Modal extension of ideal paraconsistent four-valued logic and its subsystem

Norihiro Kamide[a], Yoni Zohar[b]

[a] *Teikyo University, Faculty of Science and Engineering, Department of Information and Electronic Engineering, Toyosatodai 1-1, Utsunomiya-shi, Tochigi 320-8551, Japan.*
[b] *Stanford University, Computer Science Department, 353 Serra Mall, Stanford, CA 94305, USA.*

Abstract

This study aims to introduce a modal extension M4CC of Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic 4CC as a Gentzen-type sequent calculus and prove the Kripke-completeness and cut-elimination theorems for M4CC. The logic M4CC is also shown to be decidable and embeddable into the normal modal logic S4. Furthermore, a subsystem of M4CC, which has some characteristic properties that do not hold for M4CC, is introduced and the Kripke-completeness and cut-elimination theorems for this subsystem are proved. This subsystem is also shown to be decidable and embeddable into S4.
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1. Introduction

In this study, a modal extension M4CC of *Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic*, known as 4CC [5, 6, 7], is introduced. We prove theorems for syntactically and semantically embedding M4CC into the normal modal logic S4. We prove the Kripke-completeness and cut-elimination theorems for M4CC using these embedding theorems. We also obtain the

Email addresses: `drnkamide08@kpd.biglobe.ne.jp` (Norihiro Kamide), `yoniz@cs.stanford.edu` (Yoni Zohar)

decidability result of M4CC and the finite model property for M4CC using these embedding theorems. Furthermore, we introduce a subsystem M4CC* of M4CC and prove theorems for syntactically and semantically embedding M4CC* into S4 and vice versa. Similarly, we prove the Kripke-completeness and cut-elimination theorems for M4CC* as well as the decidability result of M4CC* and the finite model property for M4CC* using these embedding theorems.

The proposed logic M4CC is introduced as a Gentzen-type sequent calculus and is a modal extension of the Gentzen-type sequent calculus EPL, which was introduced by us in [20, 23]. The calculus EPL was shown in [23] to be theorem-equivalent to Arieli and Avron's original Gentzen-type sequent calculus $\mathrm{G}_{4\mathrm{CC}}$ [5, 6] for 4CC. Another proposed logic, M4CC*, is obtained from M4CC by deleting some initial sequents, which correspond to *the principle* $(\sim\alpha \land -\alpha)\to\beta$ *of quasi-explosion* and *the law* $\sim\alpha \lor -\alpha$ *of quasi-excluded middle*, where $\sim$ and $-$ are a paraconsistent negation connective and conflation connective, respectively. The logic M4CC* is a modal extension of the Gentzen-type sequent calculus PL, which was introduced by us in [20, 23], as an embeddable subsystem of 4CC.

The original non-modal logic 4CC is an extension of *Belnap and Dunn's useful four-valued logic* (also called *first-degree entailment logic*) [8, 9, 12], and is a variant of *the logic of logical bilattices* [3, 4]. The logic 4CC is also a specific type of *paraconsistent logics* [35] with multiple names: they are called *paradefinite logics* by Arieli and Avron [5, 6], *non-alethic logics* by da Costa, and *paranormal logics* by Béziau [10]. Regardless of their names, paradefinite logics incorporate the properties of both *paraconsistency*, which rejects *the principle* $(\alpha \land \sim\alpha)\to\beta$ *of explosion*, and *paracompleteness*, which rejects *the law* $\alpha \lor \sim\alpha$ *of excluded middle*. By these properties, paradefinite logics are known to be appropriate for handling inconsistent and incomplete information [5].

Moreover, 4CC is known to be one of the most important *ideal paraconsistent (or paradefinite) logics* that have natural many-valued semantics. The logic 4CC is maximal relative to classical logic, which means that any attempt to add to it a tautology of classical logic, which is not provable in 4CC, should necessary result in classical logic. The exact definition and motivation for introducing this property was described in [7]. The logic 4CC is also related to *connexive logics* [2, 26, 43] because its Hilbert-style axiom system contains a characteristic axiom scheme corresponding to Boethius' theses. For more information on the relation between 4CC and connexive logics, see [23].

Although 4CC is an important ideal paraconsistent (or paradefinite) logic, a modal extension of 4CC, which is suitable for actual applications concerning, for example, some additional situations on knowledge (belief) and/or time (any time in the future), has not been studied yet. Therefore, this study aims to propose the modal extension M4CC of 4CC and show the completeness theorem with respect to a Kripke semantics for M4CC as well as the finite model property for M4CC. Since adding S4-type modality allows one to formulate naturally both Gentzen-type sequent calculus and Kripke semantics for the resulting system that can also suitably handle knowledge and/or time, we develop M4CC as a system combining 4CC with S4. However, we can also combine 4CC and one of the other normal modal logics such as K. By imposing some appropriate modifications, a similar method of proof can also be used to show the Kripke-completeness, cut-elimination, and finite model property for such extensions.

It was argued in [20, 23] that the embedding-based proof method used in [20, 23] for proving the completeness and cut-elimination theorems for another propositional non-modal paradefinite logic, PL, seems insufficient for 4CC because it is unclear how a translation function can be defined for 4CC. However, in the present paper, we have shown that this argument is not true. Namely, we can use the embedding-based proof method for proving the completeness and cut-elimination theorems for M4CC and hence for the subsystem 4CC.

This work is regarded as a continuation of the previous work [23]. In [23], an alternative Gentzen-type sequent calculus (called EPL) for 4CC and its subsystem (called PL) were introduced. The proposed logics, i.e., M4CC and M4CC*, are also regarded as modal extensions of EPL and PL, respectively. The differences between PL and EPL are explained as follows. The logic EPL (i.e., 4CC) has the *negative symmetry property*, which represents a type of symmetry between $-$ and $\sim$, but PL has no such property. Similarly, the extended logic M4CC has this property, but M4CC* has no such property. On the contrary, PL has the *quasi-paraconsistency* and *quasi-paracompleteness* properties that reject the law of quasi-explosion and the law of quasi-excluded middle, but EPL has no such properties (i.e., EPL has these axioms). Similarly, the logic M4CC* has these properties, but M4CC has no such properties. In other words, the S4-type modalities in M4CC and M4CC* are formalized on the basis of preserving these characteristic properties.

The logic PL was introduced to obtain a good paradefinite logic that can simulate classical logic. Such logic is required in application areas that use both paraconsistent and classical negations. Some paraconsistent logics that can simulate classical logic were studied in [21, 23], where it was shown that some *bidirectional* embeddings (i.e., embeddings from the underlying paraconsistent logic into classical logic and vice versa) characterize such logic. We believe that the existence of such bidirectional embeddings is important in formalizing good paradefinite logic because such bidirectional embeddings are regarded as an analogue of the concept "bisimulation" (i.e., ones can simulate each other) which is known as an important concept in the concurrency theory in computer science. Thus, our motivation for introducing M4CC*, which is another aim of this study, is to extend this idea to also apply to modal logic. Although we have obtained such extended bidirectional embeddings from M4CC* into S4 and vice versa, we have not yet obtained extended bidirectional embeddings from M4CC into S4 and vice versa. We have only obtained *single-directional* embeddings from M4CC into S4. Nevertheless, using such extended bidirectional and single-directional embeddings, we can easily prove the Kripke-completeness and cut-elimination theorems for M4CC* and M4CC as well as the decidabilities and finite model properties of M4CC* and M4CC. The finite model property of a slightly different version of M4CC was presented by us in [24] using a direct proof method. However, the proof in [24] had a gap, and hence, such a finite model property has not yet been proved.

The structure of this paper is summarized as follows.

In Section 2, we introduce M4CC, M4CC*, and a Gentzen-type sequent calculus GS4 for S4 and define Kripke semantics for these systems. We also obtain some basic propositions for these systems and semantics and discuss some of their properties. These systems introduced in this section have the box modal operator $\Box$ as an explicit modal operator (i.e., $\Box$ is introduced by some logical inference rules). In these systems, the diamond modal operator $\Diamond$ is also handled as an implicit modal operator (i.e., $\Diamond$ is handled as an abbreviation of a combination of $\Box$ and negations).

In Section 3, we prove some main theorems for M4CC and M4CC*. First, we prove several theorems for syntactically embedding M4CC* into (a Gentezen-type sequent calculus GS4 for) S4 and vice versa. Using such a syntactical embedding theorem, we show the cut-elimination theorem for M4CC* and the decidability of M4CC*. Then, as corollaries of the cut-elimination theorem, we obtain quasi-paraconsistency and quasi-paracompleteness for M4CC*. Next, we prove several theorems for semantically embedding M4CC* into S4 and vice versa. Using such a semantical embedding theorem, we prove the Kripke-completeness theorem for M4CC* and the finite model property of M4CC*. Similarly, we prove several syntactical and semantical embeddings of M4CC* into (a Gentezen-type sequent calculus GS4 for) S4. But, these embedding theorems are single-directional. Using these embedding theorems, we show the cut-elimination and Kripke-completeness theorems for M4CC and the decidability and finite model property of M4CC. Then, as a corollary of the cut-elimination theorem, we obtain the negative symmetry property for M4CC.

In Section 4, we introduce other versions M4CC$_\Diamond$, M4CC$^\star_\Diamond$, and GS4$_\Diamond$ and define Kripke semantics for them. These systems are extensions of M4CC, M4CC*, and GS4 by adding the diamond modal operator $\Diamond$ as an explicit modal operator (i.e., $\Diamond$ is introduced by some logical inference rules). Using a similar embedding-based method, we show that the same main theorems (except the finite model property) as those for M4CC and M4CC* also hold for M4CC$_\Diamond$ and M4CC$^\star_\Diamond$. The finite model property does seem to hold for these systems as well (even if it is not proved in this paper).

In Section 5, we present the conclusion of this paper, some remarks on K-type modal extensions, which are based on the normal modal logic K, and related works on modal extensions of many-valued logics.

2. Modal extensions

2.1. Sequent calculi

Formulas of modal extensions of ideal paraconsistent four-valued logic and its relatives are constructed from countably many propositional variables by the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\sim$ (paraconsistent negation) and $-$ (conflation), and $\Box$ (box). We use the symbol $\Diamond$ (diamond) to denote the abbreviation of ${\sim}{-}\Box{\sim}{-}$, where, as will shown later, ${\sim}{-}$ is considered to be the classical negation connective (i.e., the classical negated formulas of the form $\neg\alpha$ can be defined as ${\sim}{-}\alpha$). In what follows, we use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to represent finite (possibly empty) sets of formulas. Let A be a set of symbols (i.e., alphabet). Then, the notation $A^\star$ is used to represent the set of all words of finite length of the alphabet A. For any $\sharp \in \{\sim, -, \Box\}^\star$, we use an expression $\sharp\Gamma$ to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. We use the symbol Φ to denote the set of all propositional variables, the symbol Φ^* to denote the set of all formulas, and the symbols $\Phi^\sim$ and Φ^- to denote the sets $\{{\sim}p \mid p \in \Phi\}$ and $\{-p \mid p \in \Phi\}$, respectively. We use the symbol $\equiv$ to denote the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. We use an expression $\alpha \Leftrightarrow \beta$ to represent the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. An expression $L \vdash S$ means that a sequent S is provable in a sequent calculus L. If L of $L \vdash S$ is clear from the context, we omit L in it. We say that two sequent calculi L_1 and L_2 are *theorem-equivalent* if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: For any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Moreover, R is said to be *derivable* in L if there is a derivation from $S_1, \cdots, S_n$ to S in L. Note that a rule R of inference is admissible in a sequent calculus L if and only if two sequent calculi L and $L + R$ are theorem-equivalent.

A Gentzen-type sequent calculus M4CC for a modal extension of the ideal paraconsistent four-valued logic 4CC is defined as follows.

Definition 2.1 (M4CC). *The initial sequents of* M4CC *are of the following form, for any propositional variable* p,

$$p \Rightarrow p \qquad {\sim}p \Rightarrow {\sim}p \qquad -p \Rightarrow -p \qquad {\sim}p, -p \Rightarrow \qquad \Rightarrow {\sim}p, -p.$$

The structural inference rules of M4CC *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The non-negated logical inference rules of M4CC *are of the form:*

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\rightarrow}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta}\ ({\rightarrow}\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box\text{left}) \quad \frac{\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi \Rightarrow \alpha}{\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi \Rightarrow \Box\alpha}\ (\Box\text{right}).$$

The negated logical inference rules of M4CC *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\alpha}\ ({\sim}{\sim}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{{\sim}{-}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{-}\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}{-}\alpha}\ ({\sim}{-}\text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta \quad {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ ({\sim}\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta)}\ ({\sim}\wedge\text{right})$$

$$\frac{{\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim}\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta)}\ ({\sim}\vee\text{right})$$

$$\frac{\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}\Diamond\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\Diamond\text{left}) \quad \frac{\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi \Rightarrow {\sim}\alpha}{\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi \Rightarrow {\sim}\Diamond\alpha}\ ({\sim}\Diamond\text{right}).$$

The conflated logical inference rules of M4CC *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{--\alpha, \Gamma \Rightarrow \Delta}\ (--\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, --\alpha}\ (--\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{-{\sim}\alpha, \Gamma \Rightarrow \Delta}\ (-{\sim}\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -{\sim}\alpha}\ (-{\sim}\text{right})$$

$$\frac{-\alpha, -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (-\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)}\ (-\wedge\text{right})$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta \quad -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (-\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, -\alpha, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \vee \beta)}\ (-\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-{\rightarrow}\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha{\rightarrow}\beta)}\ (-{\rightarrow}\text{right})$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta}{-\Box\alpha, \Gamma \Rightarrow \Delta}\ (-\Box\text{left}) \quad \frac{\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi \Rightarrow -\alpha}{\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi \Rightarrow -\Box\alpha}\ (-\Box\text{right}).$$

A Gentzen-type sequent calculus M4CC* for a modal paradefinite logic, which is a subsystem of M4CC, is defined as follows.

