Homogeneity and h-homogeneity

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Abstract

We give a criterion when a zero-dimensional homogeneous space is h-homogeneous. Using it, we show that X^ω is h-homogeneous for every metric space X with $\dim X = 0$. Moreover, every homogeneous zero-dimensional separable metric space, which is non-locally compact, is h-homogeneous. We generalize the Motorov theorem about h-homogeneity of the product of two compacta. In particular, we prove that the product $\mathcal{C} \times Y$ is an h-homogeneous space if $\mathcal{C}$ is the Cantor set and Y is a homogeneous zero-dimensional compact space.

Keywords: homogeneous space, h-homogeneous space, zero-dimensional, clopen set, pseudocompact, compact space
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We investigate the relations between homogeneous and h-homogeneous spaces. The properties of h-homogeneous spaces has been studied by several authors. Van Mill [1] obtained that every zero-dimensional h-homogeneous first countable space is homogeneous. Terada [2] proved the following statement.

Theorem 1. *Let X be a non-pseudocompact zero-dimensional space. If X has a π-base consisting of clopen subsets that are homeomorphic to X, then X is h-homogeneous.*

Medini [3] showed that the condition $\mathrm{ind} X = 0$ can be dropped in Theorem 1. We suggest a criterion of h-homogeneity for homogeneous paracompact spaces (see Theorem 2). Using it, we show that the notions of homogeneity and h-homogeneity are equivalent for a weight-homogeneous metric

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non-locally compact space X with $\mathrm{Ind}X = 0$ (see Corollary 1). This implies that X^ω is an h-homogeneous space for any metric space with $\mathrm{Ind}X = 0$. The last statement was proved by Medini [3] under the additional assumption that X is non-separable.

We describe two cases when the product $X \times Y$ is an h-homogeneous space. Theorem 7 states that this is valid if X is an h-homogeneous non-pseudocompact zero-dimensional space and Y is a homogeneous zero-dimensional Lindelöf space. Medini [3] proved that the product $X \times Y$ is an h-homogeneous space if X, Y are h-homogeneous Tyconoff spaces such that $X \times Y$ is pseudocompact. Motorov [4] announced that the product $X \times Y$ of two zero-dimensional first countable compacta X and Y will be h-homogeneous provided X is h-homogeneous and Y is homogeneous. We improve the last statement in the following way (see Theorem 8). $X \times Y$ is an h-homogeneous space if X is an infinite h-homogeneous pseudocompact space and Y is a homogeneous zero-dimensional compact space. This implies (see Corollary 6) that the product $\mathcal{C} \times Y$ is a homogeneous and h-homogeneous space if $\mathcal{C}$ is the Cantor set and Y is a homogeneous zero-dimensional compact space. Hence, $\mathcal{C} \times Y$ is n-divisible for each positive integer n.

Thus, the multiplication of a compact space Y by the Cantor set improves the homogeneous properties of Y. A homogeneous compact space which is not h-homogeneous was constructed by van Douwen [5]. Motorov [6] gives an example of a homogeneous zero-dimensional compact which is divisible by 3 but not divisible by 2.

1. Notation

For all undefined terms and notation see [7]. $X \approx Y$ means that X and Y are homeomorphic spaces. Let $\mathcal{P}$ be a topological property. Then a space X is *nowhere* $\mathcal{P}$ if no non-empty open subset of X has property $\mathcal{P}$.

We identify cardinals with initial ordinals; in particular, $\omega = \{0, 1, 2, \ldots\}$. A space X is called *weight-homogeneous* if all non-empty open subspaces of X have the same weight. A family $\mathcal{B}$ consisting of non-empty open subsets of a space X is a π-*base* if for every non-empty open subset $U \subseteq X$ there exists $V \in \mathcal{B}$ such that $V \subseteq U$.

A *clopen* set is a set which is both closed and open. A space X is *homogeneous* if for any two points $x, y \in X$ there exists a homeomorphism $f : X \to X$ with $f(x) = y$. A space X is called *h-homogeneous* if every non-empty clopen subset of X is homeomorphic to X.

The smallest cardinal number k such that every open cover of a space X has an open refinement of cardinality $\leq k$ is called the *Lindelöf number* of the space X and is denoted by $l(X)$.

We say that a space X is *k-divisible* (or *divisible by k*) if $X \approx D_k \times X$, where D_k is the discrete space of cardinality k.

Lemma 1. 1) *Let a space X be k-divisible for an infinite cardinal k. Then X is k_1-divisible for any cardinal $k_1 \leq k$.*

2) *Let X be an h-homogeneous zero-dimensional space containing at least two points. Then X is n-divisible for each positive integer n.*

3) *Let X be an h-homogeneous ω-divisible space. Then X is non-pseudocompact.*

Proof. 1) Consider the cardinal k as a discrete space. Then

$X \approx k \times X \approx (k_1 \times k) \times X \approx k_1 \times (k \times X) \approx k_1 \times X$.

