On weaker forms of the chain (F) condition for metacompactness-like covering properties

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We say that $\mathcal{W}$ satisfies (F) when it satisfies:

Definition (Definition of (F) $-$condition)

If $x \in U$ and U is open, then there exists an open set $V = V(x, U)$ containing x such that $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$ whenever $y \in V$.

The Collins-Roscoe structuring mechanism

Condition (F) was originally defined by Collins and Roscoe in [**1**], and its also referred as the "**Collins-Roscoe structuring mechanism**."

[**1**] P. J. Collins and A. W. Roscoe, *Criteria for metrisability*, Proc. Amer. Math. Soc. **90** (1984), 631-640.

The Collins-Roscoe structuring mechanism

If $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfies (F) and

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- each $\mathcal{W}(x)$ is a chain with respect to inclusion then we say that $\mathcal{W}$ satisfies chain (F).

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- each $\mathcal{W}(x)$ is a chain with respect to inclusion then we say that $\mathcal{W}$ satisfies chain (F) .
- each $\mathcal{W}(x)$ is well-ordered by reverse inclusion then we say that $\mathcal{W}$ satisfies well-ordered (F) .

We note that

any topological space X clearly has a $\mathcal{W}$ satisfying open (F).
We can just put, for each $x \in X$,

$$\mathcal{W}(x) = \{W : W \subseteq X \text{ open and } x \in W\}.$$

The concepts of rank and Noetherianness

Let P be a partially ordered set.

Definition (Noetherianness)

P is said to be Noetherian if every strictly increasing subset of P is finite.

The concepts of rank and Noetherianness

The rank of a partially ordered set P is defined by

Definition (Rank of a partially ordered set)

$$rank(P) = \min\{\kappa : \forall A \subseteq P, |A| \geq \kappa\ \exists a, b \in A : a \text{ and } b \text{ are comparable}\}.$$

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- If $rank(P) < \omega$, we will say that P is of finite rank.

The concepts of rank and Noetherianness

The concept of rank of a family of sets was introduced by Nagata in

[**7**] J. Nagata, *On dimension and metrization*, General Topology and its Relations to Modern Analysis and Algebra, (Academic Press, New York, 1962).

Since the family $\mathcal{W}(x)$ is partially ordered by inclusion for each x, we can mention rank and Noetherianness of $\mathcal{W}(x)$.

The concepts of rank and Noetherianness

Since the family $\mathcal{W}(x)$ is partially ordered by inclusion for each x, we can mention rank and Noetherianness of $\mathcal{W}(x)$.

- If $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfies (F), and each $\mathcal{W}(x)$ is Noetherian and of sub-κ-rank,

then we say that $\mathcal{W}$ is Noetherian and of sub-κ-rank (F).

Previously established results

The following results were obtained in

[**2**] P.J. Collins, G. M. Reed, A. W. Roscoe, M. E. Rudin, *A lattice of conditions on topological spaces*, Proc. Amer. Math. Soc. **94** (1985), 487-496.

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- If the space X has a $\mathcal{W}$ satisfying chain (F), then X is monotonically normal.

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- If the space X has a $\mathcal{W}$ satisfying well-ordered (F), then X is paracompact.

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- If the space X has a $\mathcal{W}$ satisfying well-ordered (F), then X is paracompact.
- If the space X has a $\mathcal{W}$ satisfying chain, neighborhood (F), then X is paracompact.

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The following result was obtained in

[**3**] P.M. Gartside and P. J. Moody, *Well-ordered* (F) spaces, Topology Proc. **17** (1992), 111-130.

- If the space X has a $\mathcal{W}$ which is Noetherian of sub-infinite rank (F), then X is metacompact.

Previously established results

The following result was obtained in

[**6**] P. J. Moody, G. M. Reed, A. W. Roscoe and P. J. Collins, *A lattice of conditions on topological spaces II*, Fund. Math. **138** (1991), 69-81.

Theorem (in [6], Theorem 14)

If the space X has a $\mathcal{W}$ satisfying chain (F) and for each x,

$$\mathcal{W}(x) = \mathcal{W}_1(x) \cup \mathcal{W}_2(x)$$

where $\mathcal{W}_1(x)$ consists of neighborhoods of x and $\mathcal{W}_2(x)$ is well-ordered by reverse inclusion, then X is paracompact.

