Pontryagin reflexivity in precompact groups

Mikhail Tkachenko
Universidad Autónoma Metropolitana, Mexico City
mich@xanum.uam.mx

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Dual group

Let G be a topological Abelian group. We denote by $\mathbb{T}$ the circle group considered as a (compact) topological subgroup of the complex plane $\mathbb{C}$.

A character of G is a continuous homomorphism $\chi\colon G \to \mathbb{T}$.

The dual group $G^\wedge$ is the group of all characters of G.

Pointwise mutiplication in $G^\wedge$:

$$(\chi_1 \cdot \chi_2)(x) = \chi_1(x) \cdot \chi_2(x) \text{ for all } \chi_1, \chi_2 \in G^\wedge \text{ and } x \in G.$$

$G^\wedge$ carries the compact-open topology:

Given a compact set $K \subset G$, the polar of K is the set

$$K^\triangleright = \{\chi \in G^\wedge : \chi(K) \subset \mathbb{T}_+\},$$

where $\mathbb{T}_+ = \{e^{i\alpha} : -\pi/2 \leq \alpha \leq \pi/2\}$ is the right half of the circle.

Dual group

A neighborhood base of $G^\wedge$ at the neutral element e (the trivial homomorphism of G to $\mathbb{T}$) is the family

$$\{K^\triangleright : K \subset G,\ K \text{ is compact}\}.$$

In other words, $G^\wedge$ carries the topology of uniform convergence on compact subsets of G, say τ_c.

The group $(G^\wedge, \tau_c)$ is Hausdorff for any topological Abelian group G.

Pontryagin-van Kampen duality

The second dual group $G^{\wedge\wedge} = (G^{\wedge})^{\wedge}$ carries the compact-open topology (determined by compact subsets of $G^{\wedge}$).

The evaluation homomorphism $\alpha_G \colon G \to G^{\wedge\wedge}$ is defined by

$$\alpha_G(x)(\chi) = \chi(x),$$

for all $\chi \in G^{\wedge}$ and $x \in G$.

If α_G is a topological isomorphism of G onto $G^{\wedge\wedge}$, then G is said to be (Pontryagin) reflexive.

Theorem 1.1 (Pontryagin–van Kampen, 1934-1935).

Every locally compact Abelian (LCA) topological group is reflexive. The dual of a discrete group is compact and the dual of a compact group is discrete.

Extensions of Pontryagin-van Kampen's duality theorem

Theorem 2.1 (S. Kaplan, 1948, 1950).

a) The product of an arbitrary family of reflexive groups is reflexive.
b) The projective limit of a sequence of LCA groups is reflexive.

In particular, the groups $\mathbb{R}^\kappa$ and D^κ are reflexive for each κ, where $\mathbb{R}$ is the (locally compact) group of reals and D is an arbitrary discrete Abelian group.

Theorem 2.2 (F. M. Smith, 1952).

The additive group of every Banach space is (Pontryagin) reflexive.

Definition 2.3.

A topological Abelian group G is called strongly reflexive if all closed subgroups and all Hausdorff quotients of G and of $G^\wedge$ are reflexive.

Strong reflexivity $\Rightarrow$ reflexivity

Every infinite-dimensional Banach space is a reflexive, but not strongly reflexive group.

Extensions of Pontryagin-van Kampen's duality theorem

Theorem 2.4 (Banaszczyk, 1986, 1991).
Every complete metrizable nuclear group G is strongly reflexive. Furthermore, if $\{A_n : n \in \omega\}$ is a sequence of LCA groups, then the product $G \times \prod_{n\in\omega} A_n$ is strongly reflexive.

Theorems 2.1 and 2.2 $\Rightarrow$ reflexive groups need not be locally compact. However, all reflexive groups that one can construct starting from LCA groups and applying Theorems 2.1, 2.2, and 2.4 are complete.

Question. *Are there reflexive groups that fail to be complete*?

Completeness of reflexive groups

Theorem 2.5 (Y. Komura, 1984).

There exist non-complete linear topological spaces which are (Pontryagin) reflexive.

Theorem 2.6 (Chasco, Außenhoffer, 1998).

Every reflexive metrizable topological group is complete.

Subgroups of compact topological groups are called precompact.

Corollary 2.7.

Every precompact reflexive metrizable group is compact.

The linear topological spaces in Komura's theorem are very far from precompact groups since they contain subgroups topologically isomorphic to the real line $\mathbb{R}$.

Question (M. J. Chasco and E. Martín-Peinador, 2007). *Is every precompact reflexive group compact (equiv., complete)?*

Precompact reflexive groups

A space X is pseudocompact if every continuous real-valued function on X is bounded.

compact $\Rightarrow$ countably compact $\Rightarrow$ pseudocompact

Comfort and Ross, 1966: Every pseudocompact topological group is precompact.

Theorem 2.8 (Ardanza-Chasco-Domínguez-Tkachenko and Galindo-Macario, 2008).

Every pseudocompact topological Abelian group without infinite compact subsets is reflexive.

The existence of pseudocompact groups without infinite compact subsets:

Theorem 2.9 (Tk., 1988).

The group $\mathbb{Z}_2^{\mathfrak{c}}$ contains a dense pseudocompact subgroup without infinite compact subsets, where $\mathbb{Z}_2 = \{0, 1\}$ and $\mathfrak{c} = 2^{\omega}$.

