Compact-like properties, normality and C^*-embeddedness of the hyperspace of compact sets

Yasser Fermán Ortiz Castillo
Center of Mathematical Ciences
Universidad Nacional Autónoma de México Campus Morelia

Working with Juan Angoa, Nobuyuki Kemoto, Reynaldo Rojas and Angel Tamariz.

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

$\mathcal{CL}(X)$ denote the hyperspace of non-empty closed sets of X with the Vietoris topology. $\mathcal{K}(X)$ is the subspace of compact sets.

The Vietoris topology has the sets of the form

$$V^+ = \{A \in CL(X) : A \subseteq V\} \text{ and } V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}$$

like a subbase, when V is an open set of X.

Given open sets of X, $U_1, \ldots U_n$, define

$$< U_1, \ldots U_n >= \{T \in CL(X) : T \in \cup_{1 \leq k \leq n} U_k^+,\ T \in U_k^-\}.$$

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

Theorem

(M.) $\mathcal{CL}(X)$ *is:*

1. T_2 *iff* X *is* T_3,
2. T_3 *iff* $\mathcal{CL}(X)$ *is Tychonoff iff* X *is* T_4,
3. T_4 *iff* $\mathcal{CL}(X)$ *is compact iff* X *is compact.*

Theorem

(M.) $\mathcal{K}(X)$ *is:*

1. T_2 *iff* X *is* T_2,
2. T_3 *iff* X *is* T_3,
3. *Tychonoff iff* X *is Tychonoff.*

Theorem

(M.) $\mathcal{CL}(X)$ *is:*

1. T_2 *iff* X *is* T_3,
2. T_3 *iff* $\mathcal{CL}(X)$ *is Tychonoff iff* X *is* T_4,
3. T_4 *iff* $\mathcal{CL}(X)$ *is compact iff* X *is compact.*

Theorem

(M.) $\mathcal{K}(X)$ *is:*

1. T_2 *iff* X *is* T_2,
2. T_3 *iff* X *is* T_3,
3. *Tychonoff iff* X *is Tychonoff.*

About the normality of $\mathcal{K}(X)$

Theorem

(M.) $\mathcal{K}(X)$ is metrizable iff X is it.

Note that $\mathcal{CL}(X)$ is metrizable iff X is compact metrizable.

Theorem

(Moresco and Artico) If L is the Sorgenfrey line then $\mathcal{K}(L)$ is not normal.

About the normality of $\mathcal{K}(X)$

Theorem

(M.) $\mathcal{K}(X)$ is metrizable iff X is it.

Note that $\mathcal{CL}(X)$ is metrizable iff X is compact metrizable.

Theorem

(Moresco and Artico) If L is the Sorgenfrey line then $\mathcal{K}(L)$ is not normal.

About the normality of $\mathcal{K}(X)$

Theorem

(M.) $\mathcal{K}(X)$ is metrizable iff X is it.

Note that $\mathcal{CL}(X)$ is metrizable iff X is compact metrizable.

Theorem

(Moresco and Artico) If L is the Sorgenfrey line then $\mathcal{K}(L)$ is not normal.

Theorem

Let γ an ordinal number.

1. *if cof$(\gamma) = \omega$ then $\mathcal{K}([0,\gamma))$ is normal.*
2. *(K.) if cof$(\gamma) > \omega$ then $\mathcal{K}([0,\gamma))$ is normal iff γ is regular.*
3. *(K. Hirata) if cof$(\gamma) > \omega$ then $\mathcal{K}([0,\gamma))$ is orthocompact iff γ is regular.*

Questions:

1. For which other class of spaces the hyperspace $\mathcal{K}$ is normal?.
2. Are there conditions $\mathcal{C}$ such that: $\mathcal{K}(X)$ is normal iff X has $\mathcal{C}$?.

Theorem

Let γ an ordinal number.

