Cofinal types of abelian topological groups: coproducts and free groups

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Cofinal type of directed sets

$(X, \leq)$ partially ordered set (poset).

- $C \subseteq X$ is *cofinal* if $\forall x \in X\ \exists c \in C\ (c \leq x)$.
- $(X, \leq)$ is a *directed set* if $\forall x_1, x_2\ \exists x_3\ (x_1 \leq x_3) \wedge (x_2 \leq x_3)$.

X and Y directed sets.

- X and Y are *cofinally similar* if there is a poset Z such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z$, and X and Y are cofinal in Z. Notation: $X \approx Y$.
- $X \preccurlyeq Y$ if there is a map $f : Y \to X$ such that $\forall x \in X\ \exists y_0 \in Y\ \forall y \geq y_0\ (f(y) \geq x)$.
- $X \approx Y \iff X \preccurlyeq Y$ and $Y \preccurlyeq X$.
- *Cofinal types:* equivalence classes with respect to $\approx$.

Cofinal type of topological groups

M an abelian topological group.

- $\mathcal{N}(M) :=$ filter of nbhds of zero in M, ordered with respect to reverse-inclusion ($\supseteq$).
- *Cofinal type of M:* the cofinal type of $\mathcal{N}(M)$.
- $\chi(M) = \mathrm{cof}\, \mathcal{N}(M)$.

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X is a P*-space* if every G_δ-set is open.

- M is not a P-space if and only if $\omega \preccurlyeq \mathcal{N}(M)$.

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Proof. ($\Leftarrow$) If $\{U_n\} \subseteq \mathcal{N}(M)$ is descending and $\bigcap U_n \notin \mathcal{N}(M)$, then put $U_0 := M$, and set $f(U) := \max\{n \in \omega \mid U \subseteq U_n\}$.
($\Rightarrow$) If $f : \mathcal{N}(M) \to \omega$ demonstrates $\omega \preccurlyeq \mathcal{N}(M)$, then $\forall n \; \exists U_n \; \forall U \subseteq U_n (f(U) \geq n)$.
If $\bigcap U_n \in \mathcal{N}(M)$, then $\forall n (f(\bigcap U_n) \geq n)$.

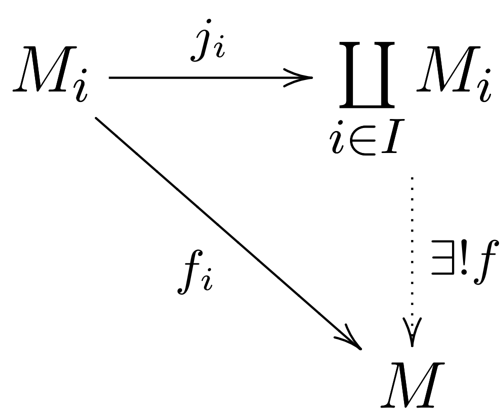
Coproducts of abelian top. groups

$\{M_i\}_{i\in I}$ abelian topological groups.

$\bigoplus_{i\in I} M_i := \{(x_i)\in \prod_{i\in I} M_i \,|\, x_i = 0$ for all but finitely many $i\}$.

$\coprod_{i\in I} M_i := \bigoplus_{i\in I} M_i$ equipped with a group topology such that:

- the inclusions $j_i\colon M_i \to \coprod_{i\in I} M_i$ are cts;
- $\forall$ family $\{f_i\colon M_i \to M\}_{i\in I}$ of cts homomorphisms,
  $\exists!$ cts homomorphism $f\colon \coprod_{i\in I} M_i \to M$ with $\forall i\in I, f_i = f j_i$.



Known properties of coproducts

$\{M_i\}_{i\in I}$ abelian topological groups.

- $\coprod_{i\in I} M_i = \prod_{i\in I} M_i$ if I is finite.
- $\coprod_{i\in I} M_i = (\bigoplus_{i\in I} M_i, \Box)$ if $|I| = \omega$.

X is a P-*space* if every G_δ-set is open.

Theorem. (Nickolas, 2002)
$\coprod_{i\in I} M_i = (\bigoplus_{i\in I} M_i, \Box)$ if and only if $|I \setminus I_P| \leq \omega$, where
$I_P := \{i \in I \mid M_i$ is a P-space$\}$.

- If $|I \setminus I_P| \leq \omega$, then $\mathcal{N}(\coprod_{i\in I} M_i) \approx \prod_{i\in I} \mathcal{N}(M_i)$.

