Isometrical embeddings of finite metric spaces

A. Oblakova

Moscow State University

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1 Uniform class.

1.1 Definition.

A class $\mathbb{K}$ of metric spaces is said to be uniform, if it has two following properties: (*a*) diameters of its elements bounded by number $\mathbf{d}$ and (*b*) for every $\varepsilon > 0$ there exist an integer $\mathbf{n}(\varepsilon)$ such that every element of this class has an ε-net the number of elements of which is less than or equal to $\mathbf{n}(\varepsilon)$.

1.2 Uniformity and isometrical embeddings

It was proved in [2][a] (see also [1][b]) that if a class $\mathbb{K}$ is uniform, then there is a totally bounded metric space (and consequently a compact metric space), which isometrically contains all elements of this class. So, properties (*a*) and (*b*) of a class $\mathbb{K}$ are necessary and sufficient conditions for the existence of a compact metric space containing isometrically every element of this class.

[a] J. Tits, *Groupes à croissance polynomiale*, Séminaire Bourbaki, 23 (1980-1981), Expposé No. 572.

[b] S.D. Iliadis, *Universal Spaces and Mappings*, North-Holland Mathematics Studies, 198, Elsevier Science B.V., Amsterdam, 2005, xvi+559 pp.

1.3 Uniformity and dimension

It was proved in [1] that if a class $\mathbb{K}$ is uniform and every its element has dimension $\leqslant n \in \omega$, then there is a totally bounded metric space of dimension $\leqslant n$, which isometrically contains every space from $\mathbb{K}$. In general, this totally bounded space is not compact.

1.4 Classes $\mathbb{F}_1$ and $\mathbb{F}_1^n$

The class $\mathbb{F}_1$ of all finite metric spaces with diameter $\leqslant 1$ is not uniform. So, for this class there is no compact metric space, which isometrically contains every element of $\mathbb{F}_1$. On the other hand, its subclass $\mathbb{F}_1^n$, $n \in \mathbb{N}$ of all finite metric spaces containing $\leqslant n$ points is uniform, and, therefore, there is a totally bounded zero-dimensional space T, isometrically containing every element of $\mathbb{F}_1^n$. However, it is not known if the completion of T is zero-dimensional.

1.5 Main result

It will be proved, without using construction from [1], the existence of a metric on the Cantor set C such that every element of $\mathbb{F}_1^n$ is isometrically embeded in C. So, this gives a positive answer to the question posed by S. Iliadis.

2 Definitions and notation.

2.1 The map φ

The set of natural numbers is denoted by $\mathbb{N}$. We fix $n \in \mathbb{N}$.
Let N be the number of pairs (i, j), $1 \leqslant i < j \leqslant n$, that is $N = \frac{n \cdot (n-1)}{2}$. We fix a map

$$\varphi\colon \ \{(i, j) | \, 1 \leqslant i, j \leqslant n\} \to \{1, 2, \dots, N\},$$

which is one-to-one on the set $\{(i, j) | \, 1 \leqslant i < j \leqslant n\}$ and satisfies the following property:

$$\varphi(i, j) = \varphi(j, i).$$

(Such a map exists by definition of N.)

2.2 Metric on $\mathbb{R}^m$

The set of real numbers is denoted by $\mathbb{R}$. For $m \in \mathbb{N}$, $\mathbb{R}^m$ denotes the set:

$$\mathbb{R}^m = \{(x_1, \cdots, x_m) | x_1, \cdots, x_m \in \mathbb{R}\},$$

on which we consider the metric $\rho_{\mathbb{R}^m} \colon \mathbb{R}^m \to \mathbb{R}$, such that for every two points $\bar{x} = (x_1, \cdots, x_m)$, $\bar{y} = (y_1, \cdots, y_m)$,

$$\rho_{\mathbb{R}^m}(\bar{x}, \bar{y}) = \sum_{i=1}^{m} |x_i - y_i|.$$

(In what follows, $m = N$ and the metric $\rho_{\mathbb{R}^N}$ will be denoted by ρ.)

