the second line, the formulas on both sides of the conjunction are in GDNF and have a number of generalized literals strictly less than that of ψ , so they are equivalent by induction hypothesis to formulas ψ_1 and ψ_2 in GCNF. To summarize, ψ is then equivalent to $\psi_1 \wedge \psi_2$, which is itself in GCNF, and so φ is equivalent to a formula in GCNF. \square

Lemma 11. *If Σ is a split signature, each of its formulas is equivalent to a formula in GDNF.*

Proof. It is well known that any first-order Σ -formula φ is equivalent to a formula in prenex normal form (PNF), that is, to a formula

$$Q_1 x_1 \cdots Q_n x_n \cdot \varphi,$$

where $Q_i \in \{\forall, \exists\}$ and φ is quantifier free; without loss of generality, let us assume that all Σ -formulas are in PNF, and we write the proof by induction on n .

If $n = 0$, φ is itself quantifier-free: writing φ in disjunctive normal form (DNF), and using the commutativity of conjunction to place literals of the same signature Σ_λ together (notice every literal on Σ is a literal of one of the Σ_λ because Σ is split), we obtain φ is equivalent to a formula in GDNF.

Now, assume the result holds for $n \geq 1$, and then it is true that

$$\varphi = Q_1 x_1 \cdots Q_{n+1} x_{n+1} \cdot \varphi = Q_1 x_1 \cdot (Q_2 x_2 \cdots Q_{n+1} x_{n+1} \cdot \varphi) = Q_1 x_1 \cdot \bigvee_{i=1}^p \bigwedge_{j=1}^q \varphi_j^i$$

by induction hypothesis, where φ_j^i are Σ_{λ_j} -formulas.¹⁶ Now, we have two cases to consider.

1. If $Q_1 = \exists$ and x_1 is of sort, without loss of generality, in Σ_{λ_q} , we have that

$$\varphi = \exists x_1 \cdot \bigvee_{i=1}^p \bigwedge_{j=1}^q \varphi_j^i = \bigvee_{i=1}^p \exists x_1 \cdot \bigwedge_{j=1}^q \varphi_j^i = \bigvee_{i=1}^p \exists x_1 \cdot \varphi_q^i \wedge \bigwedge_{j=1}^{q-1} \varphi_j^i,$$

since φ_j^i for $1 \leq j \leq q-1$ cannot have the variable x_1 . Of course, we are then done.

2. Now, suppose $Q_1 = \forall$. Because of Lemma 10 and our induction hypothesis, we know that we can rewrite the formula $Q_2 x_2 \cdots Q_{n+1} x_{n+1} \cdot \varphi$ as $\bigwedge_{i=1}^P \bigvee_{j=1}^Q \psi_j^i$ for some Σ_{λ_j} -formulas ψ_j^i . Then, assuming again without loss of generality that x_1 is of sort in Σ_{λ_Q} ,

$$\varphi = \forall x_1 \cdot \bigwedge_{i=1}^P \bigvee_{j=1}^Q \psi_j^i = \bigwedge_{i=1}^P \forall x_1 \cdot \bigvee_{j=1}^Q \psi_j^i = \bigwedge_{i=1}^P \forall x_1 \cdot \varphi_Q^i \vee \bigvee_{j=1}^{Q-1} \varphi_j^i,$$

which is in GCNF. Once again applying Lemma 10, we obtain φ may be written in GDNF, as we wanted to prove. \square

Theorem 8 (Downward). *Fix a first-order many-sorted signature Σ split by Λ . Suppose we have a Σ -structure \mathbb{A} , a cardinal κ_λ such that $\max\{\aleph_0, |\Sigma_\lambda|\} \leq \kappa_\lambda \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)\}$ for each $\lambda \in \Lambda$, and sets $A_\sigma \subseteq \sigma^\mathbb{A}$ with $|A_\sigma| \leq \kappa_\lambda$ for each $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$. Then, there is an elementary substructure \mathbb{B} of \mathbb{A} that satisfies $|\sigma^\mathbb{B}| = \kappa_\lambda$ and $\sigma^\mathbb{B} \supseteq A_\sigma$ for $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, and $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$.*