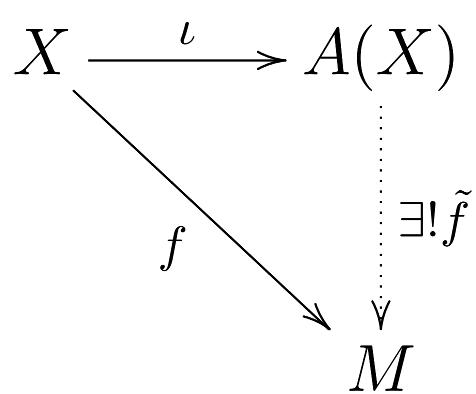
Free abelian topological groups

X a topological space (no separation axioms assumed).

$\mathbb{Z}^{(X)} := \bigoplus_X \mathbb{Z}$.

$A(X) := \mathbb{Z}^{(X)}$ equipped with a group topology such that:

- $\iota\colon X \to A(X)$ defined by $\iota(x) = 1 \cdot x$ is cts;
- $\forall f\colon X \to M$ cts into an abelian topological group M, $\exists!$ cts homomorphism $\tilde{f}\colon A(X) \to M$ with $f = \tilde{f}\iota$.



Well-known properties of $A(X)$

X a topological space.

- $\iota\colon X \to A(X)$ is an embedding if and only if X is completely regular.
- If X is Tychonoff, then so is $A(X)$.
- if X is an abelian topological group, then $X \cong A(X)/K$, where K is a subgroup of $A(X)$.

Constructing group topologies

A an abelian group.

- $B^s := B \cup (-B) \cup \{0\}$ for $B \subseteq A$.
- $\sum_{n=1}^{\infty} B_n := \bigcup_{n=1}^{\infty} (B_1^s + \cdots + B_n^s)$ for a family $\{B_n\}_{n=1}^{\infty}$ of subsets of A.

Theorem. (Zelenyuk-Protasov, 1990)
Let $\mathcal{B}$ be a filter-base on A.

- There is a finest group topology $\mathcal{T}_{\mathcal{B}}$ such that $\mathcal{B} \xrightarrow{\mathcal{T}_{\mathcal{B}}} 0$.
- The sets $\sum_{n=1}^{\infty} B_n$, where $B_n \in \mathcal{B}$ for every n, form a nbhd base at 0 for $\mathcal{T}_{\mathcal{B}}$.

Basic properties of ZP topologies

A an abelian group, $\mathcal{B}$ a filter-base on A.
M an abelian topological group.

- $M = (M, \mathcal{T}_{\mathcal{N}(M)})$.
- A homo. $\varphi\colon (A, \mathcal{T}_B) \to M$ is cts $\iff \varphi(\mathcal{B}) \longrightarrow 0$ in M.
- $\mathcal{N}(A, \mathcal{T}_{\mathcal{B}}) \preccurlyeq \mathcal{B}^\omega$. [Consider $\sum\colon \mathcal{B}^\omega \to \mathcal{N}(A, \mathcal{T}_{\mathcal{B}})$.]
- If $\mathcal{B}$ is countable (e.g., generated by a sequence in A), then $\chi(A, \mathcal{T}_{\mathcal{B}}) \leq \mathfrak{d} = \operatorname{cof} \omega^\omega$.
- If M is metrizable, $\chi(M, \mathcal{T}_{\mathcal{N}(M)}) < \mathfrak{d}$.

Special properties of ZP topologies

A an abelian group, $\mathbf{a}=(a_n)$ a sequence in A.
$\mathcal{B}_{\mathbf{a}} := \{\{a_n \mid n \geq m\} \mid m \in \mathbb{N}\}$.
$\mathcal{T}_{\mathbf{a}} := \mathcal{T}_{\mathcal{B}_{\mathbf{a}}}$, the finest group topology on A such that $a_n \longrightarrow 0$.

Theorem. (Zelenyuk-Protasov, 1990)
Suppose that $\mathcal{T}_{\mathbf{a}}$ is Hausdorff.

- $(A, \mathcal{T}_{\mathbf{a}})$ is complete (= every Cauchy filter converges).
- $(A, \mathcal{T}_{\mathbf{a}})$ is a sequential space (= every sequentially closed set is closed).
- $(A, \mathcal{T}_{\mathbf{a}})$ is not Fréchet-Urysohn (= sequential closure coincides with the closure).

