The small inductive dimension in Alexandroff spaces

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Introduction: considered spaces

This is a joint talk with S.-E. Han and Y. Hattori.

Definition (Johnstone, 1982)

A topological space X is called *Alexandroff* if for each point $x \in X$ there is the minimal open set $V(x)$ containing x.

Alexandroff spaces were first considered by Alexandroff in 1937 under the name of "discrete spaces".

Remark

If X is a T_0-space and $x, y \in X$ then $V(x) = V(y)$ iff $x = y$.

Moreover, if X is a T_1-space then $V(x) = \{x\}$ for each point $x \in X$, i.e. an Alexandroff space X is a T_1-space iff X is discrete.

Introduction: Alexandroff spaces in digital topology

Alexandroff spaces appear by a natural way in studies of topological models of digital images (Kronheimer, 1992).
They are quotient spaces of the Euclidean spaces $\mathbb{R}^n$ defined by special decompositions.

Problem

What is the dimension of such spaces ?

Remark

Since Alexandroff spaces are non-trivial only with the axioms lower than T_1, the standard dimensions ind, Ind, dim can not be used.

Introduction, considered dimension

Definition (Pears, Dimension theory of general spaces, 1975)

Let X be a space and n an integer ≥ 0. Then

(a) $\text{ind } X = -1$ iff $X = \emptyset$;

(b) $\text{ind } X \leq n$ iff for each point $x \in X$ and each open set V containing x there is an open set W such that $x \in W \subset V$ and $\text{ind } Bd_X W < n$;

(c) $\text{ind } X = \infty$ iff $\text{ind } X \leq n$ does not valid for each integer $n \geq 0$.

It is easy to see that if $\text{ind } X = n$ for some integer $n \geq 0$ then the cardinality of X is greater than n.
Recall also that for any space X and any subspace Y of X we have $\text{ind } Y \leq \text{ind } X$.

Introduction: some history

In 1993 P. Wiederhold and R. G. Wilson started to study the behavior of the small inductive dimension ind in the Alexandroff T_0-spaces. In particular,

• they proved the product theorem: ind $(X \times Y) \leq$ ind $X+$ ind Y for non-empty Alexandroff T_0-spaces X, Y;

• they showed that if (X, τ) is an Alexandroff T_0-space and $\leq_\tau$ is its specification order (i.e. $x \leq y$ iff $x \in Cl_X(\{y\})$) then ind (X, τ) is equal to the partial order dimension of (X, τ) defined as the supremum of all lengths of chains in $(X, \leq_\tau)$.

• they established that the quotient spaces of the Euclidean spaces $\mathbb{R}^n$ defined by some standard decompositions based on the model of Kronheimer have the dimension ind equal to n.

Sum and product theorems for Alexandroff spaces

Proposition 1

Let X be an Alexandroff space and $X = X_1 \cup X_2$, where X_i, $i = 1, 2$, is closed in X. Then ind $X = \max\{$ ind X_1, ind $X_2\}$.

Proposition 2

Let X and Y be non-empty Alexandroff spaces. Then ind $(X \times Y) =$ ind $X+$ ind Y.

Remark

Unlike to P. Wiederhold and R. G. Wilson product theorem we have equality.

Khalimsky line

Definition (Khalimsky, 1969)

The Khalimsky line K is the topological space $(\mathbb{Z}, \tau)$, where $\mathbb{Z}$ is the set of integers and τ is the topology of $\mathbb{Z}$ generated by the base $\mathcal{B} = \{\{2k+1\}, \{2k-1, 2k, 2k+1\} : k \in \mathbb{Z}\}$.

Note that K is a connected Alexandroff T_0-space with ind $K = 1$.

Remark

Khalimsky spaces $K^n, n \geq 1$, are widely considered in the digital topology.

An application of the sum and the product theorems

It is well known that ind $K^n = n, n \geq 1$.

We present the following statement.

Theorem (A characterization of k-dimensional sets of K^n, $k \leq n$.)

Let A be a subspace of K^n for some natural number n and k a natural number with $k \leq n$. Then ind $A \geq k$ iff A contains an element of the family $\mathbb{S}(k, n)$.

The family $\mathbb{S}(k, n)$ will be defined below.

The family $\mathbb{S}(k, n)$, $k \leq n$

- First let us consider the family $\mathbb{S}(n)$, $n \geq 1$, which consists of subsets P of K^n of cardinality $n+1$, described as follows.

For each $P \in \mathbb{S}(n)$ there exist a sequence $a_1, \ldots, a_n$ of n even integers, a sequence $b_1, \ldots, b_n$ of n odd integers and a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that

(a) $|a_i - b_i| = 1$ for each $i \leq n$,

(b) $P = \{x_1, \ldots, x_{n+1}\}$, where $x_1 = (a_1, \ldots, a_n)$, $x_{n+1} = (b_1, \ldots, b_n)$ and for each $i \leq n$ the point x_{i+1} is obtained from the point x_i through replacing in the $\sigma(i)$-th coordinate the even number $a_{\sigma(i)}$ by the odd number $b_{\sigma(i)}$.

- Let k be any integer with $0 \leq k \leq n$.

Put $\mathbb{S}(k, n) = \{P : P \subset S,\ S \in \mathbb{S}(n) \text{ and } |P| = k+1\}$.

A comparison with the Euclidean spaces $\mathbb{R}^n$

Recall that a subset A of the Euclidean space $\mathbb{R}^n$ is n-dimensional iff A contains a non-empty open subset of $\mathbb{R}^n$. Since the Euclidean topology is regular this equivalence can be rewritten as follows: a subset A of the Euclidean space $\mathbb{R}^n$ is n-dimensional iff A contains the closure of a non-empty open subset of $\mathbb{R}^n$.
In the space K^n, the closure $\mathrm{Cl}_{K^n} V(x)$ of the minimal open neighborhood $V(x)$ of $x = (x_1, \ldots, x_n) \in K^n$ with m odd coordinates contains exactly $2^{2n-m} \cdot n!$ different n-dimensional in the sense of ind subsets of cardinality $n+1$.

A decomposition of K^n into $n+1$ discrete subsets

- For each $0 \le j \le n$ put
$Z_j = \{(x_1, \ldots, x_n) \in K^n : |\{i \le n : x_i \text{ is an even number}\}| = j\}$.

- Note that the sets Z_j, $0 \le j \le n$, are disjoint, and $K^n = \cup_{i=0}^{n} Z_i$.

- Furthermore, since each Z_i contains no elements of family $\mathbb{S}(1, n)$, it follows from the Theorem that ind $Z_i = 0$ for each $j \le n$.

We have also the following additional property of the sets Z_i, $i = 0, 1, \ldots$

Corollary

Let A be a subset of K^n with ind $A = n$ and $A_i = A \cap Z_i$. Then, ind $(\cup_{i=1}^{k} A_{j_i}) = k - 1$ for any $0 \le j_1 < \cdots < j_k \le n$.

The concluding question and remarks

Question

Let A be a subspace of K^n with ind $A = k < n, k \geq 1$. Are there disjoint subsets $A_0, \ldots, A_k$ of A such that ind $(\cup_{i=1}^{m} A_{j_i}) = m - 1$ for any $0 \leq j_1 < \cdots < j_m \leq k$?

Remark

• One can consider a natural transfinite extension of ind the dimension trind for which the corresponding sum and product (however, the inequality) theorems are valid.

• Moreover, for each ordinal α there is an Alexandroff space $X(\alpha)$ such that trind $X(\alpha) = \alpha$.

Thank you!