On Normality of the Wijsman Topology

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IV Workshop on Coverings, Selections, and Games in Topology, 2012

Outline

1. Introduction
2. Normality of the Wijsman Topology
3. Cardinal invariants of the Wijsman Topology

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What is a hyperspace?

A hyperspace is a space $(CL(X), \tau)$, where $CL(X)$ is the set of all nonempty closed subsets of a Hausdorff space X. τ is called a hypertopology.

Definition

A topology generated by all sets of the form $U^- = \{A \in CL(X); A \subset U\}$, $V^+ = \{A \in CL(X); A \cap V \neq \emptyset\}$, where U, V are open subsets of X, is called the Vietoris topology. If V runs only through complements of compact sets, the corresponding topology is called the Fell topology.

Definition

Let (X, ρ) be a metric space. The topology on $CL(X)$ generated by all functions of the form $\rho(x, \cdot) : CL(X) \to R$ is called the Wijsman topology.

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Question of this section

Question from [Di Maio and Meccariello, 1998, Problem I]:

It is known that (X, ρ) is a separable metric space, iff $(CL(X), W_\rho)$ is metrizable; and then it is normal.

Is the opposite true?

Is $(CL(X), W_\rho)$ normal if and only if it is metrizable?
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Simple results

Theorem

Suppose GCH holds. If $(CL(X), W_\rho)$ is normal then $d(CL(X)) = d(X)$.

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Let (X, ρ) be a discrete metric space with 0-1 metric. If $(CL(X), W_\rho)$ is normal then X is countable.

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More general results I

Theorem

Let (X, ρ) be a metric space. If for every metric δ uniformly equivalent to ρ the space $(CL(X), W_\delta)$ is normal, then X is separable.

Idea of the Proof

If X is not separable, then there is an $\epsilon-$discrete set $Y \subset X$ with $|Y| > \aleph_0$. Put $\delta(x, y) = min(\rho(x, y), \epsilon)$, which is uniformly equivalent to ρ. One can prove that $(CL(Y), W_{\delta|Y})$ can be embedded as a closed subset of $(CL(X), W_\delta)$, so it has to be normal. Since $\delta|Y$ is $0 - \epsilon$ metric, then by previous result we have that Y is countable, a contradiction.

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More general results II

Theorem

Let (X, ρ) be a linear metric space. $(CL(X), W_\rho)$ is normal iff X is separable.

Idea of the Proof

If X is not separable, then no point has a compact neighborhood. Then one can prove that $\omega^{\aleph_1}$ can be embedded as a closed subset of $(CL(X), W_\rho)$. Since $\omega^{\aleph_1}$ is not normal, we have a contradiction.

Theorem (J. Cao, H.J. K. Junnila, W.B. Moors)

Let (X, ρ) be a metric space. $(CL(X) \setminus \{X\}, W_\rho)$ is normal iff X is separable.

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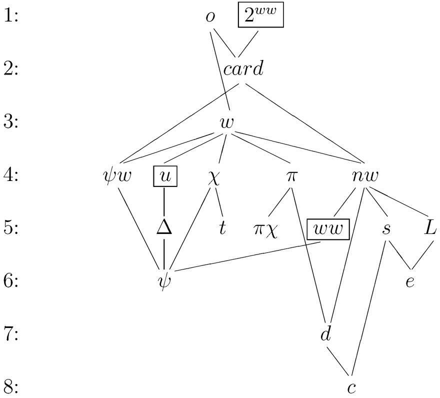
Cardinal invariants equal to d(X)

Theorem

$d(X) = f(CL(X)) = hf(CL(X))$, where f is any function from: spread (s), extent (e), netweight (nw), weight (w), pseudo weight (ψw), $\pi-$weight (π), character (χ), pseudocharacter (ψ), $\pi-$character ($\pi\chi$), tightness (t), Lindelöf number (L), diagonal degree (Δ), weak weight (ww), uniform weight (u) and hereditary density (hd).