Proof. Given a formula φ and free variable $x \in \text{vars}(\varphi)$, let f_φ^x be a Skolem function, meaning that $(\mathbb{A}, \nu) \models \exists x \cdot \varphi$ implies $(\mathbb{A}, \mu) \models \varphi$, where μ differs from ν at most on x , $\text{vars}(\varphi) = \{x, y_1, \dots, y_n\}$, and $\mu(x) = f_\varphi^x(\nu(y_1), \dots, \nu(y_n))$. Skolem functions can be proven to exist as in single-sorted logic. For each $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, we take a set A_σ^0 such that

¹⁶ Notice that if a generalized cube of a formula in GDNF does not include formulas of exactly the same signatures as the other generalized cubes, we can always add tautologies to make the treatment of that formula more uniform.

$A_\sigma \subseteq A_\sigma^0 \subseteq \sigma^\mathbb{A}$ and $|A_\sigma^0| = \kappa_\lambda$, which is possible given that the A_σ in the statement of the theorem have cardinality at most κ_λ ; if $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}(\lambda)$, we make $A_\sigma^0 = \sigma^\mathbb{A}$. We then define, for every $m \in \mathbb{N}$, if σ is a sort of Σ_λ ,

$$A_\sigma^{m+1} = A_\sigma^m \cup \{f_\varphi^x(a_1, \dots, a_n) : \varphi \text{ is a } \Sigma_\lambda\text{-formula, } \text{vars}(\varphi) = \{x, y_1, \dots, y_n\}, x \text{ is of sort } \sigma, y_i \text{ is of sort } \sigma_i, \text{ and } a_i \in A_{\sigma_i}^m\}.$$

We define a Σ -structure \mathbb{B} , where $\sigma^\mathbb{B} = \bigcup_{n \in \mathbb{N}} A_\sigma^m$; $f^\mathbb{B}$, with f of arity $(\sigma_1, \dots, \sigma_n, \sigma)$, equals $f^\mathbb{A}$ restricted to $\sigma_1^\mathbb{B} \times \dots \times \sigma_n^\mathbb{B}$; and $P^\mathbb{B}$, with P of arity $(\sigma_1, \dots, \sigma_n)$, equals $P^\mathbb{A} \cap (\sigma_1^\mathbb{B} \times \dots \times \sigma_n^\mathbb{B})$. We claim that if $\sigma \in \Sigma_\lambda$, then $|\sigma^\mathbb{B}| = \kappa_\lambda$. Since $\kappa_\lambda \geq \aleph_0$, it suffices to show that $|A_\sigma^m| = \kappa_\lambda$ for each $m \in \mathbb{N}$. This is true for $m = 0$ by hypothesis. The cardinality of the set $\text{For}(\Sigma_\lambda)$ of formulas on the signature Σ_λ is at most $\max\{|\Sigma_\lambda|, \aleph_0\} \leq \kappa_\lambda$. Thus,

$$\kappa_\lambda = |A_\sigma^0| \leq |A_\sigma^{m+1}| \leq |A_\sigma^m| + \sum_{n \in \mathbb{N}} |\text{For}(\Sigma_\lambda)| \times |A_\sigma^m|^n \leq \kappa_\lambda,$$

so $|A_\sigma^{m+1}| = \kappa_\lambda$, as desired.

Now, it remains for us to show that \mathbb{B} is an elementary substructure of \mathbb{A} , clearly being a substructure. We wish to apply Lemma 2, so take a formula φ , a free variable x in φ (the other variables of φ being y_1 through y_n , of sorts, respectively, σ_1 through σ_n), and suppose that $(\mathbb{A}, \nu) \models \exists x. \varphi$. Because of Lemma 11, we know we can write φ as $\bigvee_{i=1}^p \bigwedge_{j=1}^q \varphi_j^i$, where φ_j^i is a Σ_{λ_j} -formula: without loss of generality, suppose x is of sort σ in λ_q ; then

$$\exists x. \varphi = \bigvee_{i=1}^p \exists x. \varphi_q^i \wedge \bigwedge_{j=1}^{q-1} \varphi_j^i,$$

and thus $(\mathbb{A}, \nu) \models \exists x. \varphi_q^i \wedge \bigwedge_{j=1}^{q-1} \varphi_j^i$ for some $1 \leq i \leq p$. Then the function $f_{\varphi_q^i}^x$, calculated on $\nu(y_1) \in A_\sigma^{m_1}, \dots, \nu(y_n) \in A_\sigma^{m_n}$, returns a witness for $\exists x. \varphi_q^i$ in $A_\sigma^{\max\{m_1, \dots, m_n\}+1}$, a subset of \mathbb{B} , which finishes our proof. \square