2) Fix $n > 1$. Take two different points a and b. Let U be a clopen neighborhood of a such that $b \notin U$. Since X is h-homogeneous, $X \approx U$. Then U contains at least two different points. Repeating the division, we conclude that X contains at least n different points $x_1, \ldots, x_n$. Choose pairwise disjoint clopen neighborhoods $U_1, \ldots, U_{n-1}$ of the points $x_1, \ldots, x_{n-1}$ such that $x_n \notin U_i$ for $i < n$. Let $U_n = X \setminus (U_1 \cup \ldots \cup U_{n-1})$. Then $X \approx U_1 \oplus \ldots \oplus U_n \approx n \times X$.

3) Since$X$ is ω-divisible, $X \approx \omega \times U$ for some clopen $U \subset X$. Define the function $f : X \to R$ by the rule $f(x) = i$ if $x \in \{i\} \times U$. Then f is continuous and unbounded. Hence, X is a non-pseudocompact space. □

Example. Let $Z \subset \mathbb{R}^2$ be the union of the square $[0, 1] \times [0, 1]$ and the interval $[1, 2] \times \{0\}$. Clearly, the product $Z \times \mathcal{C}$ is not homogeneous, where $\mathcal{C}$ is the Cantor set. Nevertheless, $Z \times \mathcal{C}$ is n-divisible for each positive integer n.

2. Indications of h-homogeneity

Theorem 2. *Let X be a homogeneous paracompact space of weight k such that* $\dim X = 0$ *and X contains a discrete closed subset D of cardinality k. Then the following statements are equivalent:*

$(a1)$ *X is an h-homogeneous space,*

($a2$) *for every non-empty clopen subset U of X there are a non-empty open $V \subseteq X$ and a discrete family $\{V_\alpha : \alpha \in k\}$ such that $\cup\{V_\alpha : \alpha \in k\} \subseteq U$ and each $V_\alpha \approx V$.*

PROOF. ($a1$) $\Rightarrow$ ($a2$). Since every paracompact space is collectionwise normal [7, Theorem 5.1.18], there exists a discrete family $\{E_\alpha : \alpha \in k\}$ of open subsets of X every member of which contains a one point from D. From the Vedenissoff theorem [7, Theorem 6.2.5] it follows that ind$X$ = Ind$X = 0$ because every paracompact space is normal. Then we may assume that every E_α is clopen in X. Put $V = X$. h-Homogeneity of X implies that each $E_\alpha \approx V$.

Take a non-empty clopen subset $U \subseteq X$ and a homeomorphism $f : X \to U$. Set $V_\alpha = f(E_\alpha)$. Then the family $\{V_\alpha : \alpha \in k\}$ satisfies ($a2$).

($a2$) $\Rightarrow$ ($a1$). Choose a clopen base $\mathcal{B}$ for X. Take a $U \in \mathcal{B}$. Fix the set V and the family $\{V_\alpha : \alpha \in k\}$ according to the condition ($a2$).

Choose a point $a \in V$. From homogeneity of X it follows that for every point $x \in X$ there exists a homeomorphism $f_x : X \to X$ such that $f_x(a) = x$. Since dim$X = 0$, the open cover $\{f_x(V) : x \in X\}$ of X has a discrete open refinement $\{W_\alpha : \alpha \in k_1\}$. By construction, every W_α is clopen in X and homeomorphic to a clopen subset of V. Clearly, $k_1 \leq w(X) = k$. Then X is homeomorphic to a clopen subset U^* of $\cup\{V_\alpha : \alpha \in k_1\}$.

Since every $U^* \subseteq U$, the family $\{U^* : U \in \mathcal{B}\}$ forms a π-base for X.

One can check that if X is a finite space, then X is a one-point set. In this case the theorem is trivial.

Let the cardinal k be infinite. Then D contains a closed subset $\{d_i : i \in \omega\}$ such that $d_i \neq d_j$ whenever $i \neq j$. By the Tietze-Urysohn theorem there exists a continuous function $f : X \to R$ such that $f(d_i) = i$ for $i \in \omega$. Since f is not bounded, the space X is not pseudocompact. By virtue of Theorem 1, the space X is h-homogeneous. □

Theorem 3. *Let X be a homogeneous, non-locally compact metric space of weight k with* dim$X = 0$. *Then the following statements are equivalent*:

(b1) *X is an h-homogeneous space,*

(b2) *for every non-empty open $U \subseteq X$ there are a non-empty open $V \subseteq X$ and a discrete family $\{V_\alpha : \alpha \in k\}$ such that $\cup\{V_\alpha : \alpha \in k\} \subseteq U$ and each $V_\alpha \approx V$.*

PROOF. By Theorem 2, it suffices to show that X contains a discrete closed subset D of cardinality k. This is valid for $k = \omega$ because X is not compact. When $k > \omega$ this follows from the weight-homogeneity of X. □

The following statement was proved in [8] under the additional assumption $\mathrm{cf}(w(X)) > \omega$.