Definition 2.2 (M4CC*). M4CC* *is obtained from* M4CC *by deleting the initial sequents of the following form, for any propositional variable p,*

$$\sim p, -p \Rightarrow \qquad \Rightarrow \sim p, -p.$$

Remark 2.3.

1. $(-{\rightarrow}\text{left})$ *and* $(-{\rightarrow}\text{right})$ *correspond to the Hilbert-style axiom scheme* $-(\alpha{\rightarrow}\beta) \leftrightarrow \alpha{\rightarrow}-\beta$, *which is a characteristic axiom scheme for* connexive logics [2, 26, 43].
2. *Based on the use of* $({\sim}-\text{left})$, $({\sim}-\text{right})$, $(-{\sim}\text{left})$, $(-{\sim}\text{right})$, *we can define the classical negation* $\neg\alpha$ *(i.e., the negation of classical logic) by* ${\sim}-\alpha$ *or* $-{\sim}\alpha$. *In the later section, we will also use the symbol* $\neg$ *to denote* ${\sim}-$.

3. *The* $\Box$*-free fragment of* M4CC *is theorem-equivalent to the Gentzen-type sequent calculus* $\mathrm{G}_{4\mathrm{CC}}$ *which was originally introduced by Arieli and Avron in* [5, 6] *for the ideal paraconsistent logic* 4CC [5, 6, 7]. *See* [23] *for the detail of the equivalence among related systems.*

4. $\mathrm{G}_{4\mathrm{CC}}$ [5, 6] *is obtained from the* $\Box$*-free fragment of* M4CC *by replacing* $(p \Rightarrow p)$, $({\sim}p \Rightarrow {\sim}p)$, $(-p \Rightarrow -p)$, $({\sim}p, -p \Rightarrow)$, $(\Rightarrow {\sim}p, -p)$, $(-\wedge\mathrm{left})$, $(-\wedge\mathrm{right})$, $(-\vee\mathrm{left})$, $(-\vee\mathrm{right})$, $(-{\rightarrow}\mathrm{left})$, $(-{\rightarrow}\mathrm{right})$, $(--\mathrm{left})$, $(--\mathrm{right})$, $(-{\sim}\mathrm{left})$, *and* $(-{\sim}\mathrm{right})$ *with* $\alpha \Rightarrow \alpha$, $(-\mathrm{left})$, *and* $(-\mathrm{right})$.

5. *The* $\Box$*-free fragment of* M4CC *is theorem-equivalent to the system which is obtained from* $\mathrm{G}_{4\mathrm{CC}}$ *by adding* $({\sim}\alpha, -\alpha \Rightarrow)$, $(\Rightarrow {\sim}\alpha, -\alpha)$, $({\sim}\mathrm{left})$ *and* $({\sim}\mathrm{right})$.

6. *The rules* $(\Box\mathrm{right})$, $({\sim}\Diamond\mathrm{right})$ *and* $(-\Box\mathrm{right})$ *are just generalizations of the standard S4-type rule* $(\Box\mathrm{right}^{\mathrm{S4}})$. *Indeed, the sequents* ${\sim}\Diamond\alpha \Leftrightarrow \Box{\sim}\alpha$ *and* $-\Box\alpha \Leftrightarrow \Box-\alpha$ *are provable in cut-free* M4CC. *Hence, the context* $\Box\Gamma, {\sim}\Diamond\Sigma, -\Box\Pi$ *in these rules can be interpreted as* $\Box\Gamma, \Box{\sim}\Sigma, \Box-\Pi$, *revealing thus its structure as a genuine generalization of* $\Box\Gamma$ *to formulas with* $\sim$ *and* $-$.

Proposition 2.4. *Let* L *be* M4CC *or* M4CC*. *Then, the following sequents are provable in cut-free* L *for any formulas* α *and* β*:*

1. $\alpha \Rightarrow \alpha$,
2. ${\sim}{\sim}\alpha \Leftrightarrow \alpha$,
3. ${\sim}-\alpha \Leftrightarrow -{\sim}\alpha$,
4. ${\sim}(\alpha \wedge \beta) \Leftrightarrow {\sim}\alpha \vee {\sim}\beta$,
5. ${\sim}(\alpha \vee \beta) \Leftrightarrow {\sim}\alpha \wedge {\sim}\beta$,
6. ${\sim}(\alpha{\rightarrow}\beta) \Leftrightarrow \alpha \wedge {\sim}\beta$,
7. $--\alpha \Leftrightarrow \alpha$,
8. $-(\alpha \wedge \beta) \Leftrightarrow -\alpha \wedge -\beta$,
9. $-(\alpha \vee \beta) \Leftrightarrow -\alpha \vee -\beta$,
10. $-(\alpha{\rightarrow}\beta) \Leftrightarrow \alpha{\rightarrow}-\beta$.

Proof. Straightforward. One can prove 1 by induction on α. **Q.E.D.**

Proposition 2.5. *The following sequents are provable in cut-free* M4CC *for any formulas α and β:*

1. ${\sim}\alpha, -\alpha \Rightarrow$,
2. $\Rightarrow {\sim}\alpha, -\alpha$,
3. ${\sim}\alpha \wedge -\alpha \Rightarrow \beta$ *(the principle of quasi-explosion),*
4. $\Rightarrow {\sim}\alpha \vee -\alpha$ *(the law of quasi-excluded middle).*

Proof. Straightforward. We can prove 1 and 2 by induction on α. **Q.E.D.**

Proposition 2.6. *Let $\Diamond\alpha$ be the abbreviation of ${\sim}{-}\Box{\sim}{-}$. The following sequents are provable in cut-free* M4CC *for any formulas α and β:*

1. ${\sim}\Diamond\alpha \Leftrightarrow \Box{\sim}\alpha$,
2. $-\Box\alpha \Leftrightarrow \Box{-}\alpha$.

Proof. We show only (1) below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Proposition } 2.5(2)\\ \Rightarrow {\sim}\alpha, -\alpha\end{array}}{-{\sim}{-}\alpha \Rightarrow {\sim}\alpha}\ (-{\sim}\text{left})}{-\Box{\sim}{-}\alpha \Rightarrow {\sim}\alpha}\ (-\Box\text{left})}{-\Box{\sim}{-}\alpha \Rightarrow \Box{\sim}\alpha}\ (\Box\text{right})}{{\sim}{\sim}{-}\Box{\sim}{-}\alpha \Rightarrow \Box{\sim}\alpha}\ ({\sim}{\sim}\text{left}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Proposition } 2.5(1)\\ {\sim}\alpha, -\alpha \Rightarrow\end{array}}{{\sim}\alpha \Rightarrow -{\sim}{-}\alpha}\ (-{\sim}\text{right})}{\Box{\sim}\alpha \Rightarrow -{\sim}{-}\alpha}\ (\Box\text{left})}{\Box{\sim}\alpha \Rightarrow -\Box{\sim}{-}\alpha}\ (-\Box\text{right})}{\Box{\sim}\alpha \Rightarrow {\sim}{\sim}{-}\Box{\sim}{-}\alpha}\ ({\sim}{\sim}\text{right})$$

Q.E.D.

Proposition 2.7. *The following rules are derivable in* M4CC *using* (cut)*:*

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha}{{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\text{left}) \quad \dfrac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ ({\sim}\text{right})$$

$$\dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{-\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}) \quad \dfrac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\alpha}\ (-\text{right}).$$

Proof. Straightforward. We show only the derivability of (${\sim}$left) as follows.

$$\dfrac{\Gamma \Rightarrow \Delta, -\alpha \quad \begin{array}{c}\vdots\ \text{Proposition. } 2.4\ (1)\\ {\sim}\alpha, -\alpha \Rightarrow\end{array}}{{\sim}\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut}).$$

Q.E.D.

Remark 2.8. *Proposition 2.7 will be used to show the negative symmetry property for* M4CC*. Proposition 2.7 can also be strengthened to the cut-free version after showing the cut-elimination theorem for* M4CC *(i.e., these rules are indeed admissible in cut-free* M4CC*).*

In order to show some syntactical embedding theorems, we introduce a Gentzen-type sequent calculus GS4 for the normal modal logic S4. Formulas of GS4 are constructed from countably many propositional variables by logical connectives $\wedge$, $\vee$, $\rightarrow$, $\Box$, and $\neg$ (classical negation). We use the symbol $\Diamond$ in GS4 to denote the abbreviation of $\neg\Box\neg$ (i.e., $\Diamond\alpha$ is the abbreviation of $\neg\Box\neg\alpha$).

Definition 2.9 (GS4). *The system* GS4 *is obtained from the* $\{\sim, -\}$*-free part of* M4CC* *by adding the modal inference rule of the form:*

$$\frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\mathrm{right}^{\mathrm{S4}})$$

and adding the classical negation inference rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\mathrm{right}).$$

Note that the modal inference rule ($\Box$left) *in* M4CC* *is also included in* GS4*.*

Remark 2.10. *We have the following well-known theorems for* GS4*. See e.g.,* [33, 34, 42]*.*

1. *(Cut-elimination for* GS4*): The rule* (cut) *is admissible in cut-free* GS4*.*
2. *(Decidability for* GS4*): The system* GS4 *is decidable.*

2.2. Kripke semantics

In what follows, we use the symbol $\neg$ to denote $\sim-$. We assume the commutativity of $\wedge$ or $\vee$. We have the following fact: for any formulas $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$,

$$\vdash \alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n \ \text{ iff } \ \vdash \alpha_1 \wedge \cdots \wedge \alpha_m \Rightarrow \beta_1 \vee \cdots \vee \beta_n.$$

Let Γ be a set $\{\alpha_1, ..., \alpha_m\}$ ($m \geq 0$). Then, we use an expression Γ^* to denote $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, or otherwise $\neg(p \rightarrow p)$ where p is a fixed propositional variable. We also use an expression Γ_* to denote $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, or otherwise $p{\to}p$ where p is a fixed propositional variable.

We now introduce Kripke semantics for M4CC and M4CC*.

Definition 2.11 (Kripke frame). *A structure $\langle M, R\rangle$ is called a Kripke frame if*

1. *M is a non-empty set,*
2. *R is a transitive and reflexive binary relation on M.*

Definition 2.12 (Paraconsistent M4CC-valuation). *A* paraconsistent M4CC-valuation $\models^*$ *on a Kripke frame $\langle M, R\rangle$ is a mapping from the set $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to the power set 2^M of M such that*

(*) $x \in \models^* (-p)$ *iff* $x \notin \models^* ({\sim}p)$.

We will write $x \models^* p$, $x \models^* {\sim}p$, and $x \models^* -p$ for $x \in \models^* (p)$, $x \in \models^* ({\sim}p)$, and $x \in \models^* (-p)$, respectively. We will also use the same notation as $x \models^* \alpha$ for an extended paraconsistent M4CC-valuation for any formula α. The paraconsistent M4CC-valuation $\models^*$ is extended to the mapping from the set Φ^* of all formulas to 2^M by:

1. $x \models^* \alpha \wedge \beta$ *iff* $x \models^* \alpha$ *and* $x \models^* \beta$,
2. $x \models^* \alpha \vee \beta$ *iff* $x \models^* \alpha$ *or* $x \models^* \beta$,
3. $x \models^* \alpha{\to}\beta$ *iff* $x \models^* \alpha$ *implies* $x \models^* \beta$,
4. $x \models^* \Box\alpha$ *iff* $\forall y \in M$ $[xRy$ *implies* $y \models^* \alpha]$,
5. $x \models^* {\sim}{\sim}\alpha$ *iff* $x \models^* \alpha$,
6. $x \models^* {\sim}{-}\alpha$ *iff* $x \not\models^* \alpha$,
7. $x \models^* {\sim}(\alpha \wedge \beta)$ *iff* $x \models^* {\sim}\alpha$ *or* $x \models^* {\sim}\beta$,
8. $x \models^* {\sim}(\alpha \vee \beta)$ *iff* $x \models^* {\sim}\alpha$ *and* $x \models^* {\sim}\beta$,
9. $x \models^* {\sim}(\alpha{\to}\beta)$ *iff* $x \models^* \alpha$ *and* $x \models^* {\sim}\beta$,
10. $x \models^* {\sim}\Box\alpha$ *iff* $\exists y \in M$ $[xRy$ *and* $y \models^* {\sim}\alpha]$,
11. $x \models^* {-}{-}\alpha$ *iff* $x \models^* \alpha$,
12. $x \models^* {-}{\sim}\alpha$ *iff* $x \not\models^* \alpha$,

13. $x \models^* -(\alpha \wedge \beta)$ *iff* $x \models^* -\alpha$ *and* $x \models^* -\beta$,

14. $x \models^* -(\alpha \vee \beta)$ *iff* $x \models^* -\alpha$ *or* $x \models^* -\beta$,

15. $x \models^* -(\alpha \to \beta)$ *iff* $x \models^* \alpha$ *implies* $x \models^* -\beta$,

16. $x \models^* -\Box\alpha$ *iff* $\forall y \in M$ $[xRy$ *implies* $y \models^* -\alpha]$.