I Proof of Theorem 9

Theorem 9 (Upward). *Suppose Σ is split by Λ . Given a Σ -structure \mathbb{A} , pick a cardinal $\kappa_\lambda \geq \max\{|\Sigma_\lambda|, \aleph_0, \sup\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)\}\}$ for each $\lambda \in \Lambda$. Then, there is a Σ -structure \mathbb{B} containing \mathbb{A} as an elementary substructure that satisfies $|\sigma^\mathbb{B}| = \kappa_\lambda$ for $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, and $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$.*

Proof. For each $\sigma \in \mathcal{S}_\Sigma$, let P_σ be a set of new constants of sort σ , where $|P_\sigma|$ has cardinality $|\sigma^\mathbb{A}|$. For each $\lambda \in \Lambda$ and $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, let Q_σ be a set of new constants of sort σ , where $|Q_\sigma|$ has cardinality κ_λ . Let Σ_P be the signature obtained by adding the sets P_σ to Σ , and let Σ_Q be the signature obtained by adding the sets Q_σ to Σ_P . We extend \mathbb{A} into a Σ_P -structure $\hat{\mathbb{A}}$ by interpreting the sorts, functions, and predicates in Σ in the same way as in \mathbb{A} , and defining $b_\sigma^\hat{\mathbb{A}}$, for $b_\sigma \in P_\sigma$, so that the mapping $b_\sigma \in P_\sigma \mapsto b_\sigma^\hat{\mathbb{A}} \in \sigma^\hat{\mathbb{A}}$ is bijective.

Now, let Γ be the set of all Σ_P -sentences satisfied by $\hat{\mathbb{A}}$, and let

$$\bar{\Gamma} = \Gamma \cup \{\neg(c_\sigma = d_\sigma) : c_\sigma, d_\sigma \in Q_\sigma, \text{ with } c_\sigma \neq d_\sigma, \text{ for all } \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}.$$

By Theorem 1, $\bar{\Gamma}$ is consistent. Let \mathbb{C} be a model of $\bar{\Gamma}$. Then, for every $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$, we have $|\sigma^\mathbb{C}| = |\sigma^\mathbb{A}|$, and for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, we have $|\sigma^\mathbb{C}| \geq \kappa_\lambda$.

By Theorem 8, there is an elementary substructure \mathbb{B} of \mathbb{C} with $|\sigma^\mathbb{B}| = \kappa_\lambda$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$ and $\sigma^\mathbb{B} = \sigma^\mathbb{C}$ for each $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$. Since $\mathbb{B} \models \Gamma$, \mathbb{A} is isomorphic to an elementary substructure of \mathbb{B} , where the isomorphism is given by $b_\sigma^\hat{\mathbb{A}} \mapsto b_\sigma^\mathbb{B}$ for $b_\sigma \in P_\sigma$. Identifying these elements of \mathbb{B} with the corresponding elements of \mathbb{A} completes the proof. \square

J Proof of Corollary 4

Corollary 4 (Combined). *Suppose Σ is split by Λ . Given a Σ -structure \mathbb{A} , pick a cardinal $\kappa_\lambda \geq \max\{|\Sigma_\lambda|, \aleph_0\}$ for each $\lambda \in \Lambda$. Then, there is a Σ -structure \mathbb{B} elementarily equivalent to \mathbb{A} with $|\sigma^\mathbb{B}| = \kappa_\lambda$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$, and also $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$.*

