Connectedness and the Stone-Čech Ordered Compactifications

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(set, topology, partial order)

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$T_{3.5}$-ordered:

(a) $x \not\leq y \Rightarrow \exists$ contin increasing $f : X \to \mathbb{R}$, $f(x) > f(y)$

(b) F closed, $a \notin F \Rightarrow \exists$ contin incr $f : X \to [0, 1]$, contin decr $g : X \to [0, 1]$ such that $f(a) = g(a) = 1$ and $f(x) \wedge g(x) = 0 \ \forall x \in F$.

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$(X, \tau, \leq)$ has an ordered compactification if and only if it is $T_{3.5}$-ordered.

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$$e : X \to \prod_{f \in C^*(X)} I_f \subseteq \mathbb{R}^{C^*(X)}$$

defined by $e(x) = \prod_{f \in C^*(X)} f(x)$ is an embedding iff X is $T_{3.5}$

$$e : X \to cl(e(X)) = \beta X \subseteq \prod_{f \in C^*(X)} I_f$$

The ordered Stone-Čech compactification $\beta_o X$:
$C^{*\uparrow}(X) = \{\text{contin bounded increasing functions } f : X \to \mathbb{R}\}$
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Thm: A is C^*-embedded in X iff $cl_{\beta X} A = \beta A$.

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Thm: A is $C^{*\uparrow}$-embedded in X iff $cl_{\beta_o X} A = \beta_o A$.

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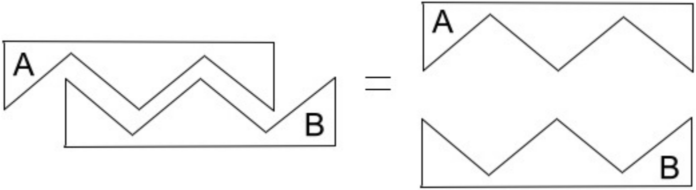

Proof: If A, B is a separation of X, $\exists$ contin surjection $s : X \to \{0, 1\}$ with $s(A) = 0, s(B) = 1$. Extend s to $\hat{s} : \beta X \to \{0, 1\}$. Then $\hat{s}(cl_{\beta X} A) = 0$ and $\hat{s}(cl_{\beta X} B) = 1$, so $cl_{\beta X} A$ and $cl_{\beta X} B$ from a separation of
$\beta X = cl_{\beta X}(X) = cl_{\beta X}(A \cup B) = cl_{\beta X} A \cup cl_{\beta X} B$.
Also, A and B are C^*-embedded in X:

For $f \in C^*(A)$, $f : A \to [-M, M]$, extend it to X by taking $\hat{f}(B) = M + 1$.

Thus, $cl_{\beta X} A = \beta A$ and $cl_{\beta X} B = \beta B$, so $\beta A, \beta B$ is a separation of βX.

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This is too restrictive to be useful.

Thm: If A, B form a separation of a locally convex *totally ordered* topological space X and $cl_{\beta_o X} A \cap cl_{\beta_o X} B = \emptyset$, then $\beta_o A$ and $\beta_o B$ form a separation of $\beta_o X$.

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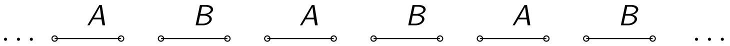
Example: $cl_{\beta_o X} A \cap cl_{\beta_o X} B$ need not be $\emptyset$.



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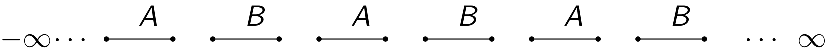

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$\hat{g}$ is a contin increasing extension of g, so A is $C^{*\uparrow}$-embedded.

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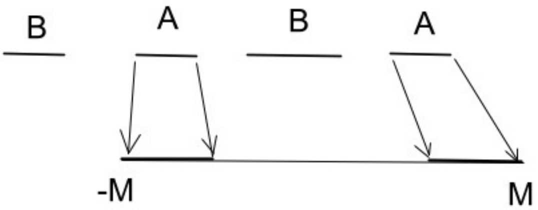

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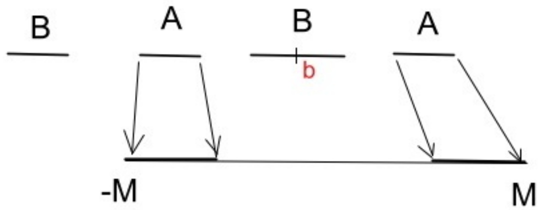

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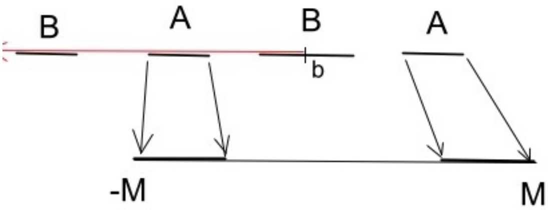

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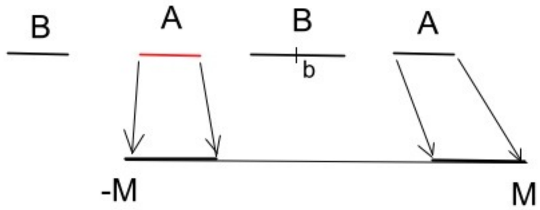