Definition 2.13 (Paraconsistent M4CC*-valuation). *A* paraconsistent M4CC*-valuation $\models^*$ *on a Kripke frame* $\langle M, R\rangle$ *is a mapping from* $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ *to* 2^M. *Note that a paraconsistent M4CC*-valuation has no condition (*) in Defintion 2.12. The paraconsistent M4CC*-valuation* $\models^*$ *is extended to the mapping from* Φ^* *to* 2^M *by the clauses 1–16 in Definition 2.12.*

Definition 2.14 (Paraconsistent Kripke M4CC- and M4CC*-models). *Let* L *be* M4CC *or* M4CC*. *A* paraconsistent Kripke L-model *is a structure* $\langle M, R, \models^*\rangle$ *such that*

1. $\langle M, R\rangle$ *is a Kripke frame,*
2. $\models^*$ *is a paraconsistent L-valuation on* $\langle M, R\rangle$.

A formula α *is* true *in a paraconsistent Kripke L-model* $\langle M, R, \models^*\rangle$ *iff* $x \models^* \alpha$ *for any* $x \in M$, *and is* L-valid *(in a Kripke frame) iff it is true for every paraconsistent L-valuation* $\models^*$ *(on the Kripke frame). A sequent* $\Gamma \Rightarrow \Delta$ *is called* L-valid *(denoted as* $L \models \Gamma \Rightarrow \Delta$*) iff the formula* $\Gamma_* \to \Delta^*$ *is L-valid.*

Next, we show a characteristic theorem for the paraconsistent M4CC-valuation.

Theorem 2.15. *In Definition 2.12, the requirement* $(*)$, *together with clauses 11–16, can be replaced with the following single requirement:*

(**) $x \models^* -\alpha$ *iff* $x \not\models^* {\sim}\alpha$.

Proof.

$(\Longrightarrow)$: We prove that $(**)$ holds in every paraconsistent M4CC-valuation $\models^*$ on every Kripke frame $\langle M, R\rangle$, every $x \in M$ and every formula α. We do so by induction on α, and show some cases. We show only some cases for the first condition of (**). The cases for the second condition of (**) can be shown similarly.

1. Case $\alpha \equiv p \in \Phi$: If α is a propositional variable, then $(**)$ directly follows from $(*)$.
2. Case$\alpha \equiv \alpha_1 \wedge \alpha_2$: If $\alpha \equiv \alpha_1 \wedge \alpha_2$, then by clause 13 we have $x \models^* -\alpha$ iff both $x \models^* -\alpha_1$ and $x \models^* -\alpha_2$. By the induction hypothesis, the latter holds iff $x \not\models^* \sim\alpha_1$ and $x \not\models^* \sim\alpha_2$, which by clause 7 holds iff $x \not\models^* \sim\alpha$.
3. Case $\alpha \equiv \sim\beta$: If $\alpha \equiv \sim\beta$, then by clause 12 we have $x \models^* -\alpha$ iff $x \not\models^* \beta$, which, by clause 6 holds iff $x \not\models^* \sim\sim\beta = \sim\alpha$.
4. Case $\alpha \equiv -\beta$: If $\alpha \equiv -\beta$, then by clause 11 we have that $x \models^* -\alpha$ iff $x \models^* \beta$, which by clause 6 holds iff $x \not\models \sim-\beta \equiv \sim\alpha$.
5. Case $\alpha \equiv \Box\beta$: If $\alpha \equiv \Box\beta$, then by clause 16 we have $x \models^* -\alpha$ iff for every $y \in M$, xRy implies $y \models^* -\beta$. By the induction hypothesis, the latter holds iff for every $y \in M$, xRy implies $y \not\models^* \sim\beta$. By clause 10, this holds iff $x \not\models^* \sim\Box\beta \equiv \sim\alpha$.

$(\Longleftarrow)$: We prove that in every paraconsistent M4CC-valuation $\models^*$ on every Kripke frame $\langle M, R\rangle$, every $x \in M$ and every formula α, clauses $(*)$ and 11–16 hold, provided that $(**)$ holds. We explicitly show $(*)$, 11, 12, 13, and 16, leaving the rest to the reader.

$(*)$: $(*)$ is a particular instance of $(**)$ for the case of propositional variables.

(11): Using $(**)$ and 6, $x \models^* --\alpha$ iff $x \not\models^* \sim-\alpha$ iff $x \models^* \alpha$.

(12): Using $(**)$ and 5, $x \models^* -\sim\alpha$ iff $x \not\models^* \sim\sim\alpha$ iff $x \not\models \alpha$.

(13): Using $(**)$ and 7, $x \models^* -(\alpha_1 \wedge \alpha_2)$ iff $x \not\models^* \sim(\alpha_1 \wedge \alpha_2)$ iff $x \not\models^* \sim\alpha_1$ and $x \not\models^* \sim\alpha_2$ iff $x \models^* -\alpha_1$ and $x \models^* -\alpha_2$.

(16): Using $(**)$ and 10, $x \models^* -\Box\alpha$ iff $x \not\models^* \sim\Box\alpha$ iff $y \not\models^* \sim\alpha$ for every $y \in M$ such that xRy, iff $y \models^* -\alpha$ for every $y \in M$ such that xRy. **Q.E.D.**

In particular, we have the following corollary.

Corollary 2.16. *For any paraconsistent M4CC-valuation $\models^*$ on a Kripke frame $\langle M, R\rangle$, any $x \in M$, and any formula α,*

1. $x \models^* \sim\alpha$ *iff* $x \not\models^* -\alpha$,
2. $\models^* (\sim\alpha) \cap \models^* (-\alpha) = \emptyset$.

We can also obtain the following proposition.

Proposition 2.17. *The following formulas are M4CC-valid for any formulas α and β:*

1. $(\sim\alpha \wedge -\alpha) \to \beta$ *(the principle of quasi-explosion),*
2. $\sim\alpha \vee -\alpha$ *(the law of quasi-excluded middle).*

Proof. By using Corollary 2.16. **Q.E.D.**

In order to show some semantical embedding theorems, we introduce the standard Kripke semantics for GS4.

Definition 2.18 (Valuation for GS4). *A* valuation $\models$ *on a Kripke frame* $\langle M, R\rangle$ *is a mapping from* Φ *to* 2^M. *We will write* $x \models p$ *for* $x \in \models (p)$. *The valuation* $\models$ *is extended to a mapping from* Φ^* *to* 2^M *by:*

1. $x \models \alpha \wedge \beta$ *iff* $x \models \alpha$ *and* $x \models \beta$*,*
2. $x \models \alpha \vee \beta$ *iff* $x \models \alpha$ *or* $x \models \beta$*,*
3. $x \models \alpha \to \beta$ *iff* $x \models \alpha$ *implies* $x \models \beta$*,*
4. $x \models \neg\alpha$ *iff* $x \not\models \alpha$*,*
5. $x \models \Box\alpha$ *iff* $\forall y \in M$ $[xRy$ *implies* $y \models \alpha]$.

Proposition 2.19. *The following condition holds for* $\models$*:*

6. $x \models \Diamond\alpha$ *iff* $\exists y \in M$ $[xRy$ *and* $y \models \alpha]$.

Proof. Straightforward. **Q.E.D.**

Definition 2.20 (Kripke model for GS4). *A* Kripke model *is a structure* $\langle M, R, \models\rangle$ *such that*

1. $\langle M, R\rangle$ *is a Kripke frame,*
2. $\models$ *is a valuation on* $\langle M, R\rangle$.

A formula α is true *in a Kripke model $\langle M, R, \models\rangle$ iff $x \models \alpha$ for any $x \in M$, and is* GS4-valid *in a Kripke frame $\langle M, R\rangle$ iff it is true for every valuation $\models$ on the Kripke frame. A sequent $\Gamma \Rightarrow \Delta$ is called* GS4-valid *(denoted as* GS4 $\models \Gamma \Rightarrow \Delta$*) iff the formula $\Gamma_* \to \Delta^*$ is GS4-valid.*

Remark 2.21. *We have the following well-known theorems for* GS4*. See e.g.,* [25, 18].

1. *(Completeness for* GS4*): The following completeness theorem holds for* GS4 *for any sequent $\Gamma \Rightarrow \Delta$:* GS4 $\vdash \Gamma \Rightarrow \Delta$ *iff* GS4 $\models \Gamma \Rightarrow \Delta$.
2. *(Finite model property for* GS4*): The following finite model property holds for* GS4 *for any sequent $\Gamma \Rightarrow \Delta$: $\Gamma \Rightarrow \Delta$ is GS4-valid in any finite Kripke frame iff* GS4 $\vdash \Gamma \Rightarrow \Delta$.

3. Main theorems

3.1. Syntactical embedding and cut-elimination theorems for M4CC*

Next, we introduce a GS4-translation function for formulas of M4CC*, and by using this translation, we show several theorems for embedding M4CC* into GS4.

Definition 3.1. *We fix a set Φ of propositional variables, and define the sets $\Phi^n := \{p^n \mid p \in \Phi\}$ and $\Phi^c := \{p^c \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{M4CC}^\star}$ of* M4CC* *is defined using Φ, $\wedge, \vee, \to, \Box, \sim$, and $-$. The language $\mathcal{L}_{\mathrm{GS4}}$ of* GS4 *is defined using Φ, Φ^n, Φ^c, $\wedge$, $\vee$, $\to$, $\Box$, and $\neg$. A mapping f from $\mathcal{L}_{\mathrm{M4CC}^\star}$ to $\mathcal{L}_{\mathrm{GS4}}$ is defined inductively by:*

1. *For any $p \in \Phi$, $f(p) := p$, $f({\sim}p) := p^n \in \Phi^n$ and $f(-p) := p^c \in \Phi^c$,*
2. $f(\alpha \wedge \beta) := f(\alpha) \wedge f(\beta)$,
3. $f(\alpha \vee \beta) := f(\alpha) \vee f(\beta)$,
4. $f(\alpha \to \beta) := f(\alpha) \to f(\beta)$,
5. $f(\Box\alpha) := \Box f(\alpha)$,
6. $f({\sim}(\alpha \wedge \beta)) := f({\sim}\alpha) \vee f({\sim}\beta)$,

7. $f(\sim(\alpha \vee \beta)) := f(\sim\alpha) \wedge f(\sim\beta)$,
8. $f(\sim(\alpha \rightarrow \beta)) := f(\alpha) \wedge f(\sim\beta)$,
9. $f(\sim\sim\alpha) := f(\alpha)$,
10. $f(\sim--\alpha) := f(\sim\alpha)$,
11. $f(\sim-\alpha) := \neg f(\alpha)$,
12. $f(\sim\Box\alpha) := \Diamond f(\sim\alpha)$,
13. $f(-(\alpha \wedge \beta)) := f(-\alpha) \wedge f(-\beta)$,
14. $f(-(\alpha \vee \beta)) := f(-\alpha) \vee f(-\beta)$,
15. $f(-(\alpha \rightarrow \beta)) := f(\alpha) \rightarrow f(-\beta)$,
16. $f(--\alpha) := f(\alpha)$,
17. $f(-\sim\sim\alpha) := f(-\alpha)$,
18. $f(-\sim\alpha) := \neg f(\alpha)$,
19. $f(-\Box\alpha) := \Box f(-\alpha)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. Analogous notation is used for the other mapping g discussed later.

Remark 3.2. *A similar translation has been used by Gurevich* [17]*, Rautenberg* [37]*, and Vorob'ev* [41] *to embed Nelson's constructive logic* [1, 27] *into intuitionistic logic. Some similar translations have also recently been used, for example, in* [21, 22, 23] *to embed some paraconsistent logics into classical logic.*

Proposition 3.3. *Let* $\Diamond$ *in* M4CC* *be the abbreviation of* $\sim-\Box\sim-$*. Then, the following condition holds for* f*:*

$f(\sim\Diamond\alpha) := \Box f(\sim\alpha)$.

Proof. We show this proposition as follows. $f(\sim\Diamond\alpha) = f(\sim\sim-\Box\sim-\alpha) = f(-\Box\sim-\alpha) = \Box f(-\sim-\alpha) = \Box\neg f(-\alpha) = \Box f(\sim--\alpha) = \Box f(\sim\alpha)$. Note that the last equivalence is derived from condition 10 of f. **Q.E.D.**

We now show a weak theorem for syntactically embedding M4CC* into GS4.