Proof. First, apply Theorem 8 to \mathbb{A} with the cardinals $\theta_\lambda = \max\{\aleph_0, |\Sigma_\lambda|\}$ to get a structure \mathbb{C} elementarily equivalent to \mathbb{A} satisfying $|\sigma^\mathbb{C}| = \max\{\aleph_0, |\Sigma_\lambda|\}$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$ and $\sigma^\mathbb{C} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$. Then, apply Theorem 9 to \mathbb{C} with the cardinals κ_λ to get a structure \mathbb{B} elementarily equivalent to \mathbb{A} with $|\sigma^\mathbb{B}| = \kappa_\lambda$ for every $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}(\lambda)$ and $\sigma^\mathbb{B} = \sigma^\mathbb{A}$ for every $\sigma \in \mathcal{S}_{<\aleph_0}^\mathbb{A}$. \square

K Proof of Corollary 5

Corollary 5. *Let Σ be a signature with $|\mathcal{S}_\Sigma| < \aleph_0$. Suppose \mathcal{T} is a Σ -theory, all of whose models \mathbb{A} satisfy $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$. Suppose further that for some cardinal $\kappa \geq |\Sigma|$, \mathcal{T} has exactly one model \mathbb{A} (up to isomorphism) such that $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} = \kappa$. Then, \mathcal{T} is complete.*

Proof. The translation of \mathcal{T} into a single-sorted Σ^\dagger -theory \mathcal{T}^\dagger is described in [4]. We informally recapitulate the translation here. First, we may assume without loss of generality that Σ has no function symbols, since function symbols can be eliminated in favor of suitably axiomatized predicate symbols. Let Σ^\dagger have the components

$$\begin{aligned}\mathcal{S}_{\Sigma^\dagger} &= \{\sigma^\dagger\} \\ \mathcal{F}_{\Sigma^\dagger} &= \emptyset \\ \mathcal{P}_{\Sigma^\dagger} &= \{P_\sigma : \sigma \in \mathcal{S}_\Sigma\} \cup \mathcal{P}_\Sigma.\end{aligned}$$

Now, a Σ -formula φ can be translated to Σ^\dagger -formula φ^\dagger by using the predicates P_σ to relativize quantifiers to their respective sorts, as in the proof of Theorem 4. Then, let

$$\mathcal{T}^\dagger = \mathcal{T}_1^\dagger \cup \mathcal{T}_2^\dagger \cup \mathcal{T}_3^\dagger,$$

where

- \mathcal{T}_1^\dagger is the set of translated axioms of \mathcal{T} ,
- \mathcal{T}_2^\dagger is a set of sentences asserting that every element belongs to exactly one sort (this is where we use the assumption that Σ has finitely many sorts), and every sort is nonempty, and
- \mathcal{T}_3^\dagger is a set of sentences asserting that predicates are true only if they are applied to elements of the appropriate arity.

Now, there is a one-to-one correspondence between models of \mathcal{T} and models of \mathcal{T}^\dagger , such that a \mathcal{T} -model \mathbb{A} corresponds to a \mathcal{T}^\dagger -model \mathbb{B} with $|\sigma^{\dagger\mathbb{B}}| = \sum_{\sigma \in \mathcal{S}_\Sigma} |\sigma^\mathbb{A}|$. Indeed, given a \mathcal{T} -model \mathbb{A} , construct a \mathcal{T}^\dagger -model \mathbb{B} by letting

$$\begin{aligned}\sigma^{\dagger\mathbb{B}} &= \bigcup_{\sigma \in \mathcal{S}_\Sigma} \sigma^\mathbb{A} \\ P_\sigma^\mathbb{B} &= \sigma^\mathbb{A} \quad \text{for each } \sigma \in \mathcal{S}_\Sigma \\ P^\mathbb{B} &= P^\mathbb{A} \quad \text{for each } P \in \mathcal{P}_\Sigma.\end{aligned}$$

Conversely, given a \mathcal{T}^\dagger -model \mathbb{B} , construct a \mathcal{T} -model \mathbb{A} by letting

$$\begin{aligned}\sigma^\mathbb{A} &= P_\sigma^\mathbb{B} \quad \text{for each } \sigma \in \mathcal{S}_\Sigma \\ P^\mathbb{A} &= P^\mathbb{B} \quad \text{for each } P \in \mathcal{P}_\Sigma.\end{aligned}$$