Precompact reflexive groups

Here is a considerably more general existence result:

Theorem 2.10 (Galindo and Macario, 2011).

If an Abelian group G admits a pseudocompact Hausdorff topological group topology, then in "almost all cases" G admits another pseudocompact Hausdorff topological group topology in which all compact subsets of G are finite.

Theorem 2.11 (A-C-D-T., 2012).

Every pseudocompact Abelian group is a quotient of a pseudo-compact reflexive group (w.r.t. a closed pseudocompact subgroup).

Concluding:

Corollary 2.12.

There are lots of pseudocompact non-compact (hence non-complete) reflexive Abelian groups.

Duality in P-groups

Question. *Do there exist countably compact non-compact reflexive groups?*

Theorem 3.1 (Ardanza-Chasco-Domínguez-Tk., 2012).

Under CH, there exists a proper, dense, countably compact, reflexive subgroup G of $\mathbb{Z}_2^{\omega_1}$, where $\mathbb{Z}_2 = \{0, 1\}$. In addition, G is hereditarily normal and hereditarily separable.

Our aim is to answer the above question in *ZFC* alone.

Definition 3.2.

A topological group H is called a P-group if every G_δ-set in H is open. The P-modification PH of a topol. group H is the same underlying group H endowed with the topology whose base consists of G_δ-sets in H.

$$H \text{ is } P\text{-group} \iff PH = H.$$

Duality in P-groups

Fact. *Open subgroups form a base at the identity of every P-group, i.e., every P-group carries a linear topology.*

Lemma 3.3.

Let G be an Abelian P-group. Then the following hold:

(a) *the evaluation homomorphism $\alpha_G \colon G \to G^{\wedge\wedge}$ is an open (not necessarily continuous) isomorphism of G onto $G^{\wedge\wedge}$;*

(b) *α_G is continuous iff every compact set $K \subseteq G^{\wedge}$ is constant on an open subgroup of G.*

Thus a P-group G is reflexive $\iff$ α_G is continuous.

Definition 3.4.

A subset A of a space X is subseparable if there exists a separable subspace B of X such that $A \subseteq B$.

Duality in P-groups

Theorem 3.5 (Galindo-Recoder-Tk., 2011).

Let $D = \prod_{i \in I} D_i$ be a product of discrete Abelian groups. Then every compact subset K of the dual group $(PD)^\wedge$ is subseparable. Hence K is constant on an open subgroup of PD. Thus the group PD is reflexive.

Does this help to find countably compact reflexive groups?

Lemma 3.6.

If G is a P-group, then the dual group $G^\wedge$ is ω-bounded, i.e., the closure of every countable subset of $G^\wedge$ is compact. Hence $G^\wedge$ is countably compact.

Corollary 3.7.

There exist countably compact (in fact, ω-bounded) non-compact reflexive groups.

For example, take the group PD, where $D = \mathbb{Z}^\kappa$ or $D = \mathbb{Z}_2^\kappa$ with $\kappa > \omega$. Then $G = (PD)^\wedge$ is as required.

Permanence properties of reflexive P-groups

Theorem 3.8 (Galindo-Recoder-Tk., 2011).

(a) *Let G be a a dense subgroup of a topological group $\widetilde{G}$. If G is a reflexive P-group, so is $\widetilde{G}$.*

(b) *Let $\pi\colon G \to H$ be a continuous open epimorphism of topological groups. If G is a reflexive P-group, so is H.*

(c) *Let H be a closed subgroup of a topological Abelian group G. If both H and G/H are reflexive P-groups, so is G.*

Neither of the above items (a), (b), (c) is valid outside the class of P-groups.

Problem 3.9.
Let $G = \prod_{i\in I} G_i$ be a product of reflexive P-groups. Is the P-modification PG of G reflexive?

It suffices to consider the case when $|I| = \omega$!

Reflexivity is not stable w.r.t. P-modification

Example 3.10.

There exists a reflexive, σ-compact, complete group G such that the P-modification PG of G fails to be reflexive.

Let X be a one-point compactification of an uncountable discrete space and $A(X)$ the free Abelian topological group over X. Then $G = A(X)$ is σ-compact, complete, reflexive (Pestov's theorem), but PG is not reflexive:

The evaluation homomorphism $\alpha_{PG} \colon PG \to (PG)^{\wedge\wedge}$ is discontinuous.

Sometimes P-modification "respects" reflexivity

Let us strengthen σ-compactness to compactness:

Theorem 3.11 (Galindo-Recoder-Tk., 2011).

If G is an arbitrary compact Abelian group, then the P-modification PG of G is reflexive.

Sketch of the proof.

Fact 1. Every compact Abelian group is a continuous homomorphic image of a product of compact metrizable Abelian groups (Vilenkin's theorem, 1958).

Let $f\colon K = \prod_{i\in I} K_i \to G$ be a continuous onto homomorphism, where each K_i is a compact metrizable group. Clearly, f is open.

Fact 2. $f\colon PK \to PG$ is continuous and open.

Fact 3. [Theorem 3.5] The group $PK \cong P(\prod_{i\in I} PK_i)$ is reflexive as the P-modification of the product of discrete groups PK_i's.

Fact 4. [Theorem 3.8] Quotients of reflexive P-groups are reflexive.

It remains to apply Facts 2 and 3.

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