Filter weight

Definition.
The *filter weight* of an (abelian) topological group M is the smallest possible cardinality for a filter-base $\mathcal{B}$ on M such that M carries the finest group topology w.r.t. $\mathcal{B} \longrightarrow 0$.
[In other words, $M = (M, \mathcal{T}_{\mathcal{B}})$.]

Examples.

- If M is metrizable, then it is a countable filter weight.
- If A is an abelian group and $\mathbf{a} = (a_n)$ is a sequence in A, then $(A, \mathcal{T}_{\mathbf{a}})$ has a countable filter weight.
- If M has a countable filter weight, then $\mathcal{N}(M) \preccurlyeq \omega^\omega$, and if in addition M is non-discrete, then $\mathcal{N}(M)^\omega \approx \omega^\omega$.

Coproducts using ZP topologies

$\{M_i\}_{i\in I}$ abelian topological groups.
We identify M_i with its image in $\bigoplus_{i\in I} M_i$.

- $\bigwedge\mathcal{N}(M_i) := \{\bigcup_{i\in I} U_i \,|\, U_i \in \mathcal{N}(M_i)\}$.
- Since $j_i\colon M_i \to \coprod_{i\in I} M_i$ are cts, $\mathcal{N}(M_i) \longrightarrow 0$ in $\coprod_{i\in I} M_i$.
- Thus, $\bigwedge\mathcal{N}(M_i) \longrightarrow 0$ in $\coprod_{i\in I} M_i$.

Theorem. (Tsaban-GL, 2011)
$\coprod_{i\in I} M_i = (\bigoplus_{i\in I} M_i, \mathcal{T}_{\bigwedge\mathcal{N}(M_i)})$.

Properties of coproducts

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Corollary. (Chasco-Domínguez, 2003) The sets

$$\sum_{n=1}^{\infty} \bigcup_{i\in I} U_{i,n} = \bigcup_{n=1}^{\infty} \bigcup_{i\in I^n} \sum_{k=1}^{n} U_{i_k,k},$$

where $(U_{i,n}) \in \prod_{i\in I} \mathcal{N}(M_i)^\omega$, form a nbhd base at 0 for $\coprod_{i\in I} M_i$.

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Theorem. (Tsaban-GL, 2011)

$$\prod_{i\in I} \mathcal{N}(M_i) \preccurlyeq \mathcal{N}\Big(\coprod_{i\in I} M_i\Big) \preccurlyeq \prod_{i\in I} \mathcal{N}(M_i)^{\omega}.$$

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- $I_\omega := \{i\in I \mid \mathcal{N}(M_i)^\omega \approx \mathcal{N}(M_i)\}$.
- $I_\infty := \{i\in I \mid \mathcal{N}(M_i)\approx\mathcal{N}(M_j)$ for infinitely many $j\in I\}$.
- $I_{\text{fin}} := I\setminus(I_\omega\cup I_\infty)$.

Theorem. (Tsaban-GL, 2011)

$$\mathcal{N}(\coprod_{i\in I}M_i)\approx\prod_{i\in I\setminus I_{\text{fin}}}\mathcal{N}(M_i)\times\mathcal{N}(\coprod_{i\in I_{\text{fin}}}M_i).$$

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Corollary. (Tsaban-GL, 2011)
If $|I_{\text{fin}}| \leq \omega$, then $\mathcal{N}(\coprod_{i\in I} M_i) \approx \prod_{i\in I} \mathcal{N}(M_i)$.

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Corollary. (Tsaban-GL, 2011)
If each M_i is metrizable and M_i is non-discrete for κ many indices i, then $\mathcal{N}(\coprod_{i\in I} M_i) \approx \omega^\kappa$ and $\chi(\coprod_{i\in I} M_i) = \mathrm{cof}(\omega^\kappa)$.

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Corollary. (Tsaban-GL, 2012)
If each M_i has a countable filter weight and M_i is non-discrete for $\kappa \geq \omega$ many indices i, then $\mathcal{N}(\coprod_{i\in I} M_i) \approx \omega^\kappa$

and $\chi(\coprod_{i\in I} M_i) = \text{cof}(\omega^\kappa)$.

Free abelian groups using ZP topology

X a topological space (no separation axioms assumed).

- $\mathcal{N}_\Delta(X) :=$ nbhds of $\Delta_X := \{(x,x) \mid x \in X\}$ in $X \times X$.

M an abelian topological group.