Theorem 4. *Let X be a homogeneous, weight-homogeneous metric space such that* $\mathrm{Ind}X = 0$ *and X is not locally compact. Then X is h-homogeneous.*

PROOF. Let $w(X) = k$. Using [7, Theorem 7.3.15], we can consider X as a dense subset of the Baire space $B(k)$. By definition, $B(k) = \prod\{D_i : i \in \omega\}$, where each D_i is the discrete space of cardinality k. Take a non-empty open subset U of X. Then U contains a clopen (in X) subset W that can be represented as

$$W = X \bigcap (\{d_0\} \times \ldots \times \{d_{n-1}\} \times D_n \times D_{n+1} \times \ldots)$$

for some $n \in \omega$ and points $d_i \in D_i$ provided $i < n$. Put $D_n = \{\alpha : \alpha \in k\}$. Define

$$V_\alpha = X \bigcap (\{d_0\} \times \ldots \times \{d_{n-1}\} \times \{\alpha\} \times D_{n+1} \times \ldots)\,.$$

Let $V = V_0$. Clearly, $V_\alpha \approx V$ and $V_\alpha \subset U$ for every $\alpha \in k$. The family $\{V_\alpha : \alpha \in k\}$ is discrete in X. Note that $\dim X = \mathrm{Ind}X = 0$ by virtue of the Katětov-Morita theorem. By Theorem 3, the space X is h-homogeneous. □

The last theorem, together with the van Mill [1] result, yields

Corollary 1. *Let X be a weight-homogeneous metric space such that* $\mathrm{Ind}X = 0$ *and X is not locally compact. Then X is h-homogeneous if and only if X is homogeneous.*

Corollary 2. *Let X be a homogeneous zero-dimensional separable metric space which is not locally compact. Then X is h-homogeneous.*

Corollary 3. *Let G be a separable zero-dimensional metrizable topological group which is not locally compact. Then G is an h-homogeneous space.*

Theorem 5. *Let X be a metric space with* $\mathrm{Ind}X = 0$. *Then X^ω is an h-homogeneous space.*

Proof. If X is a compact space, then the theorem is true because X is homeomorphic either to the Cantor set or to a one-point space.

Now, suppose X is not a compact space. By virtue of [7, Theorem 3.3.13], the product X^ω is nowhere locally compact. According to the Dow and Pearl theorem [9], X^ω is a homogeneous space. Clearly, X^ω is a weight-homogeneous space. It remains to apply Theorem 4. □

3. h-Homogeneity of the product $X \times Y$

Theorem 6. *Let X be an h-homogeneous k-divisible zero-dimensional space, where $k \geq \omega$. Let Y be a homogeneous paracompact space such that $l(Y) \leq k$ and $\dim Y = 0$. Then $X \times Y$ is an h-homogeneous space.*

Proof. Take a non-empty open subset W of $X \times Y$. Then W contains a product $U \times V$, where U is a clopen subset of X and V is a clopen subset of Y. Fix a point $a \in V$. From homogeneity of Y it follows that for every point $y \in Y$ there exists a homeomorphism $f_y : Y \to Y$ such that $f_y(a) = y$. Since $l(Y) \leq k$, the cover $\{f_y(V) : y \in Y\}$ of Y has a locally finite open refinement of cardinality $\leq k$. By the Dowker theorem [7, Theorem 7.2.4], from $\dim Y = 0$ it follows that the last cover of Y has a discrete open refinement $\{V_\alpha^* : \alpha \in k_1\}$, where $k_1 \leq k$ and each V_α^* is homeomorphic to a non-empty clopen subset V_α of V. Then $Y = \oplus\{V_\alpha^* : \alpha \in k_1\}$. By Lemma 1, we have $X \approx k_1 \times X$. From h-homogeneity of X it follows that $U \approx X \approx \oplus\{U_\alpha : \alpha \in k_1\}$, where each $U_\alpha \approx X$ and $U_\alpha \subset U$. Hence,

$$\begin{aligned} X \times Y &\approx X \times (\oplus\{V_\alpha : \alpha \in k_1\}) \approx \oplus\{X \times V_\alpha : \alpha \in k_1\} \\ &\approx \oplus\{U_\alpha \times V_\alpha : \alpha \in k_1\} = W^*. \end{aligned}$$

The set $W^* \subseteq U \times V$ and W^* is a clopen subset of W.

