Graph-theoretic aspects of continuous weak selections

Dmitri Shakhmatov

first half of the talk is joint work with Masato Nagao

A *weak selection* on a set X is a function φ which assigns to each two-point subset A of X one of its elements $\varphi(A)$.

Clearly, φ defines a tournament (= a directed complete graph) on the set X in which a team $a \in X$ wins over a team $b \in X$ if and only if $\varphi(\{a, b\}) = b$.

A weak selection φ on a topological space X is *continuous* provided that, for every $a, b \in X$ with $a \neq b$ and each open neighbourhood W of $\varphi(\{a, b\})$, there exist an open neighbourhood U of a and an open neigbourhood V of b such that $\varphi(\{a', b'\}) \in W$ whenever $a' \in U$, $b' \in V$ and $a' \neq b'$; in other words, φ is continuous when the set $[X]^2$ of all two-point subsets of X is equipped with the Vietoris topology.

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In our terminology, the classical "king chicken" theorem of Landau about the existence of kings in finite tournaments (= finite directed complete graphs) reads as follows.

Theorem

Let φ be a weak selection on a non-empty finite set X. Then there exists $z \in X$ such that, for every $x \in X \setminus \{z\}$, either $\varphi(\{x, z\}) = x$ or one can find an element $y \in X$ different from both x and z such that $\varphi(\{x, y\}) = x$ and $\varphi(\{y, z\}) = y$.

An element z in the above theorem is called a *king* for the weak selection φ.

We extend this classical theorem from the finite case to all compact Hausdorff spaces equipped with a *continuous* weak selection.

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Every continuous weak selection on a non-empty compact Hausdorff space has a king.

We also show that the compactness of X is not only sufficient but often also a necessary condition for the existence of a king for every continuous weak selection on X.

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Let X be a Tychonoff space having at least one continuous weak selection. Suppose also that every continuous weak selection on X has a king. Then X is compact in each of the following cases:

(i) *X is pseudocompact;*

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Despite these positive results, a complete reversal of our theorem is not possible.

Example

There exists a connected G_δ-subspace X of the unit square admitting precisely two continuous weak selections both of which have a king, yet X is not compact. Therefore, a *completely metrizable (separable) connected space X need not be compact, even if all continuous weak selections on X have kings.*

Recall that a space is *analytic* if it is a continuous image of the space of irrational numbers. All complete metric spaces (in particular, all compact spaces) are analytic.

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There exists a connected subspace Y of the unit square admitting precisely two continuous weak selections both of which have a king, yet Y is not even analytic (much less completely metrizable or compact).

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Given a weak selection φ on a set X, one may wonder how big the set K_φ of all kings for φ could be. In particular, can one always find a continuous weak selection φ on X such that $K_\varphi = X$?

It is useful to provide a reformulation of this question in terms of the well-known notion of 3-cycles for φ.

Let φ be a weak selection on X. For $x, y \in [X]^2$, we write $x \to_\varphi y$ if either $x = y$ or $x \neq y$ and $\varphi(\{x, y\}) = y$.

A set $C \in [X]^3$ shall be called a *φ-cycle* provided that there exists an enumeration $C = \{c_1, c_2, c_3\}$ of C such that $c_1 \to_\varphi c_2$, $c_2 \to_\varphi c_3$ and $c_3 \to_\varphi c_1$.

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For a weak selection φ on a set X, the following conditions are equivalent:

(i) $K_\varphi = X$;

(ii) *for all $a, c \in X$, there exists $b \in X$ such that $a \to_\varphi b \to_\varphi c$;*

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Item (i) says that every element of X is a king for the tournament defined by φ. Item (ii) can be viewed as some sort of "pathwise connectivity" condition on φ. Finally, item (iii) says that φ has "sufficiently many" φ-cycles.

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We say that a weak selection φ on a set X is *cycling* if it satisfies any of the three equivalent conditions above.

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For a weak selection φ *on a set* X*, the following conditions are equivalent:*

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(i) $K_\varphi = X$;

(ii) *for all* $a, c \in X$, *there exists* $b \in X$ *such that* $a \to_\varphi b \to_\varphi c$;

(iii) *for every* $B \in [X]^2$ *there exists a* φ*-cycle* $C \in [X]^3$ *such that* $B \subseteq C$.

Item (i) says that every element of X is a king for the tournament defined by φ. Item (ii) can be viewed as some sort of "pathwise connectivity" condition on φ. Finally, item (iii) says that φ has "sufficiently many" φ-cycles.

Definition

We say that a weak selection φ on a set X is *cycling* if it satisfies any of the three equivalent conditions above.

Question

Given a set X, can we find a cycling weak selection on X?