Lemma.
For $f\colon X \to M$, the following statements are equivalent:

- f is cts;
- the map $X \times X \to M$ defined by $(x,y) \mapsto f(x) - f(y)$ is cts;
- the filter-base of all sets $\{f(x) - f(y) \mid (x,y) \in U\}$, where $U \in \mathcal{N}_\Delta(X)$, converges to 0 in M.

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- $U^\ominus := \{1\cdot x - 1\cdot y \mid (x,y) \in U\} \subseteq \mathbb{Z}^{(X)}$, where $U \in \mathcal{N}_\Delta(X)$.
- $\mathcal{B}^\ominus := \{U^\ominus \mid U \in \mathcal{N}_\Delta(X)\}$.

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Theorem. (Tsaban-GL, 2011)
$A(X)$ carries the finest group topology such that $B^\ominus \longrightarrow 0$:
$A(X) = (\mathbb{Z}^{(X)}, \mathcal{B}^\ominus)$.

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Corollary. (Folklore?) The sets

$$\sum U_n^\ominus = \{x_0 - y_0 + \cdots x_n - y_n \mid (x_i, y_i) \in U_i \cup U_i^{-1}, n \in \omega\},$$

where $(U_n)_{n\in\omega} \in \mathcal{N}_\Delta(X)^\omega$, form a nbhd base at 0 for $A(X)$.

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Corollary. (Tsaban-GL, 2011)
$\mathcal{N}(A(X)) \preccurlyeq \mathcal{N}_\Delta(X)^\omega \approx \mathcal{N}_\Delta(X \times \omega)$.

Remark. Nickolas and Tkachenko (2005) used
$\chi_\Delta(X) := \operatorname{cof} \mathcal{N}_\Delta(X)$ to express $\chi(A(X))$ for a Lindelöf X.

Cofinal type of $A(X)$

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Sketch of the proof. For $V \in \mathcal{N}(A)$, put $\frac{1}{2}V := \{x \in V \mid 2x \in V\}$.
Define $f : \mathcal{N}(A(X)) \to \mathcal{N}(A(X))^\omega$ by $f(V) := (\frac{1}{2^n}V)_{n\in\omega}$.

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Lemma 1. $\frac{1}{2}\left(\sum_{n=0}^{\infty} U_n^{\ominus}\right) \subseteq \sum_{n=1}^{\infty} U_n^{\ominus}$, where $(U_n)_{n\in\omega} \in \mathcal{N}_\Delta(X)^\omega$.

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Lemma 1. $\frac{1}{2}\left(\sum_{n=0}^{\infty} U_n^{\ominus}\right) \subseteq \sum_{n=1}^{\infty} U_n^{\ominus}$, where $(U_n)_{n\in\omega} \in \mathcal{N}_\Delta(X)^\omega$.

Lemma 2. Sequences of the form $(\sum_{n=k}^{\infty} U_n^{\ominus})_{k\in\omega}$, where $(U_n)_{n\in\omega} \in \mathcal{N}_\Delta(X)^\omega$, are cofinal in $\mathcal{N}(A(X))^\omega$.

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Theorem. (Tsaban-GL, 2012)
$\mathcal{N}(A(X))^\omega \approx \mathcal{N}(A(X))$.

Corollary. (Tsaban-GL, 2012)
Let $\{X_i\}_{i\in I}$ be a family of spaces, and let $X := \coprod_{i\in I} X_i$ denote their disjoint union. Then $\mathcal{N}(A(X)) \approx \prod_{i\in I} \mathcal{N}(A(X_i))$.

Application: P-space or not P-space?

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- $\omega \not\leq \mathcal{N}_\Delta(X) \iff \mathcal{N}_\Delta(X)$ is closed under countable intersections.

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- If $\omega \not\preccurlyeq \mathcal{N}_\Delta(X)$ (in particular if X is a P-space), then:
  - $\mathcal{N}_\Delta(X)^\omega \approx \mathcal{N}_\Delta(X)$;
  - $\mathcal{N}(A(X)) \preccurlyeq \mathcal{N}_\Delta(X)$;
  - $\omega \not\preccurlyeq \mathcal{N}(A(X))$;
  - $A(X)$ is a P-space;
  - $A(X)$ is linear (i.e., admits a nbhd base at 0 consisting of subgroups).