1. *if cof$(\gamma) = \omega$ then $\mathcal{K}([0,\gamma))$ is normal.*
2. *(K.) if cof$(\gamma) > \omega$ then $\mathcal{K}([0,\gamma))$ is normal iff γ is regular.*
3. *(K. Hirata) if cof$(\gamma) > \omega$ then $\mathcal{K}([0,\gamma))$ is orthocompact iff γ is regular.*

Questions:

1. For which other class of spaces the hyperspace $\mathcal{K}$ is normal?.
2. Are there conditions $\mathcal{C}$ such that: $\mathcal{K}(X)$ is normal iff X has $\mathcal{C}$?.

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

Theorem

(G.) $\mathcal{CL}(X)$ is:

1. *ω-boundded (ultrapseudocompact) iff X is it,*
2. *p-compact (p-pseudocompact) iff X is it,*
3. *α-boundded iff X is it.*

Questions: Are there conditions $\mathcal{C}$ such that: $\mathcal{CL}(X)$ is countable compact (pseudocompact) iff X has $\mathcal{C}$?

Theorem

(G.) $\mathcal{CL}(X)$ *is:*

1. ω*-boundded (ultrapseudocompact) iff* X *is it,*
2. *p-compact (p-pseudocompact) iff* X *is it,*
3. α*-boundded iff* X *is it.*

Questions: Are there conditions $\mathcal{C}$ such that: $\mathcal{CL}(X)$ is countable compact (pseudocompact) iff X has $\mathcal{C}$?

In $\mathcal{K}(X)$.

Theorem

(A.O.T.) TFSE:

1. *X is α-hyperbounded,*
2. *$\mathcal{K}(X)$ is initially α-compact.*
3. *$\mathcal{K}(X)$ is α-bounded, and*
4. *$\mathcal{K}(X)$ is α-hyperbounded.*

Milovančević made this prove for $\alpha = \omega$

Theorem

(A.O.T.) Let X be a space. Then the next statements arte equivalent:

1. *X is pseudo-ω-bounded,*
2. *$\mathcal{K}(X)$ is pseudo-ω-bounded,*
3. *$\mathcal{K}(X)$ pseudo-$\mathcal{D}$-bounded for some $\mathcal{D} \subseteq \mathbb{N}^*$,*
4. *$\mathcal{K}(X)$ is strongly-p-pseudocompact for some $p \subseteq \mathbb{N}^*$,*
5. *$\mathcal{K}(X)$ is p-pseudocompact for some $p \subseteq \mathbb{N}^*$ and*
6. *$\mathcal{K}(X)$ is pseudocompact.*

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

Pseudocompactness has a different approach. Let $I : \mathcal{CL}(X) \longrightarrow \mathcal{CL}(\beta X) : I(A) = Cl_{\beta X} A$.
When $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$? or when is $\mathcal{CL}(X)$ is natural (I) C^*-embedded in $\mathcal{CL}(\beta X)$?

Theorem

Let X be normal.

1. *(K.G.) If $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ then $\mathcal{CL}(X)$ (and so $\mathcal{CL}(X) \times \mathcal{CL}(X)$) is pseudocompact.*
2. *(G.) If $\mathcal{CL}(X) \times \mathcal{CL}(X)$ is pseudocompact then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$.*

Natsheh proved the converse but we think it is wrong.

Pseudocompactness has a different approach. Let
$I : \mathcal{CL}(X) \longrightarrow \mathcal{CL}(\beta X) : I(A) = Cl_{\beta X} A$.
When $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$? or when is $\mathcal{CL}(X)$ is natural (I) C^*-embedded in $\mathcal{CL}(\beta X)$?

Theorem

Let X be normal.

1. *(K.G.) If $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ then $\mathcal{CL}(X)$ (and so $\mathcal{CL}(X) \times \mathcal{CL}(X)$) is pseudocompact.*
2. *(G.) If $\mathcal{CL}(X) \times \mathcal{CL}(X)$ is pseudocompact then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$.*

Natsheh proved the converse but we think it is wrong.

Pseudocompactness has a different approach. Let
$I : \mathcal{CL}(X) \longrightarrow \mathcal{CL}(\beta X) : I(A) = Cl_{\beta X} A$.
When $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$? or when is $\mathcal{CL}(X)$ is natural (I) C^*-embedded in $\mathcal{CL}(\beta X)$?

Theorem

Let X be normal.