Cardinal invariants of a Tychonoff space



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see [Beer, 1993]
$d(X) \leq \psi(CL(X))$
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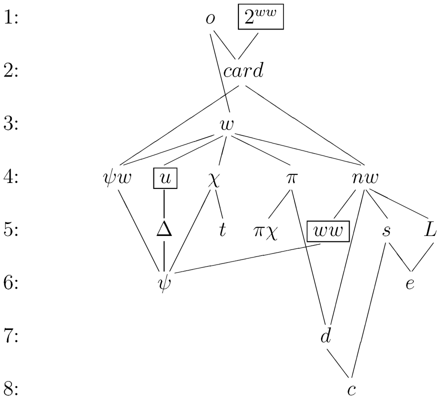

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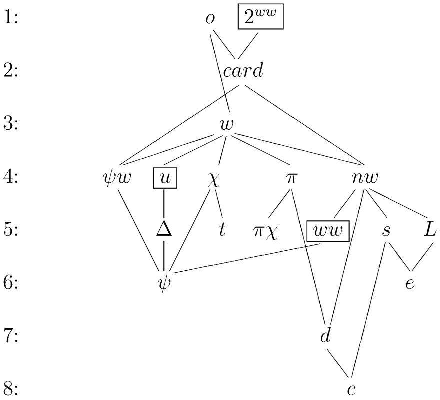

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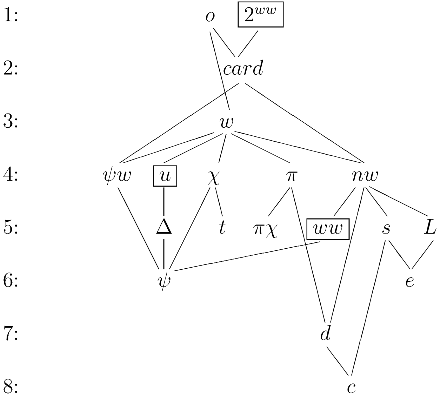

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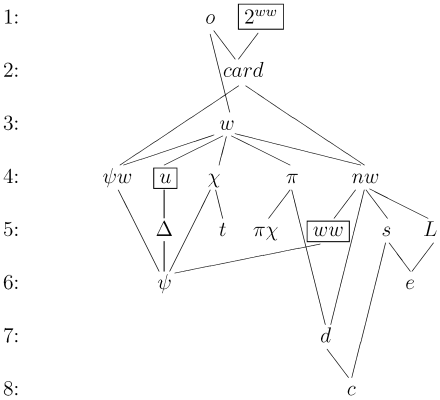

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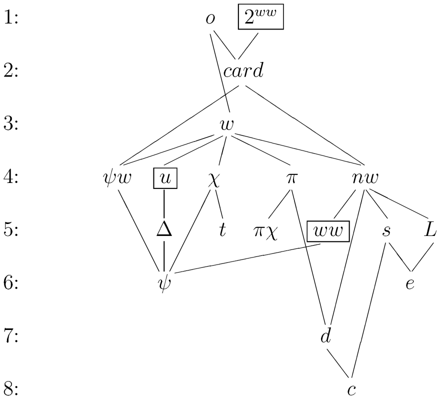

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Density and cellularity

Theorem

$\log(d(X)) \leq d(CL(X)) \leq d(X), \quad \aleph_0 \leq c(CL(X)) \leq d(CL(X))$

$log(n) = min\{m; n \leq 2^m\}$

Example

Let X be a discrete metric space with the $0-1$ metric, then $d(CL(X)) = \log(d(X))$ and $c(CL(X)) = \aleph_0$.

- $CL(X)$ with Wijsman topology is homeomorphic to $2^X \setminus \{1\}$
- Hewitt-Marczewski-Pondiczery theorem

Let $M \subset X$ fulfill $|M| = \mathfrak{m} \geq \aleph_0$, let ρ be a metric such that for every $x \in M$ there is $x' \in M$ such that $(x')' = x$ and $\rho(x, x') = 2$; $\rho(x, x) = 0$ and $\rho(x, y) = 1$ otherwise.
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Summary

- For some classes of metric spaces, the normality of the Wijsman topology (on the corresponding hyperspace) is equivalent to its metrizability;
- General question is still open. Is the normality of the Wijsman topology always equivalent to its metrizability?
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For Further Reading

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Thank You for Your Attention