For a finite set X with $|X| \geq 2$, Maurer gave a complete answer to this question by showing that there exists a weak selection φ on X such that $K_\varphi = X$ if and only if $|X| \notin \{2, 4\}$.
When X is infinite, the answer is always positive.

Proposition

Every infinite set admits a cycling weak selection.

Therefore, to make the problem meaningful, it becomes necessary to consider a *topological space* X and restrict yourself to *continuous* weak selections:

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Which infinite topological spaces have continuous cycling weak selections?

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Which infinite topological spaces have continuous cycling weak selections?

Recall that a topological space X is *totally disconnected* provided that for any two distinct points x, x' of X there exists a clopen subset U of X such that U separates x and x' in a sense that $x \in U$ and $x' \notin U$.

Proposition

A topological space having a continuous cycling weak selection is totally disconnected.

Proposition (Gutev, Nogura)

If φ is a continuous weak selection on a topological space X and C is a φ-cycle, then any two distinct elements of C can be separated by a clopen subset of X.

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Theorem

If an infinite topological space X admits a weaker zero-dimensional separable metric topology, then X has a continuous cycling weak selection.

Theorem

For an infinite space X with a countable network, the following conditions are equivalent:

(i) *X admits a continuous cycling weak selection;*

(ii) *X is totally disconnected.*

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A separable metric space X admits a continuous cycling weak selection if and only if X is totally disconnected.

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A countable regular space admits a continuous cycling weak selection.

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A separable metric space X admits a continuous cycling weak selection if and only if X is totally disconnected.

What happens when the weight of X is uncountable? In particular, can this corollary be extended to all spaces of weight ω_1?

Theorem

Let X be either ω_1 or $\omega_1 + 1$, considered with the order topology. Then the set K_φ is countable for every continuous weak selection φ on X.

Corollary

Neither ω_1 nor $\omega_1 + 1$ admits a continuous cycling weak selection.

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There exists a compact zero-dimensional (ordered) space of weight ω_1 without a continuous cycling weak selection.

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Example

A one-point compactification of a discrete space has a continuous cycling weak selection.

Question

Characterize (linearly ordered zero-dimensional) compact spaces having a cycling weak selection.

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Characterize (linearly ordered zero-dimensional) compact spaces having a cycling weak selection.

H. G. Landau, On dominance relations and the structure of animal societies, I: Effect of inherent characteristics, Bull. Math. Biol. 13 (1951), 1–19.

S. B. Maurer, The king chicken theorems, Math. Mag. 53 (1980), no. 2, 67–80.

M. Nagao, D. Shakhmatov, On the existence of kings in continuous tournaments, Topol. Appl. (2012), http://dx.doi.org/10.1016/j.topol.2012.05.021

We say that a space X is a *king space* if every continuous weak selection φ on X has a φ-king; or equivalently, if $K_\varphi \neq \emptyset$ for every continuous weak selection φ on X.

Theorem

Every non-empty compact Hausdorff space is a king space.

Let X be a non-empty compact Hausdorff space, and let φ be a continuous weak selection on X. We must prove that $K_\varphi \neq \emptyset$.

Claim

The set

$$K_{\varphi,x} = \{z \in X : z \to_\varphi y \to_\varphi x \text{ for some } y \in X\}$$

is closed in X for every $x \in X$.

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Since X is Hausdorff and φ is continuous, $F = \{(a,b) \in X^2 : a \to_\varphi b\}$ is a closed subset of X^2. Then $F \times X$ and $X \times F$ are closed subsets of X^3, and so is their intersection

$$P = (F \times X) \cap (X \times F) = \{(a,b,c) \in X^3 : a \to_\varphi b \to_\varphi c\}. \quad (1)$$

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Claim

The family $\{K_{\varphi,x} : x \in X\}$ *has the finite intersection property.*

Let S be a non-empty finite subset of X. The restriction ψ of φ to $[S]^2$ is a weak selection on S. By the Landau Theorem in the finite case, there exists a ψ-king; that is,

$$K_\psi = \bigcap\{K_{\psi,x} : x \in S\} \neq \emptyset.$$

Note that $K_{\psi,x} \subseteq K_{\varphi,x}$ for every $x \in S$, so

$$\bigcap\{K_{\varphi,x} : x \in S\} \neq \emptyset$$

as well.

$$K_{\varphi,x} = \{z \in X : \exists\ y \in X\ (z \to_\varphi y \to_\varphi x)\}$$

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The set $K_{\varphi,x} = \{z \in X : \exists\ y \in X\ (z \to_\varphi y \to_\varphi x)\}$ *is closed in* X *for every* $x \in X$.

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Since X is compact, from these two claims we conclude that

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