Theorem 3.4 (Weak syntactical embedding from M4CC* into GS4). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{M4CC}^\star}$, and f be the mapping defined in Definition 3.1.*

1. *If* M4CC* $\vdash \Gamma \Rightarrow \Delta$, *then* GS4 $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *If* GS4 $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$, *then* M4CC* $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in M4CC*. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ${\sim}p \Rightarrow {\sim}p$: The last inference of P is of the form: ${\sim}p \Rightarrow {\sim}p$ for any $p \in \Phi$. In this case, we obtain GS4 $\vdash f({\sim}p) \Rightarrow f({\sim}p)$, i.e., GS4 $\vdash p^n \Rightarrow p^n$ ($p^n \in \Phi^n$), by the definition of f.
2. Case (${\sim}{-}$left): The last inference of P is of the form:
$$\frac{\Gamma \Rightarrow \Delta, \alpha}{{\sim}{-}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{-}\text{left}).$$
By induction hypothesis, we have GS4 $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$. Then, we obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array}}{\neg f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$
where $\neg f(\alpha)$ coincides with $f({\sim}{-}\alpha)$ by the definition of f.
3. Case (${-}{\to}$left): The last inference of P is of the form:
$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha{\to}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({-}{\to}\text{left}).$$
By induction hypothesis, we have GS4 $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$ and GS4 $\vdash$ $f(-\beta), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f(-\beta), f(\Sigma) \Rightarrow f(\Pi)\end{array}}{f(\alpha){\to}f(-\beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ ({\to}\text{left})$$
where $f(\alpha){\to}f(-\beta)$ coincides with $f(-(\alpha{\to}\beta))$ by the definition of f.

4. Case ($\Box$right): The last inference of P is of the form:

$$\frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Box\alpha}\ (\Box\text{right}).$$

By induction hypothesis, we have GS4 $\vdash f(\Box\Gamma), f(\sim\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(\alpha)$ where $f(\Box\Gamma)$, $f(\sim\Diamond\Sigma)$ and $f(-\Box\Pi)$ coincide with $\Box f(\Gamma)$, $\Box f(\sim\Sigma)$ and $\Box f(-\Pi)$, respectively, by the definition of f and Proposition 3.3. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box f(\Gamma), \Box f(\sim\Sigma), \Box f(-\Pi) \Rightarrow f(\alpha)\end{array}}{\Box f(\Gamma), \Box f(\sim\Sigma), \Box f(-\Pi) \Rightarrow \Box f(\alpha)}\ (\Box\text{right}^{\text{S4}})$$

where $\Box f(\alpha)$ coincides with $f(\Box\alpha)$ by the definition of f.

5. Case ($-\Box$right): The last inference of P is of the form:

$$\frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\Box\alpha}\ (-\Box\text{right}).$$

By induction hypothesis, we have GS4 $\vdash f(\Box\Gamma), f(\sim\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(-\alpha)$ where $f(\Box\Gamma)$, $f(\sim\Diamond\Sigma)$ and $f(-\Box\Pi)$ coincide with $\Box f(\Gamma)$, $\Box f(\sim\Sigma)$ and $\Box f(-\Pi)$, respectively, by the definition of f and Proposition 3.3. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box f(\Gamma), \Box f(\sim\Sigma), \Box f(-\Pi) \Rightarrow f(-\alpha)\end{array}}{\Box f(\Gamma), \Box f(\sim\Sigma), \Box f(-\Pi) \Rightarrow \Box f(-\alpha)}\ (\Box\text{right}^{\text{S4}})$$

where $\Box f(-\alpha)$ coincides with $f(-\Box\alpha)$ by the definition of f.

• (2): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in GS4 $-$ (cut). We distinguish the cases according to the last inference of Q, and show some cases.

1. Case ($\neg$left): The last inference of Q is ($\neg$left).

   (a) Subcase (1): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha)}{f(\sim-\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$

where $f(\sim\!-\alpha)$ coincides with $\neg f(\alpha)$ by the definition of f. By induction hypothesis, we have M4CC* $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{matrix}}{\sim\!-\alpha, \Gamma \Rightarrow \Delta}\ (\sim\!-\text{left}).$$

(b) Subcase (2): The last inference of Q is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha)}{f(-\!\sim\!\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$

where $f(-\!\sim\!\alpha)$ coincides with $\neg f(\alpha)$ by the definition of f. By induction hypothesis, we have M4CC* $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{matrix}}{-\!\sim\!\alpha, \Gamma \Rightarrow \Delta}\ (-\!\sim\!\text{left}).$$

2. Case ($\Box$right$^{\text{S4}}$): The last inference of Q is ($\Box$right$^{\text{S4}}$).

(a) Subcase (1): The last inference of Q is of the form:

$$\dfrac{f(\Box\Gamma), f(\sim\!\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(\alpha)}{f(\Box\Gamma), f(\sim\!\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(\Box\alpha)}\ (\Box\text{right}^{\text{S4}})$$

where $f(\Box\Gamma)$, $f(\sim\!\Diamond\Sigma)$, $f(-\Box\Pi)$ and $f(\Box\alpha)$ coincide with $\Box f(\Gamma)$, $\Box f(\sim\!\Sigma)$, $\Box f(-\Pi)$ and $\Box f(\alpha)$, respectively, by the definition of f and Proposition 3.3. By induction hypothesis, we have M4CC* $-$ (cut) $\vdash \Box\Gamma, \sim\!\Diamond\Sigma, -\Box\Pi \Rightarrow \alpha$. We thus obtain the required fact:

$$\dfrac{\begin{matrix}\vdots\\ \Box\Gamma, \sim\!\Diamond\Sigma, -\Box\Pi \Rightarrow \alpha\end{matrix}}{\Box\Gamma, \sim\!\Diamond\Sigma, -\Box\Pi \Rightarrow \Box\alpha}\ (\Box\text{right}).$$

(b) Subcase (2): The last inference of Q is of the form:

$$\dfrac{f(\Box\Gamma), f(\sim\!\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(\sim\!\alpha)}{f(\Box\Gamma), f(\sim\!\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(\sim\!\Diamond\alpha)}\ (\Box\text{right}^{\text{S4}})$$

where $f(\Box\Gamma)$, $f(\sim\Diamond\Sigma)$, $f(-\Box\Pi)$ and $f(\sim\Diamond\alpha)$ coincide with $\Box f(\Gamma)$, $\Box f(\sim\Sigma)$, $\Box f(-\Pi)$ and $\Box f(\sim\alpha)$, respectively, by the definition of f and Proposition 3.3. By induction hypothesis, we have $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash \Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \sim\alpha$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \sim\alpha\end{array}}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \sim\Diamond\alpha}\ (\sim\Diamond\mathrm{right}).$$

(c) Subcase (3): The last inference of Q is of the form:

$$\dfrac{f(\Box\Gamma), f(\sim\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(-\alpha)}{f(\Box\Gamma), f(\sim\Diamond\Sigma), f(-\Box\Pi) \Rightarrow f(-\Box\alpha)}\ (\Box\mathrm{right}^{\mathrm{S4}})$$

where $f(\Box\Gamma)$, $f(\sim\Diamond\Sigma)$, $f(-\Box\Pi)$ and $f(-\Box\alpha)$ coincide with $\Box f(\Gamma)$, $\Box f(\sim\Sigma)$, $\Box f(-\Pi)$ and $\Box f(-\alpha)$, respectively, by the definition of f and Proposition 3.3. By induction hypothesis, we have $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash \Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\alpha$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\alpha\end{array}}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\Box\alpha}\ (-\Box\mathrm{right}).$$

Q.E.D.

Using Theorem 3.4 and the cut-elimination theorem for GS4, we obtain the following cut-elimination theorem for $\mathrm{M4CC}^\star$.

Theorem 3.5 (Cut-elimination for $\mathrm{M4CC}^\star$). *The rule* (cut) *is admissible in cut-free* $\mathrm{M4CC}^\star$*.*

Proof. Suppose $\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$. Then, we have $\mathrm{GS4} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.4 (1), and hence $\mathrm{GS4} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for GS4. By Theorem 3.4 (2), we obtain $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

Q.E.D.

Using Theorem 3.4 and the cut-elimination theorem for GS4, we obtain a strong theorem for syntactically embedding $\mathrm{M4CC}^\star$ into GS4.

Theorem 3.6 (Syntactical embedding from M4CC* into GS4). *Let* Γ*,* Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{M4CC}^\star}$*, and* f *be the mapping defined in Definition 3.1.*

1. $\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. • (1): ($\Longrightarrow$): By Theorem 3.4 (1). ($\Longleftarrow$): Suppose $\mathrm{GS4} \vdash f(\Gamma) \Rightarrow f(\Delta)$. Then we have $\mathrm{GS4} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for GS4. We thus obtain $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ by Theorem 3.4 (2). Therefore we have $\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$.

• (2): ($\Longrightarrow$): Suppose $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. Then we have $\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$. We then obtain $\mathrm{GS4} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.4 (1). Therefore we obtain $\mathrm{GS4} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for GS4. ($\Longleftarrow$): By Theorem 3.4 (2). **Q.E.D.**

Theorem 3.7 (Decidability for M4CC*). *The system* M4CC* *is decidable.*

Proof. By decidability of GS4, for each α, it is possible to decide if $f(\alpha)$ is provable in GS4. Then, by Theorem 3.6, M4CC* is also decidable. **Q.E.D.**

Using Theorem 3.5, we can obtain some characteristic properties of M4CC*, which do not hold for M4CC. Such properties are defined as follows.

Definition 3.8.

1. *A sequent system* L *is called* quasi-explosive *with respect to the combination of two different negation-like connectives* $\sharp$ *and* $\natural$ *if* $L \vdash \sharp\alpha, \natural\alpha \Rightarrow \beta$ *for any formulas* α *and* β*. A sequent system* L *is called* quasi-paraconsistent *with respect to the combination of* $\sharp$ *and* $\natural$ *if* L *is not quasi-explosive with respect to the combination of* $\sharp$ *and* $\natural$*.*
2. *A sequent system* L *is called* quasi-exclusive *with respect to the combination of two different negation-like connectives* $\sharp$ *and* $\natural$ *if* $L \vdash \Rightarrow \sharp\alpha, \natural\alpha$ *for any formula* α*. A sequent system* L *is called* quasi-paracomplete *with respect to the combination of* $\sharp$ *and* $\natural$ *if* L *is not quasi-exclusive with respect to the combination of* $\sharp$ *and* $\natural$*.*

Remark 3.9. *The quasi-paraconsistency and quasi-paracompleteness represent the relationship between* $\sim$ *and* $-$*, and are regarded as analogues of the paraconsistency and paracompleteness, which reject the axiom schemes* $(\sim\alpha \wedge \alpha)\to\beta$ *(the principle of explosion) and* $\sim\alpha \vee \alpha$ *(the law of excluded middle), respectively. The quasi-paraconsistency and quasi-paracompleteness reject the axiom schemes* $(\sim\alpha \wedge -\alpha)\to\beta$ *(the principle of quasi-explosion) and* $\sim\alpha \vee -\alpha$ *(the law of quasi-excluded middle), respectively.*

Theorem 3.10 (Quasi-paraconsistency and quasi-paracompleteness for M4CC*). *We have:*

1. *The system* M4CC* *is quasi-paraconsistent with respect to the combination of* $\sim$ *and* $-$*.*
2. *The system* M4CC* *is quasi-paracomplete with respect to the combination of* $\sim$ *and* $-$*.*

Proof. We show only (1) below. Consider sequent $\sim p, -p \Rightarrow q$ where p and q are distinct propositional variables. Then, the unprovability of this sequent is guaranteed by Theorem 3.5. **Q.E.D.**

Next, we introduce an M4CC*-translation function for formulas of GS4, and by using this translation, we show some theorems for embedding GS4 into M4CC*.

Definition 3.11. *Let* $\mathcal{L}_{\mathrm{M4CC}^\star}$ *and* $\mathcal{L}_{\mathrm{GS4}}$ *be the languages defined in Definition 3.1. A mapping* g *from* $\mathcal{L}_{\mathrm{GS4}}$ *to* $\mathcal{L}_{\mathrm{M4CC}^\star}$ *is defined inductively by:*

1. *For any* $p \in \Phi$*, any* $p^n \in \Phi^n$ *and any* $p^c \in \Phi^c$*,* $g(p) := p$*,* $g(p^n) := \sim p$ *and* $g(p^c) := -p$*,*
2. $g(\alpha \wedge \beta) := g(\alpha) \wedge g(\beta)$*,*
3. $g(\alpha \vee \beta) := g(\alpha) \vee g(\beta)$*,*
4. $g(\alpha\to\beta) := g(\alpha)\to g(\beta)$*,*
5. $g(\neg\alpha) := \sim -g(\alpha)$*,*

6. $g(\Box\alpha) := \Box g(\alpha)$.

Theorem 3.12 (Weak syntactical embedding from GS4 into M4CC*). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{GS4}}$, and g be the mapping defined in Definition 3.11.*

1. *If* $\mathrm{GS4} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{M4CC}^\star \vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. *If* $\mathrm{M4CC}^\star - (\mathrm{cut}) \vdash g(\Gamma) \Rightarrow g(\Delta)$*, then* $\mathrm{GS4} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in GS4. We distinguish the cases according to the last inference of P, and show only the following cases.