1. *(K.G.) If $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ then $\mathcal{CL}(X)$ (and so $\mathcal{CL}(X) \times \mathcal{CL}(X)$) is pseudocompact.*
2. *(G.) If $\mathcal{CL}(X) \times \mathcal{CL}(X)$ is pseudocompact then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$.*

Natsheh proved the converse but we think it is wrong.

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

Question: When is $\mathcal{K}(X)$ C^*-embedded in $\mathcal{CL}(X)$? Is there some relation betwen this problem and the problem: When is $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$?

Theorem

(H.) If $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$ then $\mathcal{K}(X)$ is ω-boundded (and so $\mathcal{K}(X)$ is C-embedded in $\mathcal{CL}(X)$).

So if $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$ then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ and the converse is not true. We don't know wath happens if $\mathcal{K}(X)$ is not normal.

Question: When is $\mathcal{K}(X)$ C^*-embedded in $\mathcal{CL}(X)$? Is there some relation betwen this problem and the problem: When is $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$?

Theorem

(H.) If $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$ then $\mathcal{K}(X)$ is ω-boundded (and so $\mathcal{K}(X)$ is C-embedded in $\mathcal{CL}(X)$).

So if $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$ then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ and the converse is not true. We don't know wath happens if $\mathcal{K}(X)$ is not normal.

Corollary

Let X be a metrizable space. Then $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$ iff X is a compact space.

Theorem

(A. O. T.) Suppose $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$. TFAE:

1. *X is τ-bounded,*
2. *X is τ-hyperbounded,*
3. *$\mathcal{K}(X)$ is τ-pseudocompact,*
4. *$\mathcal{K}(X)$ is initially τ-compact,*
5. *$\mathcal{K}(X)$ is τ-bounded, and*
6. *$\mathcal{K}(X)$ is τ-hyperbounded.*

Corollary

Let X be a metrizable space. Then $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$ iff X is a compact space.

Theorem

(A. O. T.) Suppose $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$. TFAE:

1. *X is τ-bounded,*
2. *X is τ-hyperbounded,*
3. *$\mathcal{K}(X)$ is τ-pseudocompact,*
4. *$\mathcal{K}(X)$ is initially τ-compact,*
5. *$\mathcal{K}(X)$ is τ-bounded, and*
6. *$\mathcal{K}(X)$ is τ-hyperbounded.*

Theorem

(A. O. T.) Suppose $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$. TFAE:

1. *X is compact,*
2. *X is σ-compact,*
3. *$\mathcal{K}(X)$ is compact,*
4. *$\mathcal{K}(X)$ is σ-compact,*
5. *$\mathcal{K}(X)$ is Lindelöf,*
6. *$\mathcal{K}(X)$ is paracompact,*
7. *$\mathcal{K}(X)$ is normal and metacompact,*
8. *$\mathcal{CL}(X)$ is compact, and*
9. *$\mathcal{CL}(X)$ is σ-compact.*

Our main result:

Theorem

(K. O. R.) Let γ be an ordinal number. TFAE:

1. $\mathcal{K}([0,\gamma))$ *is C-embedded in* $\mathcal{CL}([0,\gamma))$.
2. $\mathcal{K}([0,\gamma))$ *is C^*-embedded in* $\mathcal{CL}([0,\gamma))$.
3. $cof(\gamma) \neq \omega$

Our main result:

Theorem

(K. O. R.) Let γ be an ordinal number. TFAE:

1. $\mathcal{K}([0,\gamma))$ *is* C*-embedded in* $\mathcal{CL}([0,\gamma))$.
2. $\mathcal{K}([0,\gamma))$ *is* C^**-embedded in* $\mathcal{CL}([0,\gamma))$.
3. $cof(\gamma) \neq \omega$

Theorem

(K. O.) Let γ be an infinite ordinal number. TFAE:

1. $cof(\gamma) \neq \omega$,
2. $[0,\gamma)$ *is pseudocompact,*
3. $\beta(\mathcal{CL}([0,\gamma))) = \mathcal{CL}(\beta([0,\gamma)))$,
4. $\beta(\mathcal{K}([0,\gamma))) = \mathcal{K}(\beta([0,\gamma)))$,
5. $\beta(\mathcal{CL}([0,\gamma))) = \mathcal{CL}([0,\gamma])$, *and*
6. $\beta(\mathcal{K}([0,\gamma))) = \mathcal{K}([0,\gamma])$.