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In [**6**], the authors pointed out that the Sorgenfrey line has such a $\mathcal{W}$.

For each x, put

$\mathcal{W}_1(x) = \{[x-r, x+r] : r \in \mathbb{R}^+\}$ (this family consists of neighborhoods of x)

$\mathcal{W}_2(x) = \{\{x\}\}$ (this family is well-ordered by reverse inclusion)

and

$\mathcal{W}(x) = \mathcal{W}_1(x) \cup \mathcal{W}_2(x)$ (this family is a chain)

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where $\mathcal{W}_1(x)$ consists of neighborhoods of x and $\mathcal{W}_2(x)$ is well-ordered by reverse inclusion, then X is paracompact.

The argument in the proof of this theorem in [**6**] is not applicable if $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ are considered as different chains that is to say that their elements are not comparable.

We realized that the Sorgenfrey line has also a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying (F) such that, for each x, $\mathcal{W}(x) = \mathcal{W}_1(x) \cup \mathcal{W}_2(x)$ where $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ are chains of neighborhoods of x with respect to inclusion, but $\mathcal{W}(x)$ is not a chain.

Just put, for each x,

$$\mathcal{W}_1(x) = \{[x, x + r] : r \in \mathbb{R}^+\}$$

and

$$\mathcal{W}_2(x) = \{[x - r, x + r] : r \in \mathbb{R}^+\}.$$

So we can ask the following question

QUESTION: If the space X has a $\mathcal{W}$ satisfying (F) and if, for each x, $\mathcal{W}(x) = \mathcal{W}_1(x) \cup \mathcal{W}_2(x)$ where $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ are chains of neighborhoods of x with respect to inclusion, then is X metacompact?

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We found that the answer of this question is "**YES**".

MAIN RESULTS

Theorem (Main Theorem)

If the space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying (F) and for each x there exists a non-negative integer $n(x)$ such that $\mathcal{W}(x) = \bigcup_{i=0}^{n(x)} \mathcal{W}_i(x)$, where $\mathcal{W}_0(x)$ is Noetherian of sub-κ-rank and, for each i with $1 \leq i \leq n(x)$, $\mathcal{W}_i(x)$ is a chain of neighborhoods of x with respect to inclusion, then each open cover of X has a point$- < \kappa$ open refinement.

Main Theorem

To prove the theorem, we use the following

Lemma

Let τ and κ be infinite cardinal numbers with $\tau \geq \kappa$. Let n be a non-negative integer, let $(P_i, \leq)$ be a partially ordered set for each $i < n$, and let $P = \bigcup_{i=0}^{n} P_i$. Let $\{x_\alpha : \alpha < \tau\}$ be a subset of P such that $x_\rho \not\leq x_\alpha$ for each ρ, α in τ with $\alpha < \rho$. If P_0 is Noetherian of sub-κ-rank and P_i is a chain for each i with $1 \leq i \leq n$, then there exists a subset J of τ with $|J| = \kappa$ and a k with $1 \leq k \leq n$ such that $\{x_\alpha : \alpha \in J\}$ is an increasing subset of P_k.

Proof of lemma

Erdös-Dushnik-Miller Theorem

To prove the lemma, we need the Erdös-Dushnik-Miller theorem. Recall that the theorem.

Erdös-Dushnik-Miller theorem states that;

for an arbitrary set X of cardinality κ and for every $2-$partition f of X with 2 colors, either there is a homogeneous set of cardinality κ in color 0 or there is a countable homogeneous set in color 1, with respect to f.

Theorem (Erdös-Dushnik-Miller theorem)

$\forall \kappa \geq \omega, \forall |X| = \kappa, \forall f : [X]^2 \to \{0,1\}$ $(\exists A \subseteq X, |A| = \kappa$ and $f \mid_{[A]^2} \equiv 0)$ or $(\exists B \subseteq X, |B| = \omega$ and $f \mid_{[B]^2} \equiv 1)$.

Theorem (arrow notation of the Erdös-Dushnik-Miller theorem)

For every infinite cardinal κ, *we have* $\kappa \longrightarrow (\kappa, \omega)^2$.