1. Case $p^\star \Rightarrow p^\star$ with $\star \in \{n, c\}$: The last inference of P is of the form: $p^\star \Rightarrow p^\star$ for any $p^\star \in \Phi^\star$ with $\star \in \{n, c\}$. In this case, we obtain $\mathrm{M4CC}^\star \vdash g(p^n) \Rightarrow g(p^n)$ and $\mathrm{M4CC}^\star \vdash g(p^c) \Rightarrow g(p^c)$ i.e., $\mathrm{M4CC}^\star \vdash {\sim}p \Rightarrow {\sim}p$ and $\mathrm{M4CC}^\star \vdash -p \Rightarrow -p$, by the definition of g.
2. Case ($\neg$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left})$$

   By induction hypothesis, we have $\mathrm{M4CC}^\star \vdash g(\Gamma) \Rightarrow g(\Delta), g(\alpha)$. We then obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ g(\Gamma) \Rightarrow g(\Delta), g(\alpha)\end{array}}{{\sim}{-}g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ ({\sim}{-}\text{left})$$

   where ${\sim}{-}g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g.
3. Case ($\Box$right$^{\mathrm{S4}}$): The last inference of P is of the form:

$$\frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{right}^{\mathrm{S4}})$$

   By induction hypothesis, we have $\mathrm{M4CC}^\star \vdash g(\Box\Gamma) \Rightarrow g(\alpha)$ where $g(\Box\Gamma)$ coincides with $\Box g(\Gamma)$ by the definition of g. We then obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box g(\Gamma) \Rightarrow g(\alpha)\end{array}}{\Box g(\Gamma) \Rightarrow \Box g(\alpha)}\ (\Box\text{right})$$

where $\Box g(\alpha)$ coincides with $g(\Box\alpha)$ by the definition of g.

• (2): By induction on the proofs Q of $g(\Gamma) \Rightarrow g(\Delta)$ in M4CC* − (cut). We distinguish the cases according to the last inference of Q, and show only the following cases.

1. Case ($\sim\!-$left): The last inference of Q is of the form:

$$\frac{g(\Gamma) \Rightarrow g(\Delta), g(\alpha)}{\sim\!-g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ (\sim\!-\text{left})$$

   where $\sim\!-g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g. By induction hypothesis, we have GS4 − (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array}}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}).$$

2. Case ($\Box$right): The last inference of Q is of the form:

$$\frac{g(\Box\Gamma) \Rightarrow g(\alpha)}{g(\Box\Gamma) \Rightarrow \Box g(\alpha)}\ (\Box\text{right})$$

   where $g(\Box\Gamma)$ and $\Box g(\alpha)$ coincide with $\Box g(\Gamma)$ and $g(\Box\alpha)$, respectively, by the definition of g. By induction hypothesis, we have GS4 − (cut) $\vdash$ $\Box\Gamma \Rightarrow \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box\Gamma \Rightarrow \alpha\end{array}}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{right}^{\text{S4}}).$$

Q.E.D.

Theorem 3.13 (Syntactical embedding from GS4 into M4CC*). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{GS4}}$, *and* g *be the mapping defined in Definition 3.11.*

1. GS4 $\vdash \Gamma \Rightarrow \Delta$ *iff* M4CC* $\vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. GS4 − (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* M4CC* − (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$.

Proof. By using Theorems 3.12 and 3.5. Similar to Theorem 3.6. **Q.E.D.**

Next, we show a theorem for semantically embedding M4CC* into GS4. Prior to prove the semantical embedding theorem, we need to show some lemmas.

Lemma 3.14. *Let f be the mapping defined in Definition 3.1. For any paraconsistent Kripke M4CC*-model $\langle M, R, \models^* \rangle$, we can construct a Kripke model $\langle M, R, \models \rangle$ such that for any formula α and any $x \in M$,*

$x \models^* \alpha$ *iff* $x \models f(\alpha)$.

Proof. Suppose that $\langle M, R, \models^* \rangle$ is a paraconsistent Kripke M4CC*-model where $\models^*$ is a mapping from $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to 2^M. Suppose that $\langle M, R, \models \rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \Phi^n \cup \Phi^c$ to 2^M such that for any $x \in M$ and any $q \in \Phi$,

1. $x \models^* q$ iff $x \models q$,
2. $x \models^* {\sim}q$ iff $x \models q^n$,
3. $x \models^* -q$ iff $x \models q^c$.

Then, the lemma is proved by induction on α.

- Base step:

1. Case when $\alpha \equiv q$ where q is a propositional variable: $x \models^* q$ iff $x \models q$ (by the assumption) iff $x \models f(q)$ (by the definition of f).
2. Case when $\alpha \equiv {\sim}q$ where q is a propositional variable: $x \models^* {\sim}q$ iff $x \models q^n$ (by the assumption) iff $x \models f({\sim}q)$ (by the definition of f).
3. Case when $\alpha \equiv -q$ where q is a propositional variable: $x \models^* -q$ iff $x \models q^c$ (by the assumption) iff $x \models f(-q)$ (by the definition of f).

- Induction step: We show some cases.

1. Case $\alpha \equiv \beta \wedge \gamma$: $x \models^* \beta \wedge \gamma$ iff $x \models^* \beta$ and $x \models^* \gamma$ iff $x \models f(\beta)$ and $x \models f(\gamma)$ (by induction hypothesis) iff $x \models f(\beta) \wedge f(\gamma)$ iff $x \models f(\beta \wedge \gamma)$ (by the definition of f).

2. Case when $\alpha \equiv \beta \to \gamma$: $x \models^* \beta \to \gamma$ iff $x \models^* \beta$ implies $x \models^* \gamma$ iff $x \models f(\beta)$ implies $x \models f(\gamma)$ (by induction hypothesis) iff $x \models f(\beta) \to f(\gamma)$ iff $x \models f(\beta \to \gamma)$ (by the definition of f).
3. Case $\alpha \equiv \Box\beta$: $x \models^* \Box\beta$ iff $\forall y \in M[xRy$ implies $y \models^* \beta]$ iff $\forall y \in M[xRy$ implies $y \models f(\beta)]$ (by induction hypothesis) iff $x \models \Box f(\beta)$ iff $x \models f(\Box\beta)$ (by the definition of f).
4. Case $\alpha \equiv {\sim}{\sim}\beta$: $x \models^* {\sim}{\sim}\beta$ iff $x \models^* \beta$ iff $x \models f(\beta)$ (by induction hypothesis) iff $x \models f({\sim}{\sim}\beta)$ (by the definition of f).
5. Case $\alpha \equiv {\sim}(\beta \wedge \gamma)$: $x \models^* {\sim}(\beta \wedge \gamma)$ iff $x \models^* {\sim}\beta$ or $x \models^* {\sim}\gamma$ iff $x \models f({\sim}\beta)$ or $x \models f({\sim}\gamma)$ (by induction hypothesis) iff $x \models f({\sim}\beta) \vee f({\sim}\gamma)$ iff $x \models f({\sim}(\beta \wedge \gamma))$ (by the definition of f).
6. Case when $\alpha \equiv {\sim}(\beta \to \gamma)$: $x \models^* {\sim}(\beta \to \gamma)$ iff $x \models^* \beta$ and $x \models^* {\sim}\gamma$ iff $x \models f(\beta)$ and $x \models f({\sim}\gamma)$ (by induction hypotheses) iff $x \models f(\beta) \wedge f({\sim}\beta)$ iff $x \models f({\sim}(\beta \to \gamma))$ (by the definition of f).
7. Case $\alpha \equiv {\sim}\Box\beta$: $x \models^* {\sim}\Box\beta$ iff $\exists y \in M[xRy$ and $y \models^* {\sim}\beta]$ iff $\exists y \in M[xRy$ and $y \models f({\sim}\beta)]$ (by induction hypothesis) iff $x \models \Diamond f({\sim}\beta)$ (By Proposition 2.19) iff $x \models f({\sim}\Box\beta)$ (by the definition of f).
8. Case $\alpha \equiv {\sim}{-}\beta$: $x \models^* {\sim}{-}\beta$ iff $x \not\models^* (\beta)$ iff $x \not\models f(\beta)$ (by induction hypothesis) iff $x \models \neg f(\beta)$ iff $x \models f({\sim}{-}\beta)$ (by the definition of f).
9. Case when $\alpha \equiv -(\beta \wedge \gamma)$: $x \models^* -(\beta \wedge \gamma)$ iff $x \models^* -\beta$ and $x \models^* -\gamma$ iff $x \models f(-\beta)$ and $x \models f(-\gamma)$ (by induction hypothesis) iff $x \models f(-\beta) \wedge f(-\gamma)$ iff $x \models f(-(\beta \wedge \gamma))$ (by the definition of f).
10. Case $\alpha \equiv -(\beta \to \gamma)$: $x \models^* -(\beta \to \gamma)$ iff $x \models^* \beta$ implies $x \models^* -\gamma$ iff $x \models f(\beta)$ implies $x \models f(-\gamma)$ (by induction hypothesis) iff $x \models f(\beta) \to f(-\gamma)$ iff $x \models f(-(\beta \to \gamma))$ (by the definition of f).
11. Case $\alpha \equiv -\Box\beta$: $x \models^* -\Box\beta$ iff $\forall y \in M[xRy$ implies $y \models^* -\beta]$ iff $\forall y \in M[xRy$ implies $y \models f(-\beta)]$ (by induction hypothesis) iff $x \models \Box f(-\beta)$ iff $x \models f(-\Box\beta)$ (by the definition of f). **Q.E.D.**

Lemma 3.15. *Let f be the mapping defined in Definition 3.1. For any Kripke model $\langle M, R, \models\rangle$, we can construct a paraconsistent Kripke M4CC*-*

model $\langle M, R, \models^* \rangle$ *such that for any formula* α *and any* $x \in M$,

$x \models f(\alpha)$ *iff* $x \models^* \alpha$.

Proof. Similar to the proof of Lemma 3.14. **Q.E.D.**

Theorem 3.16 (Semantical embedding from M4CC* into GS4). *Let* f *be the mapping defined in Definition 3.1. For any sequent* $\Gamma \Rightarrow \Delta$,

$\mathrm{M4CC}^\star \models \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4} \models f(\Gamma) \Rightarrow f(\Delta)$.

Proof. By Lemmas 3.14 and 3.15. **Q.E.D.**

Theorem 3.17 (Kripke-completeness for M4CC*). *For any sequent* $\Gamma \Rightarrow \Delta$,

$\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{M4CC}^\star \models \Gamma \Rightarrow \Delta$.

Proof. We have the following. $\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$ iff $\mathrm{GS4} \vdash f(\Gamma) \Rightarrow f(\Delta)$ (by Theorem 3.6) iff $\mathrm{G4S} \models f(\Gamma) \Rightarrow f(\Delta)$ (by the completeness theorem for GS4) iff $\mathrm{M4CC}^\star \models \Gamma \Rightarrow \Delta$ (by Theorem 3.16). **Q.E.D.**

Theorem 3.18 (Finite model property for M4CC*). *For any sequent* $\Gamma \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta$ *is M4CC*-valid in any finite Kripke frame iff* $\mathrm{M4CC}^\star \vdash \Gamma \Rightarrow \Delta$.

Proof. We can modify Lemmas 3.14 and 3.15 for finite models. Using such modified lemmas and the finite model property for GS4, we can obtain the required property. **Q.E.D.**

Next, we show a theorem for semantically embedding GS4 into M4CC*.

Lemma 3.19. *Let* g *be the mapping defined in Definition 3.11. For any Kripke model* $\langle M, R, \models \rangle$, *we can construct a paraconsistent Kripke M4CC*-model* $\langle M, R, \models^* \rangle$ *such that for any formula* α *and any* $x \in M$,

$x \models \alpha$ *iff* $x \models^* g(\alpha)$.

Proof. Suppose that $\langle M, R, \models\rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \Phi^n \cup \Phi^c$ to 2^M. Suppose that $\langle M, R, \models^*\rangle$ is a paraconsistent Kripke M4CC*-model where $\models^*$ is a mapping from $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to 2^M such that for any $x \in M$ and any $q \in \Phi$,

1. $x \models^* q$ iff $x \models q$,
2. $x \models^* {\sim}q$ iff $x \models q^n$,
3. $x \models^* -q$ iff $x \models q^c$.

Then, the lemma is proved by induction on α.

- Base step:

1. Case $\alpha \equiv p$ where q is a propositional variable: $x \models q$ iff $x \models^* q$ (by the assumption) iff $x \models^* g(q)$ (by the definition of g).
2. Case $\alpha \equiv q^n$ where q^n is a propositional variable in Φ^n: $x \models q^n$ iff $x \models^* {\sim}q$ (by the assumption) iff $x \models^* g(q^n)$ (by the definition of g).
3. Case $\alpha \equiv q^c$ where q^c is a propositional variable in Φ^c: $x \models q^c$ iff $x \models^* -q$ (by the assumption) iff $x \models^* g(q^c)$ (by the definition of g).

- Induction step: We show some cases.