3 The Cantor set and the function ψ.

3.1 Construction of the Cantor set

We will construct the Cantor set in segment $[0,2]$, by considering system of intervals $I_k = (a_k, b_k)$, $k \in \mathbb{N}$, $b_k > a_k$, which satisfies following properties:

1. For every $k \in \mathbb{N}$, $[a_k, b_k] \subset (0,2)$.
2. For every $k \neq m$, $[a_k, b_k] \cap [a_m, b_m] = \emptyset$.
3. $\sum_{k=1}^{\infty}(b_k - a_k) = 1$.
4. The set $[0,2] \setminus (\bigcup_{k=1}^{\infty} I_k)$ doesn't contain an interval.

3.2 Lemma

The set

$$C = [0, 2] \setminus \bigcup_{k \in \mathbb{N}} I_k$$

is closed, compact, zero-dimensional and doesn't contain isolated points, consequently it is the Cantor set. ■

3.3 Function ψ

Define the function $\psi\colon C \to \mathbb{R}$ setting for every point $x \in C$

$$\psi(x) = x - \sum_{m\in\mathbb{N},\ b_m \leqslant x} (b_m - a_m).$$

3.4 Lemma

Function ψ satisfies following properties:

1. ψ is nondecreasing,
2. For every $x, y \in C$:
$$|\psi(x) - \psi(y)| \leqslant |x - y|,$$
3. ψ is continuous map of C on $[0, 1]$.

4.1 Compactum K

Consider unit cube in $\mathbb{R}^N$

$$Q^N = \{(x_1, \ldots, x_N)|\ 0 \leqslant x_s \leqslant 1,\ 1 \leqslant s \leqslant N\}$$

and its subset

$$K^0 = \{(x_1, \ldots, x_N) \in Q^N|\ x_{\varphi(i,k)} \leqslant x_{\varphi(i,j)} + x_{\varphi(j,k)},\ 1 \leqslant i, j, k \leqslant n\}.$$

Let

$$K = \{(x_1, \ldots, x_N) \in C^N|\ (\psi(x_1), \ldots, \psi(x_N)) \in K^0\}.$$

4.2 Remark

The set K is closed (as a preimage of a closed set by continuous map), bounded, and consequently compact.

4.3 Parameters

For convenience of notation, denote the number $\psi(x_s)$ by x^s, where $1 \leqslant s \leqslant N$. Numbers x^s, for $1 \leqslant s \leqslant N$, is called **parameters** of the point $x = (x_1, \ldots, x_N) \in K$.

4.4 Space X_0

Let K_i, $i = 1, \cdots, n$, be copies of the metric compactum K and π_i the corresponding isometry K_i on K. Disjoint union of this copies is denoted by X_0:

$$\mathrm{X}_0 = \bigsqcup_{1 \leqslant i \leqslant n} K_i.$$

Let π be the map of X_0 on K, which coinsides with π_i on K_i, $i = 1, \cdots, n$. Let also

$$S_K = \cup\{K_i \times K_i \subset X_0 \times X_0 | 1 \leqslant i \leqslant n\},$$

$$S_\pi = \{(a, b) \in X_0 \times X_0 |\ \exists 1 \leqslant i, j \leqslant n\colon a \in K_i,\ b \in K_j,\ \pi(a) = \pi(b)\},$$

$$S = S_K \cup S_\pi.$$

5 The pseudometric ρ_{X_0} on X_0.

5.1 The map ρ_0

Let $\rho_0 \colon S \to \mathbb{R}$ be a map defined as follows:

$$\rho_0(a,b) = \begin{cases} \rho(\pi(a), \pi(b)), & \text{if } a, b \in K_i, \\ (\pi(a))^{\varphi(i,j)}, & \text{if } a \in K_i,\ b \in K_j,\ i \neq j,\ \pi(a) = \pi(b). \end{cases}$$

5.2 Remark

By definition, the map ρ_0 is symmetrical and non-negative.