If we take a base $\mathcal{B}$ for $X \times Y$, then the family $\{W^* : W \in \mathcal{B}\}$ forms a π-base. Since $k \geq \omega$, the space $X \times Y$ is non-pseudocompact. Theorem 1 implies that $X \times Y$ is an h-homogeneous space. □

Theorem 7. *Let X be an h-homogeneous non-pseudocompact zero-dimensional space. Let Y be a homogeneous zero-dimensional Lindelöf space. Then $X \times Y$ is an h-homogeneous space.*

Proof. The theorem is a particular case of Theorem 6 when $k = \omega$. One can verify that X is ω-divisible. Next, the conditions $\mathrm{ind} Y = 0$ and $\dim Y = 0$ are equivalent for every Lindelöf space (see [7, Theorem 7.1.11]). □

Corollary 4. *Let Y be a homogeneous zero-dimensional Lindelöf space. Then $Q \times Y$ is an h-homogeneous space, where Q is the space of rationals.*

Theorem 8. *Let X be an infinite h-homogeneous pseudocompact space and Y be a homogeneous zero-dimensional compact space. Then $X \times Y$ is an h-homogeneous space.*

Proof. Take a non-empty clopen subset V of the space Y. Fix a point $a \in V$. From homogeneity of Y it follows that for every point $y \in Y$ there exists a homeomorphism $f_y : Y \to Y$ such that $f_y(a) = y$. Since Y is compact, the cover $\{f_y(V) : y \in Y\}$ of Y has a finite subcover $\{V_i^* : 0 \leq i \leq n\}$ for some n, where each V_i^* is clopen in Y. Put $\hat{V}_0 = V_0^*$ and $\hat{V}_i = V_i^* \setminus \bigcup\{V_j^* : j < i\}$ for $i > 0$. Then each $\hat{V}_i$ is homeomorphic to a clopen subset V_i of V. Clearly, $Y = \cup\{\hat{V}_i : i \leq n\}$, where $\hat{V}_i \cap \hat{V}_j = \emptyset$ whenever $i \neq j$. Without loss of generality, $\hat{V}_0 = V_0 = V$ and each $\hat{V}_i$ is non-empty.

Consider a non-empty clopen subset U of the space X. By Lemma 1, we have $X \approx n \times X$. From h-homogeneity of X it follows that $U = \cup\{U_i : i \leq n\}$, where each U_i is clopen in X, $U_i \approx X$, and $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Hence,

$$\begin{aligned} X \times Y &\approx X \times (\oplus\{V_i : i \leq n\}) \approx \oplus\{X \times V_i : i \leq n\} \\ &\approx \cup\{U_i \times V_i : i \leq n\} = Z. \end{aligned}$$

One can check that Z is a clopen subset of $U \times V$.

Let $\mathcal{A}$ be the Boolean algebra over V generated by clopen sets $V_0, \ldots, V_n$. Since $\mathcal{A}$ is finite, it must be atomic. Let $W_0, \ldots, W_m$ be the atoms of $\mathcal{A}$. Denote by n_j the number of sets $V_0, \ldots, V_n$ containing W_j for $j \leq m$. Note that each $n_j \geq 1$ because $V_0 = V$. Then

$$\begin{aligned} Z &= \cup\{U_i \times \cup\{W_j : W_j \subseteq V_i\} : i \leq n\} \approx \oplus\{X \times \cup\{W_j : W_j \subseteq V_i\} : i \leq n\} \\ &\approx \oplus\{X \times n_j \times W_j : j \leq m\} \approx \oplus\{X \times W_j : j \leq m\} \\ &\approx X \times (\cup\{W_j : j \leq m\}) \approx X \times V_0 \approx U \times V. \end{aligned}$$

Thus, $X \times Y \approx U \times V$ for every non-empty clopen subsets U and V of the spaces X and Y, respectively.

Next, take a non-empty clopen subset C of $X \times Y$. By virtue of [7, Theorem 3.10.27], the product $X \times Y$ is pseudocompact. Proposition 6 and Lemma 7 from [3] imply that C can be written as the union of finitely many pairwise disjoint clopen rectangles. Say, C is the union of l such rectangles. Then $C \approx l \times X \times Y \approx X \times Y$. Hence, $X \times Y$ is h-homogeneous. $\square$

Corollary 5. *Let X be an infinite h-homogeneous compact space and Y be a homogeneous zero-dimensional compact space. Then $X \times Y$ is an h-homogeneous space.*

Corollary 6. *Let Y be a homogeneous zero-dimensional compact space. Then $\mathcal{C} \times Y$ is a homogeneous, h-homogeneous space, where $\mathcal{C}$ is the Cantor set.*

Corollary 7. *Let Y be a homogeneous zero-dimensional compact space. Then Y is a retract of an h-homogeneous compact space Z with $w(Z) = w(Y)$.*

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