1. Case $\alpha \equiv \beta \wedge \gamma$: $x \models \beta \wedge \gamma$ iff $x \models \beta$ and $x \models \gamma$ iff $x \models^* g(\beta)$ and $x \models^* g(\gamma)$ (by induction hypothesis) iff $x \models^* g(\beta) \wedge g(\gamma)$ iff $x \models^* g(\beta \wedge \gamma)$ (by the definition of g).
2. Case $\alpha \equiv \neg\beta$: $x \models \neg\beta$ iff $x \not\models \beta$ iff $x \not\models^* g(\beta)$ (by induction hypothesis) iff $x \models^* {\sim}{-}g(\beta)$ iff $x \models^* g(\neg\beta)$ (by the definition of g).
3. Case $\alpha \equiv \Box\beta$: $x \models \Box\beta$ iff $\forall y \in M[xRy$ implies $y \models \beta]$ iff $\forall y \in M[xRy$ implies $y \models^* g(\beta)]$ (by induction hypothesis) iff $x \models^* \Box g(\beta)$ iff $x \models g(\Box\beta)$ (by the definition of g). **Q.E.D.**

Lemma 3.20. *Let g be the mapping defined in Definition 3.11. For any paraconsistent Kripke M4CC*-model $\langle M, R, \models^*\rangle$, we can construct a Kripke model $\langle M, R, \models\rangle$ such that for any formula α and any $x \in M$,*

$x \models^* g(\alpha)$ *iff* $x \models \alpha$.

Proof. Similar to the proof of Lemma 3.19. **Q.E.D.**

Theorem 3.21 (Semantical embedding from GS4 into M4CC*). *Let g be the mapping defined in Definition 3.11. For any sequent $\Gamma \Rightarrow \Delta$,*

$$\text{GS4} \models \Gamma \Rightarrow \Delta \text{ iff } \text{M4CC}^\star \models g(\Gamma) \Rightarrow g(\Delta).$$

Proof. By Lemmas 3.19 and 3.20. **Q.E.D.**

3.3. Theorems for M4CC

Next, we introduce a GS4-translation function for formulas of M4CC, and by using this translation, we show several theorems for embedding M4CC into GS4.

Definition 3.22. *We fix a set Φ of propositional variables, and define the set $\Phi^n := \{p^n \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\text{M4CC}}$ of* M4CC *is the same as that of* M4CC**, i.e., it is defined using Φ, $\wedge, \vee, \rightarrow, \Box, \sim$, and $-$. The new alternative language $\mathcal{L}^*_{\text{GS4}}$ of* GS4 *is defined using Φ, Φ^n, $\wedge$, $\vee$, $\rightarrow$, $\Box$, and $\neg$. Note that Φ^c is not used in $\mathcal{L}^*_{\text{GS4}}$, which differs from Definition 3.1 for* M4CC**.*

A mapping f from $\mathcal{L}_{\text{M4CC}}$ to $\mathcal{L}^*_{\text{GS4}}$ is defined inductively by the conditions 2–17 in Definition 3.1 and the following new condition:

1*. For any $p \in \Phi$, $f(p) := p$, $f({\sim}p) := p^n$, and $f(-p) := \neg p^n$ where $p^n \in \Phi^n$.

We now show a weak theorem for syntactically embedding M4CC into GS4.

Theorem 3.23 (Weak syntactical embedding from M4CC into GS4). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\text{M4CC}}$, and f be the mapping defined in Definition 3.22.*

1. *If* M4CC $\vdash \Gamma \Rightarrow \Delta$*, then* GS4 $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *If* GS4 $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$*, then* M4CC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. Since the proof of (2) is the same as that for M4CC*, we show only (1) by induction on the proofs P of $\Gamma \Rightarrow \Delta$ in M4CC. We distinguish the cases according to the last inference of P, and show only the following cases which are not included in or differ from the cases for M4CC*.

1. Case $\sim p, -p \Rightarrow$: The last inference of P is of the form: $\sim p, -p \Rightarrow$ for any $p \in \Phi$. In this case, using ($\neg$left), we obtain GS4 $\vdash f(\sim p), f(-p) \Rightarrow$, i.e., GS4 $\vdash p^n, \neg p^n \Rightarrow$ ($p^n \in \Phi^n$) by the definition of f.
2. Case $\Rightarrow \sim p, -p$: Similar to Case $\sim p, -p \Rightarrow$.
3. Case $-p \Rightarrow -p$: The last inference of P is of the form: $-p \Rightarrow -p$ for any $p \in \Phi$. In this case, we obtain GS4 $\vdash f(-p) \Rightarrow f(-p)$, i.e., GS4 $\vdash$ $\neg p^n \Rightarrow \neg p^n$ ($p^n \in \Phi^n$) by the definition of f. **Q.E.D.**

Theorem 3.24 (Cut-elimination for M4CC). *The rule* (cut) *is admissible in cut-free* M4CC*.*

Proof. Similar to the proof of Theorem 3.5. By using Theorem 3.23. **Q.E.D.**

Theorem 3.25 (Syntactical embedding from M4CC into GS4). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{M4CC}}$*, and* f *be the mapping defined in Definition 3.22.*

1. M4CC $\vdash \Gamma \Rightarrow \Delta$ *iff* GS4 $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. M4CC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* GS4 $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. Similar to the proof of Theorem 3.6. By using Theorem 3.23 and the cut-elimination theorem for GS4. **Q.E.D.**

Theorem 3.26 (Decidability for M4CC). *The system* M4CC *is decidable.*

Proof. Similar to the proof of Theorem 3.7. By using Theorem 3.25. **Q.E.D.**

We show the following characteristic property of M4CC.

Theorem 3.27 (Negative symmetry for M4CC). *For any formulas* α *and* β*,*

$\text{M4CC} - (\text{cut}) \vdash {\sim}\alpha \Rightarrow {\sim}\beta$ *iff* $\text{M4CC} - (\text{cut}) \vdash -\beta \Rightarrow -\alpha$.

Proof. By Theorem 3.24 and Proposition 2.7, we have the fact that (${\sim}$left), (${\sim}$right), ($-$left), and ($-$right) are admissible in cut-free M4CC. This fact implies the required fact. **Q.E.D.**

Remark 3.28.

1. *The negative symmetry property does not hold for* $\text{M4CC}^{\star}$*. Similarly to* [23]*, this can be shown by the fact that the sequent* $-p \Rightarrow -{\sim}{\sim}p$ *is not provable in* $\text{M4CC}^{\star}$*, while the sequent* ${\sim}p \Rightarrow {\sim}{\sim}{\sim}p$ *is.*
2. *We cannot introduce an M4CC-translation function* g *for formulas of* GS4*, and hence we cannot show some theorems for embedding* GS4 *into* M4CC*. This situation differs from that for* $\text{M4CC}^{\star}$*.*
3. *A reason why we cannot introduce such a function* g *from* GS4 *into* M4CC *is explained as follows. Suppose that we define the condition concerning* $-$ *as* $g(\neg p^n) = -p$*, which is in a sense natural. On the other hand, we also have:* $g(\neg p^n) = {\sim}{-}g(p^n) = {\sim}{-}{\sim}p$*. Thus, we obtain the inappropriate result* $-p = {\sim}{-}{\sim}p$*.*

Next, we show a theorem for semantically embedding M4CC into GS4. Prior to prove the semantical embedding theorem, we need to show some lemmas.

Lemma 3.29. *Let* f *be the mapping defined in Definition 3.22. For any paraconsistent Kripke M4CC-model* $\langle M, R, \models^* \rangle$*, we can construct a Kripke model* $\langle M, R, \models \rangle$ *such that for any formula* α *and any* $x \in M$*,*

$x \models^* \alpha$ *iff* $x \models f(\alpha)$.

Proof. Similar to the proof Lemma 3.14. Let $\Phi^{\neg n}$ be the set $\{\neg p^n \mid p^n \in \Phi^n\}$. Suppose that $\langle M, R, \models^* \rangle$ is a paraconsistent Kripke M4CC-model where $\models^*$ is a mapping from $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to 2^M with the valuation condition $(*)$ in Definition 2.12. Suppose that $\langle M, R, \models \rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \Phi^n \cup \Phi^{\neg n}$ to 2^M such that for any $x \in M$ and any $q \in \Phi$,

1. $x \models^* q$ iff $x \models q$,
2. $x \models^* {\sim}q$ iff $x \models q^n$,
3. $x \models^* -q$ iff $x \models \neg q^n$.

Then, the lemma is proved by induction on α. Since the proof of the induction step is the same as that for M4CC*, we show only the following proof of the base step which differs from that for M4CC*.

Case when $\alpha \equiv -q$ where q is a propositional variable: $x \models^* -q$ iff $x \models \neg q^n$ (by the assumption) iff $x \models f(-q)$ (by the definition of f).

Q.E.D.

Lemma 3.30. *Let f be the mapping defined in Definition 3.22. For any Kripke model $\langle M, R, \models \rangle$, we can construct a paraconsistent Kripke M4CC-model $\langle M, R, \models^* \rangle$ such that for any formula α and any $x \in M$,*

$x \models f(\alpha)$ *iff* $x \models^* \alpha$.

Proof. Similar to the proof of Lemma 3.29. **Q.E.D.**

Theorem 3.31 (Semantical embedding from M4CC into GS4). *Let f be the mapping defined in Definition 3.22. For any sequent $\Gamma \Rightarrow \Delta$,*

M4CC $\models \Gamma \Rightarrow \Delta$ *iff* GS4 $\models f(\Gamma) \Rightarrow f(\Delta)$.

Proof. By Lemmas 3.29 and 3.30. **Q.E.D.**

Theorem 3.32 (Kripke-completeness for M4CC). *For any sequent $\Gamma \Rightarrow \Delta$,*

M4CC $\vdash \Gamma \Rightarrow \Delta$ *iff* M4CC $\models \Gamma \Rightarrow \Delta$.

Proof. Similar to the proof of Theorem 3.17. By using Theorem 3.31 and the completeness theorem for GS4. **Q.E.D.**

Theorem 3.33 (Finite model property for M4CC). *For any sequent $\Gamma \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta$ is M4CC-valid in any finite Kripke frame iff* M4CC $\vdash \Gamma \Rightarrow \Delta$.

Proof. Similar to the proof of Theorem 3.18. Using the appropriate modifications of Lemmas 3.29 and 3.30 and the finite model property for GS4, we can obtain the required property. **Q.E.D.**

4. Adding $\Diamond$ as an explicit modal operator

4.1. Sequent calculi and Kripke semantics

We consider an extended language with the diamond modal operator $\Diamond$ as an explicit modal operator instead of the abbreviation of $\sim - \Box \sim -$.

Gentzen-type sequent calculi $\mathrm{M4CC}_\Diamond$ and $\mathrm{M4CC}^\star_\Diamond$ for the extended language with $\Diamond$ are defined as follows.

Definition 4.1 ($\mathbf{M4CC}_\Diamond$ and $\mathbf{M4CC}^\star_\Diamond$). *The systems* $\mathrm{M4CC}_\Diamond$ *and* $\mathrm{M4CC}^\star_\Diamond$ *are respectively obtained from* M4CC *and* $\mathrm{M4CC}^\star$ *by replacing* ($\Box$right)*,* ($\sim\Diamond$right)*, and* ($-\Box$right) *with the logical inference rules of the form:*

$$\frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Diamond\Delta, \sim\Box\Omega, -\Diamond\Lambda, \alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Diamond\Delta, \sim\Box\Omega, -\Diamond\Lambda, \Box\alpha}\ (\Box\mathrm{right}^*)$$

$$\frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Diamond\Delta, \sim\Box\Omega, -\Diamond\Lambda, \sim\alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Diamond\Delta, \sim\Box\Omega, -\Diamond\Lambda, \sim\Diamond\alpha}\ (\sim\Diamond\mathrm{right}^*)$$

$$\frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Diamond\Delta, \sim\Box\Omega, -\Diamond\Lambda, -\alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Diamond\Delta, \sim\Box\Omega, -\Diamond\Lambda, -\Box\alpha}\ (-\Box\mathrm{right}^*)$$

and adding the logical inference rules of the form:

$$\frac{\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{\Diamond\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}\ (\Diamond\mathrm{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}\ (\Diamond\mathrm{right})$$

$$\frac{\sim\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{\sim\Box\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}\ (\sim\Box\mathrm{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha}{\Gamma \Rightarrow \Delta, \sim\Box\alpha}\ (\sim\Box\mathrm{right})$$

$$\frac{-\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{-\Diamond\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}\ (-\Diamond\mathrm{left}) \quad \frac{\Gamma \Rightarrow \Delta, -\alpha}{\Gamma \Rightarrow \Delta, -\Diamond\alpha}\ (-\Diamond\mathrm{right}).$$

Definition 4.2. *Kripke semantics for* $\mathrm{M4CC}^\star_\Diamond$ *and* $\mathrm{M4CC}_\Diamond$ *are defined by adding the following valuation clauses to the Kripke semantics for* $\mathrm{M4CC}^\star$ *and* M4CC*, respectively:*

1. $x \models^* \Diamond\alpha$ *iff* $\exists y \in M\ [xRy$ *and* $y \models^* \alpha]$,
2. $x \models^* \sim\Diamond\alpha$ *iff* $\forall y \in M\ [xRy$ *implies* $y \models^* \sim\alpha]$,
3. $x \models^* -\Diamond\alpha$ *iff* $\exists y \in M\ [xRy$ *and* $y \models^* -\alpha]$.

We have the following propositions.