Proof of lemma

Lemma

Let τ and κ be infinite cardinal numbers with $\tau \geq \kappa$. Let n be a non-negative integer, let $(P_i, \leq)$ be a partially ordered set for each $i < n$, and let $P = \bigcup_{i=0}^{n} P_i$. Let $\{x_\alpha : \alpha < \tau\}$ be a subset of P such that $x_\rho \nleq x_\alpha$ for each ρ, α in τ with $\alpha < \rho$. If P_0 is Noetherian of sub-κ-rank and P_i is a chain for each i with $1 \leq i \leq n$, then there exists a subset J of τ with $|J| = \kappa$ and a k with $1 \leq k \leq n$ such that $\{x_\alpha : \alpha \in J\}$ is an increasing subset of P_k.

Proof of lemma

Proof:

Since $\{x_\alpha : \alpha < \tau\} \subseteq \bigcup_{i=0}^{n} P_i$ and κ is an infinite cardinal with $\tau \geq \kappa$, there exist a k with $0 \leq k \leq n$ and a subset J of τ with $|J| = \kappa$ such that $\{x_\alpha : \alpha \in J\} \subseteq P_k$.

Define the function

$$f : [J]^2 \longrightarrow \{0, 1\}$$

as follows:

Given $\{\alpha, \beta\} \in [J]^2$, we put

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & ; \quad x_\alpha \text{ and } x_\beta \text{ are not comparable} \\ 1 & ; \quad \text{otherwise} \end{cases}$$

By the Erdös-Dushnik-Miller theorem we have either

★ ... $\exists I \subseteq J : |I| = \kappa$ and $f\mid_{[I]^2} \equiv 0$
or
★★ ... $\exists M \subseteq J : |M| = \omega$ and $f\mid_{[M]^2} \equiv 1$.

We claim that $k > 0$. Assume contrary, i.e., $k = 0$.

If we have ★; this contradicts with the fact that the set P_0 is of sub-κ-rank.

If we have ★★; since $x_\rho \not\leq x_\alpha$ for each ρ, α in κ with $\alpha < \rho$, the set $\{x_\alpha : \alpha \in M\}$ is a strictly increasing countable subset of P_0. But this contradicts the Noetherianness of the set P_0.

So we have $k > 0$. Since the set $\{x_\alpha : \alpha \in J\} \subseteq P_k$ and the set P_k is a chain, the set $\{x_\alpha : \alpha \in J\}$ is increasing.

Sketch of proof of Main the Theorem

Theorem (Main Theorem)

If a topological space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying (F) and for each x there exists a non-negative integer $n(x)$ such that $\mathcal{W}(x) = \bigcup_{i=0}^{n(x)} \mathcal{W}_i(x)$, where $\mathcal{W}_0(x)$ is Noetherian of sub-κ-rank and, for each i with $1 \leq i \leq n(x)$, $\mathcal{W}_i(x)$ is a chain of neighborhoods of x with respect to inclusion, then each open cover of X has a point$- < \kappa$ open refinement.

Sketch of proof of the Main Theorem

Let $\mathcal{O} = \{O_\alpha : \alpha < \tau\}$ be an open cover for X and $P_\alpha = O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$ for each $\alpha < \tau$.

Define

$$X_i = \{x \in X : 1 \leq i \leq n(x) \text{ and } \exists W \in \mathcal{W}_i(x), \exists \alpha < \tau, W \subseteq O_\alpha\},$$

for each $i \in \omega$,

$$\gamma(x, i) = \min\{\alpha < \tau : \exists W \in \mathcal{W}_i(x), W \subseteq O_\alpha\},$$

for each x in X_i, and

$$I(x) = \{i : 1 \leq i \leq n(x) \text{ and } x \in X_i\}.$$

Sketch of proof of the Main Theorem

$X_i = \{x \in X : 1 \leq i \leq n(x) \text{ and } \exists W \in \mathcal{W}_i(x), \exists \alpha < \tau, W \subseteq O_\alpha\}$
$\gamma(x,i) = \min\{\alpha < \tau : \exists W \in \mathcal{W}_i(x), W \subseteq O_\alpha\}$
$I(x) = \{i : 1 \leq i \leq n(x) \text{ and } x \in X_i\}$

For each $x \in X_i$ and $i \in I(x)$ choose a $W(x,i) \in \mathcal{W}_i(x)$ with $W(x,i) \subseteq O_{\gamma(x,i)}$. Let

$$W_x = \bigcap_{i \in I(x)} W(x,i) \qquad \text{and} \qquad Y = \bigcup_{i \in \omega \setminus \{0\}} X_i,$$

where Y is indexed by some ordinal $\lambda : Y = \{x_\beta : \beta < \lambda\}$.