Using this correspondence, we see that \mathcal{T}^\dagger is a Σ^\dagger -theory all of whose models are infinite, and that \mathcal{T}^\dagger is κ -categorical. By Theorem 10, \mathcal{T}^\dagger is complete. Thus, $\vdash_{\mathcal{T}^\dagger} \varphi^\dagger$ or $\vdash_{\mathcal{T}^\dagger} \neg\varphi^\dagger$ for every Σ -sentence φ . Hence, $\vdash_{\mathcal{T}} \varphi$ or $\vdash_{\mathcal{T}} \neg\varphi$, so \mathcal{T} is complete. \square

L Proof of Theorem 11

Theorem 11. *Suppose Σ is completely split into $\{\Sigma_\sigma : \sigma \in \mathcal{S}_\Sigma\}$, \mathcal{T} is a Σ -theory all of whose models are strongly infinite, and \mathcal{T} is κ -categorical for some function κ such that $\kappa(\sigma) \geq |\Sigma_\sigma|$ for every $\sigma \in \mathcal{S}_\Sigma$. Then, \mathcal{T} is complete.*

Proof. Suppose \mathcal{T} is not complete. Then, for some sentence φ , the theories $\mathcal{T}_0 = \mathcal{T} \cup \{\varphi\}$ and $\mathcal{T}_1 = \mathcal{T} \cup \{\neg\varphi\}$ are consistent. Let \mathbb{A}_0 and \mathbb{A}_1 be models of \mathcal{T}_0 and \mathcal{T}_1 respectively. Since \mathbb{A}_0 and \mathbb{A}_1 are also models of \mathcal{T} , both are strongly infinite. By Corollary 4, there are models \mathbb{A}'_0 and \mathbb{A}'_1 that are elementarily equivalent to \mathbb{A}_0 and \mathbb{A}_1 respectively such that $|\sigma^{\mathbb{A}'_0}| = |\sigma^{\mathbb{A}'_1}| = \kappa(\sigma)$ for all $\sigma \in \mathcal{S}_\Sigma$. Since $\mathbb{A}'_0 \models \varphi$ and $\mathbb{A}'_1 \models \neg\varphi$, the models \mathbb{A}'_0 and \mathbb{A}'_1 are not isomorphic, contradicting the assumption that \mathcal{T} is κ -categorical. \square