Proposition 4.3. *Let L be* $\mathrm{M4CC}_{\Diamond}$ *or* $\mathrm{M4CC}^{\star}_{\Diamond}$. *The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in cut-free L.*

Proof. Straightforward. We can prove this by induction on α. **Q.E.D.**

Proposition 4.4. *The following sequents are provable in cut-free* M4CC *for any formulas α and β:*

1. $\sim\alpha, -\alpha \Rightarrow,$
2. $\Rightarrow\ \sim\alpha, -\alpha,$

Proof. Straightforward. We can prove them by induction on α. **Q.E.D.**

Proposition 4.5. *Let L be* $\mathrm{M4CC}_{\Diamond}$ *or* $\mathrm{M4CC}^{\star}_{\Diamond}$, *and let $\neg$ be the abbreviation of $\sim-$. The following sequents are provable in cut-free L for any formulas α and β:*

1. $\neg\Box\alpha \Leftrightarrow \Diamond\neg\alpha,$
2. $\neg\Diamond\alpha \Leftrightarrow \Box\neg\alpha,$
3. $\sim\Box\alpha \Leftrightarrow \Diamond\sim\alpha,$
4. $\sim\Diamond\alpha \Leftrightarrow \Box\sim\alpha,$
5. $-\Box\alpha \Leftrightarrow \Box-\alpha,$
6. $-\Diamond\alpha \Leftrightarrow \Diamond-\alpha.$

Proof. We show only the following cases.

1. Case (1):

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Proposition}\ 4.3\\ \alpha \Rightarrow \alpha\end{array}}{\Rightarrow\ \sim-\alpha, \alpha}\ (\sim-\text{right})}{\Rightarrow \Diamond\sim-\alpha, \alpha}\ (\Diamond\text{right})}{\Rightarrow \Diamond\sim-\alpha, \Box\alpha}\ (\Box\text{right}^{*})}{\sim-\Box\alpha \Rightarrow \Diamond\sim-\alpha}\ (\sim-\text{left}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Proposition}\ 4.3\\ \alpha \Rightarrow \alpha\end{array}}{\alpha, \sim-\alpha \Rightarrow}\ (\sim-\text{left})}{\Box\alpha, \sim-\alpha \Rightarrow}\ (\Box\text{left})}{\Box\alpha, \Diamond\sim-\alpha \Rightarrow}\ (\Diamond\text{left}^{*})}{\Diamond\sim-\alpha \Rightarrow\ \sim-\Box\alpha}\ (\sim-\text{right})$$

2. Case (3):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Proposition}\ 4.3\\ \sim\alpha \Rightarrow\ \sim\alpha\end{array}}{\sim\alpha \Rightarrow \Diamond\sim\alpha}\ (\Diamond\text{right})}{\sim\Box\alpha \Rightarrow \Diamond\sim\alpha}\ (\sim\Box\text{left}) \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Proposition}\ 4.3\\ \sim\alpha \Rightarrow\ \sim\alpha\end{array}}{\sim\alpha \Rightarrow\ \sim\Box\alpha}\ (\sim\Box\text{right})}{\Diamond\sim\alpha \Rightarrow\ \sim\Box\alpha}\ (\Diamond\text{left}).$$

Q.E.D.

Next, we introduce an extension $\mathrm{GS4}_{\diamond}$ of GS4 by adding some logical inference rules for $\Diamond$.

Definition 4.6 ($\mathbf{GS4}_{\diamond}$). *The system* $\mathrm{GS4}_{\diamond}$ *is obtained from* GS4 *by replacing* ($\Box$right) *with the logical inference rule of the form:*

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \alpha}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\alpha}\ (\Box\mathrm{right}^{\mathrm{S4}*})$$

and adding ($\Diamond$right) *introduced in Definition 4.1 and the logical inference rule of the form:*

$$\frac{\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}\ (\Diamond\mathrm{left}^{\mathrm{S4}*}).$$

Remark 4.7.

1. *Almost the same system as* $\mathrm{GS4}_{\diamond}$ *was originally introduced by Kripke in* [25] *(p. 91) in order to deal with* $\Box$ *and* $\Diamond$ *simultaneously. The system was introduced by modifying Ohnishi and Matsumoto's Gentzen-type sequent calculus introduced in* [33]*. This system has also recently investigated and extended by Grigoriev and Petrukhin in* [16]*.*
2. *In* $\mathrm{GS4}_{\diamond}$*, the characteristic inference rules are* ($\Box\mathrm{right}^{\mathrm{S4}*}$) *and* ($\Diamond\mathrm{left}^{\mathrm{S4}*}$)*. Using these rules, we can show that the following sequents are provable in cut-free* $\mathrm{GS4}_{\diamond}$ *for any formula* α*:*
   (a) $\neg\Box\alpha \Leftrightarrow \Diamond\neg\alpha$,
   (b) $\neg\Diamond\alpha \Leftrightarrow \Box\neg\alpha$.

   For more information on these characteristic rules, see [25] *(p. 91) and* [16] *(pp. 692-693).*
3. *It is known that the cut-elimination and Kripke-completeness theorems hold for* $\mathrm{GS4}_{\diamond}$*. In addition to these theorems, the decidability for* $\mathrm{GS4}_{\diamond}$ *can be obtained in a straightforward way. For more information on these theorems, see* [25, 16]*. On the other hand, as far as we know, the finite model property for* $\mathrm{GS4}_{\diamond}$ *is unknown in the literature. However, the finite*

model property does seem to hold for GS4$_\diamond$ *as well, because it is clear that* GS4$_\diamond$ *is definitionally equivalent to* GS4*, in which case the finite model property for* GS4$_\diamond$ *follows from that for* GS4*.*

The Kripke semantics for GS4$_\diamond$ is naturally defined as follows.

Definition 4.8. *A Kripke semantics for* GS4$_\diamond$ *is defined by adding the following valuation clause to the Kripke semantics for* GS4*:*

$$x \models \Diamond\alpha \text{ iff } \exists y \in M \ [xRy \text{ and } y \models \alpha].$$

4.2. Theorems for M4CC$^\star_\diamond$

Next, we introduce a GS4$_\diamond$-translation function for formulas of M4CC$^\star_\diamond$, and by using this translation, we show several theorems for syntactically embedding M4CC$^\star_\diamond$ into GS4$_\diamond$.

Definition 4.9. *The language* $\mathcal{L}_{\mathrm{M4CC}^\star_\diamond}$ *of* M4CC$^\star_\diamond$ *is obtained from the language* $\mathcal{L}_{\mathrm{M4CC}^\star}$ *defined in Definition 3.1 by adding* $\Diamond$*. The language* $\mathcal{L}_{\mathrm{GS4}_\diamond}$ *of* GS4 *is obtained from the language* $\mathcal{L}_{\mathrm{GS4}}$ *defined in Definition 3.1 by adding* $\Diamond$*.*

A mapping $f_\diamond$ *from* $\mathcal{L}_{\mathrm{M4CC}^\star_\diamond}$ *to* $\mathcal{L}_{\mathrm{GS4}_\diamond}$ *is obtained from the conditions of the mapping* f *defined in Definition 3.1 by adding the following conditions:*

1. $f_\diamond(\Diamond\alpha) := \Diamond f_\diamond(\alpha)$,
2. $f_\diamond({\sim}\Diamond\alpha) := \Box f_\diamond({\sim}\alpha)$,
3. $f_\diamond(-\Diamond\alpha) := \Diamond f_\diamond(-\alpha)$.

We then obtain the following theorem. We remark that the proof of this theorem does not require a similar proposition to Proposition 3.3.

Theorem 4.10 (Weak syntactical embedding from M4CC$^\star_\diamond$ into GS4$_\diamond$). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{M4CC}^\star_\diamond}$*, and* $f_\diamond$ *be the mapping defined in Definition 4.9.*

1. *If* M4CC$^\star_\diamond$ $\vdash \Gamma \Rightarrow \Delta$*, then* GS4$_\diamond$ $\vdash f_\diamond(\Gamma) \Rightarrow f_\diamond(\Delta)$*.*
2. *If* GS4$_\diamond$ $-$ (cut) $\vdash f_\diamond(\Gamma) \Rightarrow f_\diamond(\Delta)$*, then* M4CC$^\star_\diamond$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$*.*

Proof. We show only (1) by induction on the proofs P of $\Gamma \Rightarrow \Delta$ in M4CC$^\star_\Diamond$. We distinguish the cases according to the last inference of P, and show only the following cases

1. Case ($\sim\Box$left): The last inference of P is of the form:

$$\frac{\sim\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{\sim\Box\alpha, \Box\Delta, \sim\Diamond\Omega, -\Box\Lambda \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi} \ (\sim\Box\text{left}).$$

   By induction hypothesis, we have

   $\mathrm{GS4}_\Diamond \vdash f_\Diamond(\Box\alpha), f_\Diamond(\Box\Delta), f_\Diamond(\sim\Diamond\Omega), f_\Diamond(-\Box\Lambda) \Rightarrow f_\Diamond(\Diamond\Gamma), f_\Diamond(\sim\Box\Sigma),$
   $f_\Diamond(-\Diamond\Pi)$

   where $f_\Diamond(\Box\Delta), f_\Diamond(\sim\Diamond\Omega), f_\Diamond(-\Box\Lambda)$, $f_\Diamond(\Diamond\Gamma)$, $f_\Diamond(\sim\Box\Sigma)$ and $f_\Diamond(-\Diamond\Pi)$ coincide with $\Box f_\Diamond(\Delta), \Box f_\Diamond(\sim\Omega), \Box f_\Diamond(-\Lambda)$, $\Diamond f_\Diamond(\Gamma)$, $\Diamond f_\Diamond(\sim\Sigma)$ and $\Diamond f_\Diamond(-\Pi)$, respectively, by the definition of $f_\Diamond$. We then obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f_\Diamond(\sim\alpha), \Box f_\Diamond(\Delta), \Box f_\Diamond(\sim\Omega), \Box f_\Diamond(-\Lambda) \Rightarrow \Diamond f_\Diamond(\Gamma), \Diamond f_\Diamond(\sim\Sigma), \Diamond f_\Diamond(-\Pi)\end{array}}{\Diamond f_\Diamond(\sim\alpha), \Box f_\Diamond(\Delta), \Box f_\Diamond(\sim\Omega), \Box f_\Diamond(-\Lambda) \Rightarrow \Diamond f_\Diamond(\Gamma), \Diamond f_\Diamond(\sim\Sigma), \Diamond f_\Diamond(-\Pi)} \ (\Diamond\text{left}^{\mathrm{S4}*})$$

   where $\Diamond f_\Diamond(\sim\alpha)$ coincides with $f_\Diamond(\sim\Box\alpha)$ by the definition of $f_\Diamond$.

2. Case ($-\Diamond$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, -\alpha}{\Gamma \Rightarrow \Delta, -\Diamond\alpha} \ (-\Diamond\text{right}).$$

   By induction hypothesis, we have $\mathrm{GS4}_\Diamond \vdash f_\Diamond(\Gamma) \Rightarrow f_\Diamond(\Delta), f_\Diamond(-\alpha)$. We then obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f_\Diamond(\Gamma) \Rightarrow \Diamond f_\Diamond(\Delta), f_\Diamond(-\alpha)\end{array}}{f_\Diamond(\Gamma) \Rightarrow \Diamond f_\Diamond(\Delta), \Diamond f_\Diamond(-\alpha)} \ (\Diamond\text{right})$$

   where $\Diamond f_\Diamond(-\alpha)$ coincides with $f_\Diamond(-\Diamond\alpha)$ by the definition of $f_\Diamond$. **Q.E.D.**

Theorem 4.11 (Syntactical embedding from M4CC$^\star_\Diamond$ into GS4$_\Diamond$). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{M4CC}^\star_\Diamond}$, *and* $f_\Diamond$ *be the mapping defined in Definition 4.9.*

1. $\mathrm{M4CC}^{\star}_{\diamond} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4}_{\diamond} \vdash f_{\diamond}(\Gamma) \Rightarrow f_{\diamond}(\Delta)$.
2. $\mathrm{M4CC}^{\star}_{\diamond} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4}_{\diamond} - (\mathrm{cut}) \vdash f_{\diamond}(\Gamma) \Rightarrow f_{\diamond}(\Delta)$.

Proof. Similar to the proof of Theorem 3.6. By using Theorem 4.10. **Q.E.D.**

We then obtain the following theorems in a similar way as those for $\mathrm{M4CC}^{\star}$.

Theorem 4.12.

1. *(Cut-elimination for* $\mathrm{M4CC}^{\star}_{\diamond}$*): The rule* (cut) *is admissible in cut-free* $\mathrm{M4CC}^{\star}_{\diamond}$*.*
2. *(Decidability for* $\mathrm{M4CC}^{\star}_{\diamond}$*): The system* $\mathrm{M4CC}^{\star}_{\diamond}$ *is decidable.*
3. *(Quasi-paraconsistency for* $\mathrm{M4CC}^{\star}_{\diamond}$*): The system* $\mathrm{M4CC}^{\star}_{\diamond}$ *is quasi-paraconsistent with respect to the combination of* $\sim$ *and* $-$*.*
4. *(Quasi-paracompleteness for* $\mathrm{M4CC}^{\star}_{\diamond}$*): The system* $\mathrm{M4CC}^{\star}_{\diamond}$ *is quasi-paracomplete with respect to the combination of* $\sim$ *and* $-$*.*

Next, we introduce an $\mathrm{M4CC}^{\star}_{\diamond}$-translation function for formulas of $\mathrm{GS4}_{\diamond}$, and by using this translation, we can show some theorems for embedding $\mathrm{GS4}_{\diamond}$ into $\mathrm{M4CC}^{\star}_{\diamond}$.