5.3 The notion of a way

Any sequence $a_1, a_2, ...a_m \in X_0$, which satisfies condition

$$\forall 1 \leqslant i < m: \ (a_i, a_{i+1}) \in S,$$

is called a **way** between $a = a_1$ and $b = a_m$. The set of all ways between a and b is denoted by $W(a, b)$.

5.4 The pseudometric ρ_{X_0} on X_0

Let for every $a, b \in X_0$:

$$\rho_{\mathrm{X}_0}(a, b) = \inf_{(a_1,\ldots,a_l)\in W(a,b)} \sum_{i=1}^{l-1} \rho_0(a_i, a_i + 1).$$

It is obvious, that ρ_{X_0} is pseudometric.

5.5 Proposition

1. If $a, b \in K_i$, then
$$\rho_{\mathrm{X}_0}(a, b) = \rho_0(a, b) = \rho(\pi(a), \pi(b)).$$
2. If $a \in K_i, b \in K_j, (a, b) \in S_\pi$, then
$$\rho_{\mathrm{X}_0}(a, b) = \rho_0(a, b) = (\pi(a))^{\varphi(i,j)}.$$

6 The main result.

6.1 The space X_1

Let C_i, $i = 1, \cdots, n$, be copies of the Cantor cube C^N and θ_i the corresponding isometry C_i on C^N. The disjoint union of this copies is denoted by X_1:

$$\mathrm{X}_1 = \bigsqcup_{1 \leqslant \mathrm{i} \leqslant \mathrm{n}} \mathrm{C}_\mathrm{i}.$$

Let θ be the map of X_1 on C^N, which coinsides with θ_i on C_i, $i = 1, \cdots, n$. Obviously, the set X_0 naturaly contained in X_1. Then, the pseudometric ρ_{X_0} can be extended to pseudometric ρ_{X_1} on X_1 as follows: for every $a \in C_i, b \in C_j$ we set

$$\rho_{\mathrm{X}_1}(a, b) = \min_{c \in K_i, d \in K_j} \{\rho(\theta(a), \pi(c)) + \rho_{\mathrm{X}_0}(c, d) + \rho(\pi(d), \theta(b))\}.$$

6.2 The metric space T

The pseudometric ρ_{X_1} naturally defines a metric ρ_{T} on the factor space $\mathrm{T} = \mathrm{X}_1/\sim$, where

$$x \sim y \Leftrightarrow \rho_{\mathrm{X}_1}(x,\ y) = 0.$$

Let σ be the natural map X_1 on T. By construction, the space T is compact, zero-dimensional and doesn't contain isolated points. So, it is homeomorphic to the Cantor set.

6.3 Theorem

For every $(M, \rho_M) \in \mathbb{F}_1^n$ and for every numeration of elements of M:

$$M = \{M_1, ...M_m\}$$

there exist points $a_1, \ldots a_m \in T$ such that

$$\rho_M(M_i, M_j) = \rho_{\mathrm{T}}(a_i, a_j).$$

Therefore, T contains isometrically all ellements of $\mathbb{F}_1^n$.

Proof in case $m = n$

Consider collection $(d_1, \dots, d_N)$ of distances between points of a space $(M, \rho_M) \in \mathbb{F}_1^n$:

$$d_{\varphi(i,j)} = \rho_M(M_i, M_j).$$

There exists point $x \in K^o$ such that $x_i = d_i$. For this point there is one (or more) point $b \in K$ such that $b^i = x_i$. Consider different points $b_1, \dots b_n \in \mathrm{X}_1$ such that $\theta(b_i) = b$. By the definition of pseudometric on X_1:

$$\rho_{\mathrm{X}_1}(b_i, b_j) = d_{\varphi(i,j)} = \rho_M(M_i, M_j).$$

Then points $a_i = \sigma(b_i)$ are required points.

Thank you!

- S.D. Iliadis, *Universal Spaces and Mappings*, North-Holland Mathematics Studies, 198, Elsevier Science B.V., Amsterdam, 2005, xvi+559 pp.
- J. Tits, *Groupes à croissance polynomiale*, Séminaire Bourbaki, 23 (1980-1981), Exppposé No. 572.