Theorem

(K. O.) Let γ be an infinite ordinal number. TFAE:

1. $cof(\gamma) \neq \omega$,
2. $[0, \gamma)$ *is pseudocompact,*
3. $\beta(\mathcal{CL}([0, \gamma))) = \mathcal{CL}(\beta([0, \gamma)))$,
4. $\beta(\mathcal{K}([0, \gamma))) = \mathcal{K}(\beta([0, \gamma)))$,
5. $\beta(\mathcal{CL}([0, \gamma))) = \mathcal{CL}([0, \gamma])$, *and*
6. $\beta(\mathcal{K}([0, \gamma))) = \mathcal{K}([0, \gamma])$.

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

Theorem

1. *(M.)* $\mathcal{K}(X)$ *is* 0*-dimensional iff* X *is it.*
2. *(K. T.)* $\mathcal{CL}(\omega)$ *is strong* 0*-dimensional.*
3. *(K. T.)* $\mathcal{K}([0,\gamma))$ *is strong* 0*-dimensional for every* γ*.*

Theorem

1. *(O. O.) If cof*$(\gamma) \neq \omega$ *then* $\mathcal{CL}(\omega)$ *is strong* 0*-dimensional.*
2. *(O.) If cof*$(\gamma) \neq \omega$ *then* $\mathcal{K}([0,\gamma))$ *is strongly* 0*-dimensional.*

Theorem

1. *(M.)* $\mathcal{K}(X)$ *is* 0*-dimensional iff* X *is it.*
2. *(K. T.)* $\mathcal{CL}(\omega)$ *is strong* 0*-dimensional.*
3. *(K. T.)* $\mathcal{K}([0,\gamma))$ *is strong* 0*-dimensional for every* γ*.*

Theorem

1. *(O. O.) If cof*$(\gamma) \neq \omega$ *then* $\mathcal{CL}(\omega)$ *is strong* 0*-dimensional.*
2. *(O.) If cof*$(\gamma) \neq \omega$ *then* $\mathcal{K}([0,\gamma))$ *is strongly* 0*-dimensional.*

CONTENIDO

1 Introduction

2 Normality

3 Compact-like properties

4 Pseudocompactness

5 C^*-embeddedness

6 Strong 0-dimensionality

7 Bibliography.

- J. Angoa, Y. F. Ortiz-Castillo, A. Tamariz-Mascarua *Compact like properties on hyperspaces*, to appear.
- J. Ginsburg, *Some results on the countable compactness and pseudocompactness of hyperspaces*, Canad. J. Math. 27(1975), no. **6**, 1392-1399.
- J. Ginsburg, *On the Stone-Čech compactification of the space of closed sets*, Trans. Amer. Math. Soc. **215** (1976), 301?311.
- Y. Hirata, N. Kemoto, *Orthocompactness versus normality in hyperspaces*, Topology Appl. **159**, (2012) 1169-1178.
- N. Kemoto, *Normality and countable paracompactness of hyperspaces of ordinals*, Topology Appl. **154**, (2007) 358-363.

- N. Kemoto, J. Terasawa, *Strong zero-dimensionality of hyperspaces*, Topology Appl. **157**, (2010) 2376-2382.
- E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., **71** (1951), 152-182.
- D. Milovančević, *A property between compact and strongly countably compact*, Publ. Inst. Math. (Beograd) (N.S.) **38**(**52**) (1985), 193-201.
- M. A. Natsheh, *On pseudocompactness of hyperspaces*, Q. and A. in General Top. 16(1998), no. **7**, 67-71.

Gracias gracias gracias!!!

Thank you thank you thank you!!!

Grazie grazie grazie!!!

Gracias gracias gracias!!!

Thank you thank you thank you!!!

Grazie grazie grazie!!!

Gracias gracias gracias!!!

Thank you thank you thank you!!!

Grazie grazie grazie!!!