Definition 4.13. *Let* $\mathcal{L}_{\mathrm{M4CC}^{\star}_{\diamond}}$ *and* $\mathcal{L}_{\mathrm{GS4}_{\diamond}}$ *be the languages defined in Definition 4.9. A mapping* $g_{\diamond}$ *from* $\mathcal{L}_{\mathrm{GS4}_{\diamond}}$ *to* $\mathcal{L}_{\mathrm{M4CC}^{\star}_{\diamond}}$ *is obtained from the conditions of the mapping* g *defined in Definition 3.11 by adding the following condition:*

$$g_{\diamond}(\Diamond\alpha) := \Diamond g_{\diamond}(\alpha).$$

We then obtain the following theorem in a similar way as that for $\mathrm{M4CC}^{\star}$.

Theorem 4.14 (Syntactical embedding from $\mathrm{GS4}_{\diamond}$ into $\mathrm{M4CC}^{\star}_{\diamond}$). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{GS4}_{\diamond}}$*, and* $g_{\diamond}$ *be the mapping defined in Definition 4.13.*

1. $\mathrm{GS4}_{\diamond} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{M4CC}^{\star}_{\diamond} \vdash g_{\diamond}(\Gamma) \Rightarrow g_{\diamond}(\Delta)$.
2. $\mathrm{GS4}_{\diamond} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{M4CC}^{\star}_{\diamond} - (\mathrm{cut}) \vdash g_{\diamond}(\Gamma) \Rightarrow g_{\diamond}(\Delta)$.

We also have the following theorems in a similar way as these for M4CC*. We remark that the proof of Theorem 4.15 does not require a similar proposition to Proposition 2.19.

Theorem 4.15 (Semantical embedding from M4CC$^\star_\Diamond$ into GS4$_\Diamond$). *Let $f_\Diamond$ be the mapping defined in Definition 4.9. For any sequent $\Gamma \Rightarrow \Delta$,*

$$\mathrm{M4CC}^\star_\Diamond \models \Gamma \Rightarrow \Delta \text{ iff } \mathrm{GS4}_\Diamond \models f_\Diamond(\Gamma) \Rightarrow f_\Diamond(\Delta).$$

Theorem 4.16 (Kripke-completeness for M4CC$^\star_\Diamond$). *For any sequent $\Gamma \Rightarrow \Delta$,*

$$\mathrm{M4CC}^\star_\Diamond \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathrm{M4CC}^\star_\Diamond \models \Gamma \Rightarrow \Delta.$$

Theorem 4.17 (Semantical embedding from GS4$_\Diamond$ into M4CC$^\star_\Diamond$). *Let $g_\Diamond$ be the mapping defined in Definition 4.13. For any sequent $\Gamma \Rightarrow \Delta$,*

$$\mathrm{GS4}_\Diamond \models \Gamma \Rightarrow \Delta \text{ iff } \mathrm{M4CC}^\star_\Diamond \models g_\Diamond(\Gamma) \Rightarrow g_\Diamond(\Delta).$$

4.3. Theorems for M4CC$_\Diamond$

Next, we introduce a GS4$_\Diamond$-translation function for formulas of M4CC$_\Diamond$, and by using this translation, we can show several theorems for embedding M4CC$_\Diamond$ into GS4$_\Diamond$.

Definition 4.18. *We fix a set Φ of propositional variables, and define the set $\Phi^n := \{p^n \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{M4CC}_\Diamond}$ of* M4CC$_\Diamond$ *is the same as the language $\mathcal{L}_{\mathrm{M4CC}^\star_\Diamond}$ of* M4CC$^\star_\Diamond$*. The new alternative language $\mathcal{L}_{\mathrm{GS4}^\star_\Diamond}$ of* GS4$_\Diamond$ *is defined using Φ, Φ^n, $\wedge$, $\vee$, $\rightarrow$, $\Box$, $\Diamond$, and $\neg$.*

A mapping $f_\Diamond$ from $\mathcal{L}_{\mathrm{M4CC}_\Diamond}$ to $\mathcal{L}_{\mathrm{GS4}^\star_\Diamond}$ is obtained from the conditions of the mapping f defined in Definition 3.22 by adding the following conditions:

1. $f_\Diamond(\Diamond\alpha) := \Diamond f_\Diamond(\alpha)$,
2. $f_\Diamond({\sim}\Diamond\alpha) := \Box f_\Diamond({\sim}\alpha)$,
3. $f_\Diamond(-\Diamond\alpha) := \Diamond f_\Diamond(-\alpha)$.

We can obtain the following theorems in a similar way as those for M4CC.

Theorem 4.19 (Syntactical embedding from M4CC$_\diamond$ into GS4$_\diamond$). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{M4CC}_\diamond}$*, and* $f_\diamond$ *be the mapping defined in Definition 4.18.*

1. $\mathrm{M4CC}_\diamond \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4}_\diamond \vdash f_\diamond(\Gamma) \Rightarrow f_\diamond(\Delta)$.
2. $\mathrm{M4CC}_\diamond - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GS4}_\diamond - (\mathrm{cut}) \vdash f_\diamond(\Gamma) \Rightarrow f_\diamond(\Delta)$.

Theorem 4.20.

1. *(Cut-elimination for* $\mathrm{M4CC}_\diamond$*): The rule* (cut) *is admissible in cut-free* $\mathrm{M4CC}_\diamond$.
2. *(Decidability for* $\mathrm{M4CC}_\diamond$*): The system* $\mathrm{M4CC}_\diamond$ *is decidable.*
3. *(Negative symmetry for* $\mathrm{M4CC}_\diamond$*): For any formulas* α *and* β*,*

$$\mathrm{M4CC}_\diamond - (\mathrm{cut}) \vdash {\sim}\alpha \Rightarrow {\sim}\beta \ \textit{ iff } \ \mathrm{M4CC}_\diamond - (\mathrm{cut}) \vdash -\beta \Rightarrow -\alpha.$$

Theorem 4.21 (Semantical embedding from M4CC$_\diamond$ into GS4$_\diamond$). *Let* $f_\diamond$ *be the mapping defined in Definition 4.18. For any sequent* $\Gamma \Rightarrow \Delta$*,*

$$\mathrm{M4CC}_\diamond \models \Gamma \Rightarrow \Delta \ \textit{ iff } \ \mathrm{GS4}_\diamond \models f_\diamond(\Gamma) \Rightarrow f_\diamond(\Delta).$$

Theorem 4.22 (Kripke-completeness for M4CC$_\diamond$). *For any sequent* $\Gamma \Rightarrow \Delta$*,*

$$\mathrm{M4CC}_\diamond \vdash \Gamma \Rightarrow \Delta \ \textit{ iff } \ \mathrm{M4CC}_\diamond \models \Gamma \Rightarrow \Delta.$$

5. Conclusions, remarks, and related works

In this study, we introduced a modal extension M4CC of Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic known as 4CC [5, 6, 7]. We proved several theorems for syntactically embedding M4CC into a Gentzen-type sequent calculus for the normal modal logic S4. Furthermore, using such a syntactical embedding theorem, we obtained the cut-elimination theorem for M4CC and the decidability result for M4CC. We obtained the negative symmetry theorem for M4CC as a corollary of the cut-elimination theorem. We also proved several theorems for semantically embedding M4CC into S4. Furthermore, using such a semantical embedding theorem, we obtained the Kripke-completeness theorem for M4CC and the finite model property of M4CC. Moreover, we introduced another logic M4CC* that is obtained from M4CC by deleting some initial sequents which correspond to the principle of quasi-explosion and the law of a quasi-excluded middle. We proved several theorems for syntactically embedding M4CC* into a Gentzen-type sequent calculus for S4 and vice versa. Furthermore, using such a syntactical embedding theorem, we obtained the cut-elimination theorem for M4CC* and the decidability result for M4CC*. We also obtained the quasi-paraconsistency and quasi-paracompleteness for M4CC* as corollaries of the cut-elimination theorem. We also proved several theorems for semantically embedding M4CC* into S4. Furthermore, using such a semantical embedding theorem, we obtained the Kripke-completeness theorem for M4CC* and the finite model property of M4CC*. Furthermore, we introduced the extended systems M4CC$_\diamond$ and M4CC$^\star_\diamond$ by adding some logical inference rules for $\diamondsuit$. It was shown that the same theorems (except the finite model property) as those for M4CC and M4CC* hold for M4CC$_\diamond$ and M4CC$^\star_\diamond$ in a similar embedding-based method.

Next, we address some remarks on K-type modal extensions, which are based on the normal modal logic K. We can construct K-type modal extensions and can prove the same theorems as those for M4CC and M4CC*. We now address the K-type modal extensions M4CC$_{\rm K}$ and M4CC$^\star_{\rm K}$, which are analogues of M4CC and M4CC*, respectively, as follows.

Definition 5.1 (M4CC$_{\rm K}$ and M4CC$^\star_{\rm K}$). *Let $\diamondsuit$ be the abbreviation of $\sim - \Box \sim -$ or $- \sim \Box - \sim$. The systems* M4CC$_{\rm K}$ *and* M4CC$^\star_{\rm K}$ *are obtained from* M4CC *and* M4CC* *defined in Definitions 2.1 and 2.2 by replacing the logical inference rules concerning $\Box$ by the logical inference rules concerning $\Box$ of the form:*

$$\frac{\Gamma, {\sim}\Sigma, -\Pi \Rightarrow \alpha}{\Box\Gamma, {\sim}\diamondsuit\Sigma, -\Box\Pi \Rightarrow \Box\alpha}\ (\Box\text{regularity}) \quad \frac{\Gamma, {\sim}\Sigma, -\Pi \Rightarrow {\sim}\alpha}{\Box\Gamma, {\sim}\diamondsuit\Sigma, -\Box\Pi \Rightarrow {\sim}\diamondsuit\alpha}\ ({\sim}\diamondsuit\text{regularity})$$

$$\frac{\Gamma, {\sim}\Sigma, -\Pi \Rightarrow -\alpha}{\Box\Gamma, {\sim}\diamondsuit\Sigma, -\Box\Pi \Rightarrow -\Box\alpha}\ (-\Box\text{regularity}).$$

A Gentzen-type sequent calculus GK for K is presented as follows.

Definition 5.2 (GK). *The system* GK *is obtained from* GS4 *defined in Definition 2.9 by replacing the logical inference rule concerning* $\Box$ *with the logical inference rule of the form:*

$$\frac{\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{regularity}^{\mathrm{K}}).$$

The Kripke-type semantics for $\mathrm{M4CC_K}$, $\mathrm{M4CC_K^\star}$, and GK are obtained from those for M4CC, $\mathrm{M4CC^\star}$, and GS4, respectively, by deleting the reflexive and transitive conditions on the accessibility relations R used for them (i.e., R has no condition). Then, we can prove the same theorems for $\mathrm{M4CC_K}$ and $\mathrm{M4CC_K^\star}$ as those for M4CC and $\mathrm{M4CC^\star}$ by imposing some appropriate modifications. On the other hand, we do not know how to construct cut-free and Kripke-complete Gentzen-type sequent calculi for the extended logics with $\Diamond$ as an explicit modal operator.

In the remainder of this section, we address some related works on some modal extensions of many-valued logics. The idea of extending many-valued logics to modal many-valued logic is not new. Some traditional results in this respect are found, such as in [13, 14]. Nevertheless, the modal extensions of many-valued logics have not yet been studied intensively. Some many-valued modal logics over finite residuated lattices were studied by Bou et al. in [11], with special attention to some basic classes of Kripke frames and their axiomatizations. One may refer also to [16, 22, 36, 28, 30, 31, 32, 29, 38, 39], wherein some modal extensions of Belnap and Dunn's useful four-valued logic and related logics have been studied and certain properties of such logics from proof-theoretic, semantic, and algebraic viewpoints have also been analyzed. We now address some of these studies. Some three- and four-valued modal logics, which are extensions of Belnap and Dunn's four-valued logic and its three-valued variant were introduced by Odintsov and Wansing in [31], by providing them with the sound and complete tableau calculi, Kripke semantics, and modal algebras with twist structures. By considering the many-valued Kripke structures and their counterpart modal algebras in the sense of the topological duality theory, a family of four-valued modal logics, which are modal extensions of Belnap and

Dunn's four-valued logic, was studied by Rivieccio et al. in [38]. A Belnapian version BK of the least normal modal logic K with the addition of strong negation was introduced by Odintsov and Speranski in [30], and a systematic study of the lattices of logic containing BK was carried out by them. Modal multilattice logic based on S4 was studied by Kamide and Shramko in [22]. A Gentzen-type sequent calculus and a Kripke semantics for this S4 modal multilattice logic were developed by them. However, some of the results by Kamide and Shramko for the S4 modal multilattice logic were not correct. The Gentzen-type sequent calculus proposed in [22] was not Kripke-complete with respect to the Kripke semantics for the S4 modal multilattice logic. The wrong results by Kamide and Shramko were correct by Grigoriev and Petrukhin in [16]. Modal multilattice logic based on S5 was studied by Grigoriev and Petrukhin in [16]. A hyper sequent calculus and a Kripke semantics for this S5 modal multilattice logic was introduced, and the cut-elimination and Kripke-completeness theorems for the S5 modal multilattice logic were proved by them.

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