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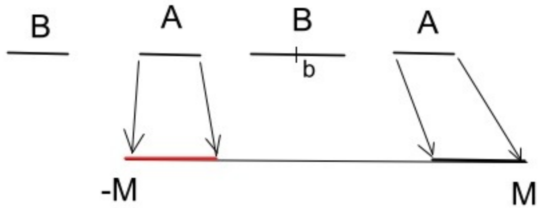

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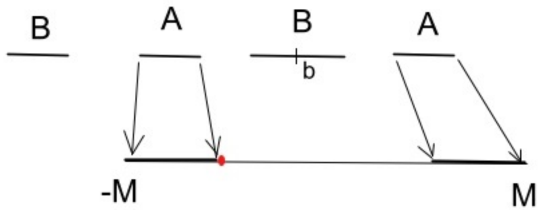

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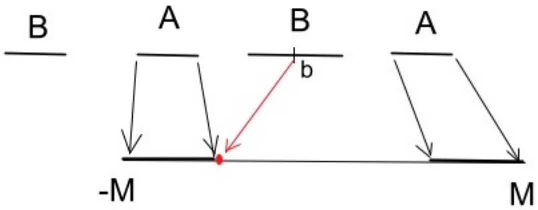

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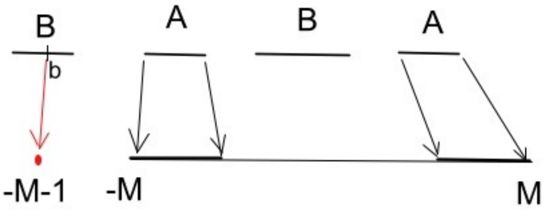

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These results do not hold for partially ordered topological spaces.

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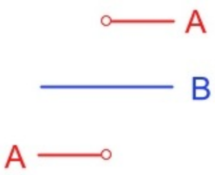

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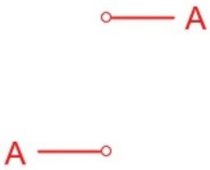

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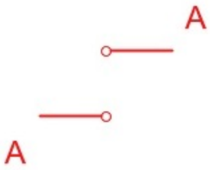

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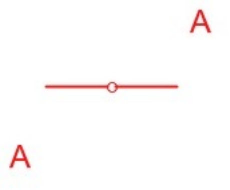

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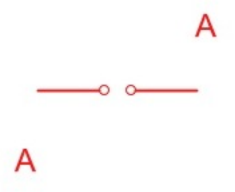

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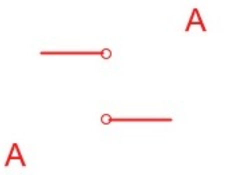

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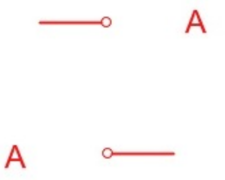

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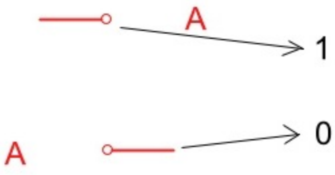

Here is a continuous increasing function f defined on A which cannot be extended to B.

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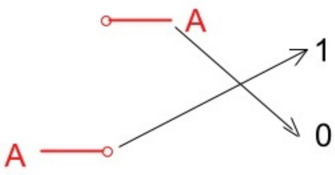

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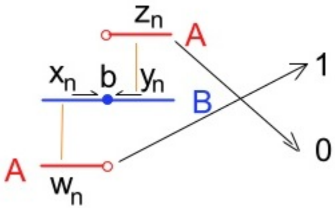

$w_n \leq x_n; \quad y_n \leq z_n$, so $1 = f(w_n) \leq f(x_n)$ and $f(y_n) \leq f(z_n) = 0$. Taking the limit shows that $1 \leq \hat{f}(b) \leq 0$ for any continuous incr extension $\hat{f}$.

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NONE. We will sketch the proof.

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If the smallest element a were in $\beta_o X - X$, then there exists a decreasing net in X converging to a, so X has no smallest element. But $X \approx \beta_o X - X$ would imply $\beta_o X - X$, and thus $\beta_o X$, has no smallest element.

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Engelking: If X is dense in T_2 space B, $\hat{f} : B \to Y$ is continuous, and $\hat{f}|_X$ is a homeomorphisms into Y, then $\hat{f}(B - X) \cap \hat{f}(X) = \emptyset$.

$C = \{x \in X : f(x) > x\}$ and $D = \{x \in X : f(x) < x\}$ is a separation of X.

$cl_{\beta_o X} C = \{x \in \beta_o X : \hat{f}(x) > x\}$ and
$cl_{\beta_o X} D = \{x \in \beta_o X : \hat{f}(x) < x\}$ are disjoint, so by a previous theorem, $\beta_o C$ and $\beta_o D$ are a separation of $\beta_o X$, and it follows that $\hat{f}|_C$ is a homeomorphism and order isomorphism from C to $\beta_o C - C$.
Thus the largest element w of $\beta_o C$ is in C, so

$$\hat{f}(w) < w \in C.$$

But $w \in C \Rightarrow \hat{f}(w) > w$. So, no such f exists.

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