Sketch of proof of the Main Theorem

$$W_x = \bigcap_{i \in I(x)} W(x,i), \qquad Y = \bigcup_{i \in \omega \setminus \{0\}} X_i, \qquad Y = \{x_\beta : \beta < \lambda\}.$$

We will construct a subset Y_β of Y for each $\beta < \lambda$.

Suppose that Y_γ has been constructed for each $\gamma < \beta$. Then define

$$Y_\beta = \begin{cases} \emptyset & ; \quad \text{if } x_\beta \in \bigcup_{\gamma<\beta} Y_\gamma \\ \{z \in Y : x_\beta \in W_z,\ z \notin \bigcup_{\gamma<\beta} Y_\gamma\} & ; \quad \text{otherwise.} \end{cases}$$

Sketch of proof of the Main Theorem

Pick any element x of X. There exists a unique $\alpha < \tau$ such that $x \in P_\alpha$.

If x belongs to Y, then there exists a unique $\beta < \lambda$ such that $x \in Y_\beta$.

Define an open neighborhood T_x of x as

$$T_x = \begin{cases} O_\alpha & ; \quad x \in X \setminus Y \\ int(W_x) \cap O_\alpha & ; \quad x \in Y_\beta \text{ and } x = x_\beta \\ \big(int(W_x) \setminus \{x_\beta\}\big) \cap O_\alpha & ; \quad x \in Y_\beta \text{ and } x \neq x_\beta. \end{cases}$$

Sketch of proof of the Main Theorem

For each $\alpha < \tau$, define

$$V_\alpha = \bigcup \{V(x, V(x, T_x)) : x \in P_\alpha\}$$

where $V(x, T_x)$ is an open set arising from the condition (F).

Using the above Lemma, one can see that the family

$$\mathcal{V} = \{V_\alpha : \alpha < \tau\}$$

is a point$- < \kappa$ open refinement of $\mathcal{O}$.

Results

Result-1

- If the space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying (F),

where

and,

then X is metacompact.

Results

Result-1

- If the space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying (F),
- and if, for each x,there exists a non-negative integer $n(x)$ such that $\mathcal{W}(x) = \bigcup_{i=0}^{n(x)} \mathcal{W}_i(x)$,

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and,

- $\mathcal{W}_i(x)$ is a chain of neighborhoods of x with respect to inclusion, for each i with $1 \leq i \leq n(x)$

then X is metacompact.

Results

Result-2

- If the space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying (F),

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Result-2

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where

- $\mathcal{W}_i(x)$ is a chain of neighborhoods of x with respect to inclusion for each i,

then X is metacompact.

Results

Result-3

- If the space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying neighborhood (F),

then X is metacompact.

Results

Result-3

- If the space X has a $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying neighborhood (F),
- and if, each $\mathcal{W}(x)$ is of finite rank

then X is metacompact.

By employing Result-3, the following claim first proved by Gruenhage and Nyikos [**4**] can be obtained in a different manner.

Result-4

If the space X has a base $\mathcal{B}$ of point-finite rank (that is, for each x, the family $\{B \in \mathcal{B} : x \in B\}$ is of finite rank), then X is metacompact.

References

1 P. J. Collins and A. W. Roscoe, *Criteria for metrisability,* Proc. Amer. Math. Soc. **90** (1984), 631-640.

2 P.J. Collins, G. M. Reed, A. W. Roscoe, M. E. Rudin, *A lattice of conditions on topological spaces*, Proc. Amer. Math. Soc. **94** (1985), 487-496.

3 P.M. Gartside and P. J. Moody, *Well-ordered* (F) spaces, Topology Proc. **17** (1992), 111-130.

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$\mathcal{T} \quad \mathcal{H} \quad \mathcal{A} \quad \mathcal{N} \quad \mathcal{K} \qquad \mathcal{Y} \quad \mathcal{O} \quad \mathcal{U}$