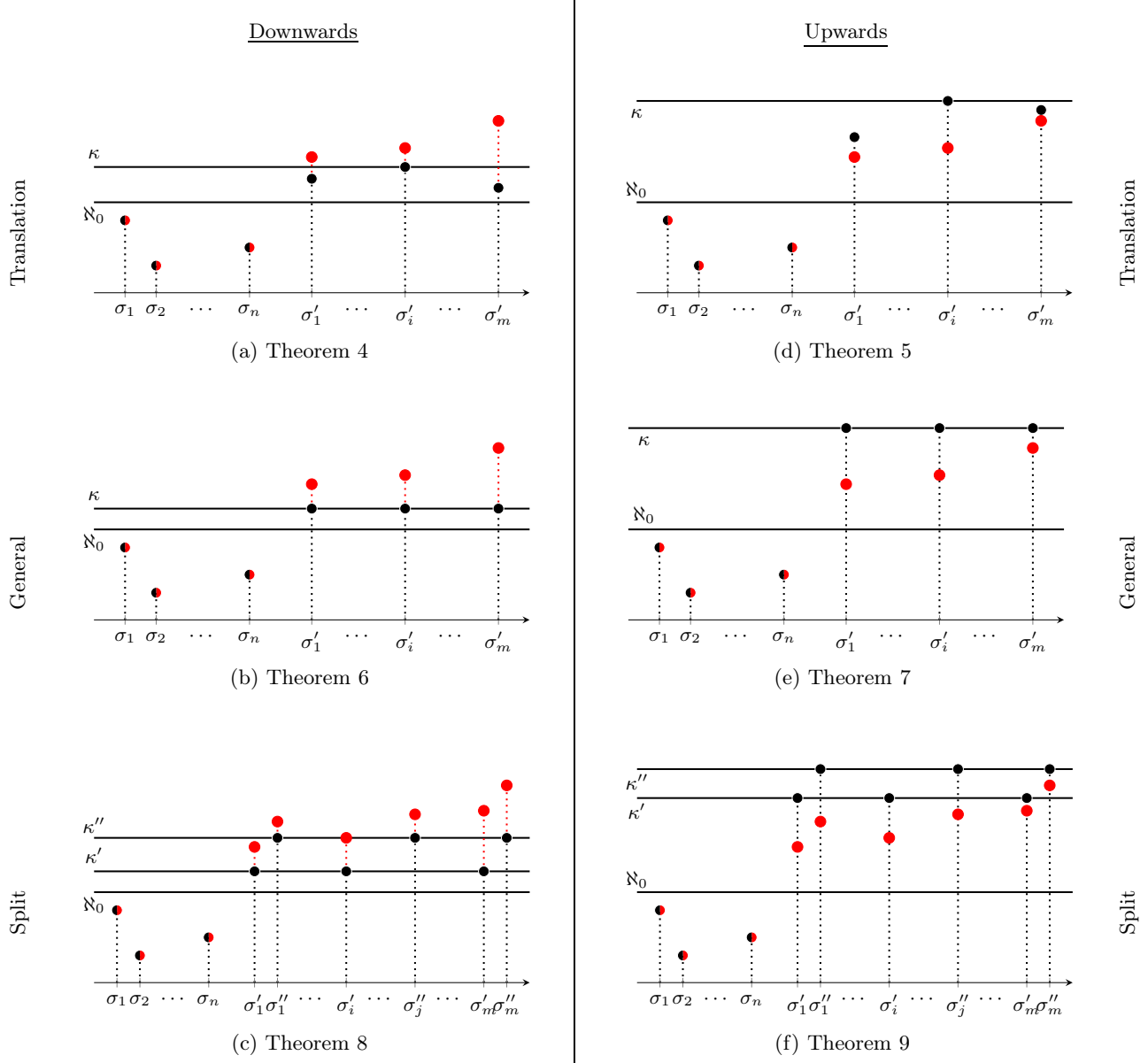


Fig. 4: Downward and Upward Löwenheim–Skolem Theorems.

M Illustrations

In the body of the paper, we included Figures 2 and 3, in order to visualize the effect of the new combined Löwenheim–Skolem theorems. We only included the diagrams for the combined theorems. In this appendix, we include similar diagrams for the downward and upward theorems as well.

Translation-based proofs Theorems 4 and 5 are proven using a translation-based approach. We visualize their effects in, respectively, Figure 4.(a) and Figure 4.(d). Notice that σ'_i is the sort in the resulting structure that gets assigned the desired cardinality κ , although more than one such sort could exist; the final cardinalities for other sorts with infinite domains are bounded from below by \aleph_0 and from above by κ , but we do not have any further control over them.

Direct proofs Theorems 6 and 7 are proven in a direct manner, by adapting the single-sorted classical proofs to the many-sorted case. We visualize Theorems 6 and 7 in Figure 4.(b) and Figure 4.(e), respectively. With these new results, we are able to set the cardinalities of all the infinite sorts, but only to the exact same cardinal.

Direct proofs for split signatures We provide examples of applying Theorems 8 and 9 in, respectively, Figure 4.(c), Figure 4.(f). We assume that our signature is split into Σ_{λ_1} and Σ_{λ_2} , where $\mathcal{S}_{\geq \aleph_0}^{\Delta}(\lambda_1) = \{\sigma'_1, \dots, \sigma'_m\}$ and $\mathcal{S}_{\geq \aleph_0}^{\Delta}(\lambda_2) = \{\sigma''_1, \dots, \sigma''_m\}$ (the sorts with finite interpretations can belong to either λ_1 or λ_2). Then, κ' is the cardinal associated with Σ_{λ_1} , and κ'' the cardinal associated with Σ_{λ_2} . Thus, for split signatures, we are able to choose a cardinality for each class of sorts.

Overall, going downward in either the left or right part of Figure 4, we see that the translation-based proofs only let us set the maximal cardinality (diagrams (a) and (d)), while the theorems proved using the direct approach allow us to set all sorts to a single cardinality (diagrams (b) and (e)). For split signatures, we can do even better, and have a dedicated cardinality for each class of sorts (diagrams (c) and (f)).