The Efimov System and Erdös-Rado Theorem

Marian Turzanski

Cardinal Stefan Wyszynski University

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Abstract

The strong sequences method was introduced by B. A. Efimov, as a useful method for proving famous theorems in dyadic spaces: Marczewski theorem on cellularity, Shanin theorem on a calibre, Esenin-Volpin theorem. The aim of this paper is to introduce a new cardinal invariant $l(X)$- a length of the strong sequence and to investigate relations between $l(X)$ and other well known invariants like: saturation, boundeness, cofinality, calibre.

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p\colon p\colon T \to \{0,1\}\}.$$

For $s \subset T$, $i\colon s \to \{0,1\}$ it will be used the following notation

$$H_s^i = \{p \in D^T\colon p|s = i\}.$$

A continuous image of a Cantor cube will be called *a dyadic space*.

Efimov defined strong sequences in the subbase $\{H^i_{\{\alpha\}} : \alpha \in T\}$ of the Cantor cube

Definition

A pair (H^i_s, H^i_v) where $card(s) < \omega$ will be called the connected pair if $H^i_s \cap H^i_v \neq \emptyset$

Definition

A sequence $\{(H^{i_\alpha}_{s_\alpha}, H^{i_\alpha}_{v_\alpha}) : \alpha < \kappa\}$ cosisting of connected pairs is called a strong sequence if $H^{i_\alpha}_{s_\alpha} \cap H^{i_\beta}_{v_\beta} = \emptyset$ whenever $\alpha > \beta$.

Theorem (Efimov [?])

Let κ be a regular, uncountable cardinal number. In the space D^T there is not a strong sequence

$$\{(H^i_{v_\xi}, H^i_{w_\xi}) \ ; \ \xi < \kappa\}$$

such that $|w_\xi| < \kappa$ and $|v_\xi| < \omega$ for each $\xi < \kappa$.

Let X be a topological space. Denote by

$c(X)$-celularity of the space X.

$d(X)$-density of the space X.

$\chi(X)$-character of the space X.

The following theorems could be proved with using strong sequences:

1. Marczewski theorem on cellularity of the Cantor cube,
2. Shanin theorem on calibres of dyadic spaces ,
3. Esenin-Volpin theorem on a weight of a dyadic space.

Let X be a set, and $B \subset P(X)$ be a family of non-empty subsets of X closed with respect to the finite intersections. Let S be a finite subfamily and $S \subseteq B$. A pair (S, H), where $H \subseteq B$, will be called *connected* if $S \cup H$ is centered.

Definition

A sequence $\{(S_\phi, H_\phi)\ ;\ \phi < \alpha\}$ consisting of connected pairs is called *a strong sequence* if $S_\lambda \cup H_\phi$ is not centered whenever $\lambda > \phi$.

Theorem

If for $B \subset P(X)$ there exists a strong sequence $S = (S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$ then the family B contains a subfamily of cardinality λ^+ consisting of pairwise disjoint sets.

Theorem

If B is a family of sets of cardinality $(2^\lambda)^+$, then the family B contains a subfamily of cardinality λ^+ consisting of pairwise disjoint sets or contains a centered subfamily of cardinality $(2^\lambda)^+$.

Let (X, r) be a set with relation r. Let $a, b \in X$. We say that elements a and b are *comparable* if $(a, b) \in r$ or $(b, a) \in r$.
We say that a is *compatible* with b (or a and b are compatible) if there exists c such that

$$(c, a) \in r \text{ and } (c, a) \in r.$$

(We say then, that a, b have a *lower bound*). We say that element a is *compatible* with element b (or elements a and b are compatible) if there exists c such that

$$(a, c) \in r \text{ and } (b, c) \in r.$$

(We say then, that a, b have an *upper bound*).
We say that $\mathcal{L} \subset X$ is a *chain* if any $a, b \in \mathcal{L}$ are comparable.

We say that a set $\mathcal{A} \subset X$ is an *antichain* if any two distinct elements $a, b \in \mathcal{A}$ are incompatible. If each of two elements in a set $A \subset X$ are compatible, then A is a *lower directed* set. A set A is κ- *lower directed* if every subset of A of cardinality less than κ has a lower bound, i.e. for each $B \subset A$ with $|B| < \kappa$ there exists $a \in A$ such that $(a, b) \in r$ for all $b \in B$.

Definition

Let (X, r) be a set with relation r.
A sequence $(S_\phi, H_\phi); \phi < \alpha$ where $S_\phi, H_\phi \subset X$ and S_ϕ is finite is called a strong sequence if
1^o $S_\phi \cup H_\phi$ is ω-lower directed (ω-upper directed)
2^o $S_\beta \cup H_\phi$ is not ω-lower directed (ω-upper directed) for $\beta > \phi$.

Consider the following notations:

$$l(X) = sup\{\kappa\colon \text{there exists a strong sequence in } X \text{ of the length } \kappa\}.$$

$$s(X) = min\{\kappa\colon \text{ there is not exist a strong sequence in } X \text{ of the length } \kappa\}.$$

Let consider the following definition of a calibre.

Definition

A cardinal κ is *a calibre* for X if κ is infinite and every set $A \in [X]^\kappa$ has κ -lower directed subset.

Definition

A cardinal κ is a *precalibre* for X if κ is infinite and every set $A \in [X]^\kappa$ has ω -lower directed subset of cardinality κ (ω -upper directed subset of cardinality κ).

Theorem

Let (X, r) be an arbitrary set with relation. Each regular cardinal number greater then $s(X)$ is precaliber of X.