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  - $A(X)$ is a P-space;
  - $A(X)$ is linear (i.e., admits a nbhd base at 0 consisting of subgroups).
- if X is completely regular, then X is a P-space if and only if $A(X)$ is a P-space. [$i\colon X \to A(X)$ is an embed.]

Application: P-space or not P-space?

X is a completely regular space.

- The following statements are equivalent:
  - X is not a P-space;
  - $A(X)$ is not P-space;
  - $\omega^\omega \preccurlyeq \mathcal{N}(A(X))$.

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Corollary. (Nickolas-Tkachenko, 2005)
If a Tychonoff space X is not a P-space, then $\mathfrak{d} \leq \chi(A(X))$.

Application: Paracompact spaces

X a topological space (no separation axioms assumed).

- $UV := \{(x,z) \mid \exists y\ ((x,y) \in U, (y,z) \in V)\}$ for $U, V \in \mathcal{N}_\Delta(X)$.
- X is *paracompact* if for every $U \in \mathcal{N}_\Delta(X)$ there is $V \in \mathcal{N}_\Delta(X)$ such that $VV \subseteq U$.

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Lemma. X is paracompact if and only if the sets $\{(x,y) \mid x - y \in U\}$, where $U \in \mathcal{N}(A(X))$, are cofinal in $\mathcal{N}_\Delta(X)$.

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Lemma. X is paracompact if and only if the sets $\{(x,y) \mid x - y \in U\}$, where $U \in \mathcal{N}(A(X))$, are cofinal in $\mathcal{N}_\Delta(X)$.

Theorem. (Tsaban-GL, 2012)
If X is paracompact, then $\mathcal{N}(A(X)) \approx \mathcal{N}_\Delta(X)^\omega \approx \mathcal{N}_\Delta(X \times \omega)$.

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X a topological space (no separation axioms assumed).

- $UV := \{(x,z) \mid \exists y\ ((x,y) \in U, (y,z) \in V)\}$ for $U, V \in \mathcal{N}_\Delta(X)$.
- X is *paracompact* if for every $U \in \mathcal{N}_\Delta(X)$ there is $V \in \mathcal{N}_\Delta(X)$ such that $VV \subseteq U$.

Lemma. X is paracompact if and only if the sets $\{(x,y) \mid x - y \in U\}$, where $U \in \mathcal{N}(A(X))$, are cofinal in $\mathcal{N}_\Delta(X)$.

Theorem. (Tsaban-GL, 2012)
If X is paracompact, then $\mathcal{N}(A(X)) \approx \mathcal{N}_\Delta(X)^\omega \approx \mathcal{N}_\Delta(X \times \omega)$.

Corollary. (Nickolas-Tkachenko, 2005)
If X is paracompact and completely regular, then $\chi(A(X)) = \chi_\Delta(X \times \omega)$.

Application: Compact spaces

Theorem. (Nickolas-Tkachenko, 2005)
If X is an infinite compact space of weight τ, then $\chi(A(X))=\mathfrak{d}\cdot\mathrm{cof}([\tau]^{\le\omega},\subseteq)$.

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Theorem. (Tsaban-GL, 2011)
If X is an infinite compact space of weight τ, then $\mathcal{N}(A(X))\approx\mathrm{Cov}_{\mathrm{fin}}(X)^\omega\preccurlyeq\mathrm{Fin}(\tau)^\omega\approx\omega^\omega\times[\tau]^\omega$.

[$\mathrm{Cov}_{\mathrm{fin}}(X):=$ finite open covers of X, ordered by the refinement relation.]

Application: k_ω-spaces

A Hausdorff space X is a k_ω-*space* if:

- $X = \bigcup_{n\in\omega} K_n$, where each K_n is compact; and
- if $O \cap K_n$ is open in K_n for every n, then O is open in X.

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- $\mathcal{N}(A(X)) \preccurlyeq \prod_{n\in\omega} \operatorname{Fin}(\kappa_n)^\omega$, where $\kappa_n := w(K_n)$.

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Theorem. (Chis-Ferrer-Hernández-Tsaban, 2011)
For a k_ω-space $X = \operatorname{colim} K_n$, put $\kappa_n := w(K_n)$ and $\tau := \sup \kappa_n$.

- $\mathcal{N}(A(X)) \preccurlyeq \mathrm{Fin}(\tau)^\omega \approx \omega^\omega \times [\tau]^\omega$.
- $\chi(A(X)) \leq \mathfrak{d} \cdot \operatorname{cof}([\tau]^\omega)$.

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