A space X is ccc if each collection of disjoint open sets is at most countable. A Souslin line is a ccc nonseparable linearly ordered topological space. It is consistent with the usual axioms for set theory that a Souslin line exists . Such a line yields an example of a perfectly normal first countable ccc nonseparable space

MURRAY G. BELL,"A Normal First Countable ccc Nonseparable Space" Proceedings of the American Mathematical Society Volume 74, Number 1, April 1979

Example

An example of a strong sequence of a lenght greater than a celurality of a family of sets.
Let us take a regular topological space with $c(X) < d(X)$ and $\chi(X) < d(X)$. Let M be a dense set with $|M| = d(X)$. Let $B(x)$ be a base in a point x such that $|B(x)| \leq \chi(X)$.) Let us start from an arbitrary point $x \in M$ and an arbitrary set $U \in B(x)$. We take $(\{U\}, B(x))$ as a first pair of a strong sequence. Suppouse that for $\alpha < \gamma < d(X)$ a strong sequence $\{(\{U_\alpha\}, B(x_\alpha)\} : \alpha < \gamma\}$ has been defined. Let us take the set $\{x_\alpha : \alpha < \gamma\}$. The set $X \setminus cl(\{x_\alpha : \alpha < \gamma\}) \cap M$ is nonempty. Take a point $x_\gamma \in X \setminus cl(\{x_\alpha : \alpha < \gamma\}) \cap M$. Since the space X is regular hence there exists a set $U_\gamma \in B(x_\gamma)$ such that $U_\gamma \cap cl(\{x_\alpha : \alpha < \gamma\})$ is empty. Hence for each $\alpha < \gamma$ we have $\{U_\gamma\} \cup B(x_\alpha)$ is not centered. Hence the strong sequence $\{(U_\gamma, B(x_\gamma) : \gamma < d(X)\}$ has been defined.

Theorem

Let X be a set and r be a relation on $[X]^{<\omega}$. Let $(S\alpha, H\alpha); \alpha < (\kappa)^{\lambda^+}$ be a strong sequence such that $|(H\alpha)| \leq \kappa$ for each $\alpha < (\kappa)^{\lambda^+}$. Then there exists a strong sequence $(S\alpha, T\alpha); \alpha < (\lambda)^+$, where $|(T\alpha) < \omega$ for each $\alpha < (\lambda)^+$.

Definition

Let φ be an ordinal. A system (φ, G, F) consisting of two functions

$$G : \varphi \to [\varphi]^{\leq m}$$

$$F : \bigcup rng(G) \to 2^{\varphi}$$

such that
$(*)$ for any condition $\beta < \alpha$ there exists $b \in G(\beta)$ such that $\alpha \in F(b)$ is said to be *the Efimov's system*.

Definition

Let $g : K \to \varphi$ where $K \subset \varphi$ is a selector of G.
The selector g is called *a selector of Efimov's system* if for any α, β a condition $\beta < \alpha$ implies $\alpha \in F(g(\beta))$.

Theorem

If $\Phi = ((\kappa)^\lambda)^+$ *and* $|G(\alpha)| \leq \kappa$ *for each* $\alpha < \Phi$, *then there exists a selector* $g : K \longrightarrow \Phi$ *of the Efimov system such that* $|K| \geq (\lambda)^+$

From Efimov's system to strong sequences

For each H_φ let

$$G(\varphi) = \{T : T \subset H_\varphi, |T| < \omega \text{ and there exists } \beta > \varphi \text{ such that } \backsim (TrS_\beta)\}.$$

Let $X = \{T : T \in G(\varphi) \text{ for some } \varphi\}$.

Let us define the functions:

$$G : (\kappa)^{\lambda})^{+} \longrightarrow 2^{X} : \varphi \longrightarrow G(\varphi)$$

and

$$F : X \longrightarrow ((\kappa)^{\lambda})^{+} : T \longrightarrow \{\beta :\backsim (TrS_{\beta})\}$$

. We shall show that (F, G) is the Efimov's system.

Theorem (Erdös-Rado)

Let λ be an infinite cardinal number and $\mathcal{F}$ be a partition of $[X]^2$ of cardinality not greater than λ. If the cardinality of the set X is greater than 2^λ, then there exists a subset $Y \subseteq X$ of the cardinality greater than λ such that the family $[Y]^2$ is contained in some element of $\mathcal{F}$.

From Erdös-Rado Theorem to Efimov's system.

Let $\varphi = (2^m)^+$. Let (φ, G, F) consisting of two functions

$$G : \varphi \to [\varphi]^{\leq m}$$

$$F : \bigcup rng(G) \to 2^\varphi$$

such that

$(*)$ for any condition $\beta < \alpha$ there exists $b \in G(\beta)$ such that $\alpha \in F(b)$ is said be the Efimov's system.

Let for $\forall_{\beta<\varphi}$ $G(\beta) = \{x_\gamma : \gamma < m\}$.
Let $P : [\varphi]^2 \longrightarrow m$ be such that

$$P(\{\alpha, \beta\}) = \gamma \Rightarrow \alpha \in F(x_\gamma(\beta))$$

. From the Erdös-Rado Theorem there exista $K \subset \varphi$, $|K| = m^+$ such that for some $\gamma < m$, $[K]^2 \subset P^{-1}(\gamma)$

From this it follows that

$$\forall_{\alpha<\beta}\ \alpha \in F(x_\gamma(\beta)),\ \alpha, \beta \in K$$

. Then

$$g(\beta) = x_\gamma(\beta)$$

is the selector of Efimov's system.