t-symmetrizable quasi-uniformities

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Definition 1 *Let* X *be a set and let* $d : X \times X \to [0, \infty)$ *be a function mapping into the set* $[0, \infty)$ *of the nonnegative reals. Then* d *is called a* quasi-pseudometric *on* X *if*

(a) $d(x, x) = 0$ *whenever* $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ *whenever* $x, y, z \in X$.

We shall say that d *is a* T_0-quasi-(pseudo)metric *provided that* d *also satisfies the following condition: For each* $x, y \in X$,

$d(x, y) = 0 = d(y, x)$ *implies that* $x = y$.

For each positive ϵ we shall set $U_{d,\epsilon} := \{(x,y) \in X \times X : d(x,y) < \epsilon\}$.

The *quasi-pseudometric quasi-uniformity induced by* d on X will be denoted by $\mathcal{U}_d$.

As usual, we say that a topological space (X,τ) has a compatible quasi-uniformity if there is a quasi-uniformity $\mathcal{U}$ on X such that $\tau = \tau_{\mathcal{U}}$, where by $\tau_{\mathcal{U}}$ we denote the topology induced by $\mathcal{U}$, and we say that (X,τ) has a compatible quasi-pseudometric if there is a quasi-pseudometric d on X such that $\tau = \tau_{\mathcal{U}_d}$.

Remark 1 *Let* d *be a quasi-pseudometric on a set* X*, then* $d^{-1} : X \times X \to [0, \infty)$ *defined by* $d^{-1}(x, y) = d(y, x)$ *whenever* $x, y \in X$ *is also a quasi-pseudometric, called the* conjugate quasi-pseudometric of d.

As usual, a quasi-pseudometric d *on* X *such that* $d = d^{-1}$ *is called a* pseudometric.

Note that for any (T_0*-)quasi-pseudometric* d*,* $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ *is a pseudometric (metric).*

According to Matthews a quasi-pseudometric q on a set X is called *weightable* provided that there is a function $|\cdot| : X \to [0,\infty)$ such that

$$q(x,y) + |x| = q(y,x) + |y|$$

whenever $x, y \in X$.

The function $|\cdot|$ will be called a *weight* for q.

More generally we shall call a quasi-uniformity $\mathcal{U}$ on a set X *weightable* if $\mathcal{U} = \bigvee_{d \in \mathcal{E}} \mathcal{U}_d$ where $\mathcal{E}$ is a family of weightable quasi-pseudometrics on X.

We shall call a quasi-uniform space $(X, \mathcal{U})$ *weightable* if $\mathcal{U}$ is a weightable quasi-uniformity on X.

For later use, we recall that a quasi-uniformity $\mathcal{U}$ is called *precompact* provided that for each $U \in \mathcal{U}$ there is a finite $F \subseteq X$ such that $U(F) = X$.

Furthermore a quasi-uniformity $\mathcal{U}$ is called *totally bounded* provided that the uniformity $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is precompact.

A quasi-uniform space $(X, \mathcal{U})$ is said to be *hereditarily precompact* provided that each subspace of $(X, \mathcal{U})$ is precompact.

For any quasi-uniformity $\mathcal{U}$, by $\mathcal{U}_\omega$ we shall denote the finest totally bounded quasi-uniformity coarser than $\mathcal{U}$.

A quasi-uniformity is called *transitive* provided that it has a base consisting of transitive entourages.

Throughout $\mathcal{D}$ will denote the discrete uniformity on a set X.

Moreover for two real numbers a, b we shall set $a \dot{-} b = \max\{a - b, 0\}$.

Example 1 *Equip the set* $[0, \infty)$ *with the quasi-pseudometric* $u(x, y) = y \dot{-} x$ *whenever* $x, y \in [0, \infty)$, *and set* $|x| = x$ *whenever* $x \in [0, \infty)$. *Then* u *is a* T_0*-quasi-pseudometric on* $[0, \infty)$ *that is weighted by the usual norm* $|\cdot|$ *on* $\mathbb{R}$ *restricted to* $[0, \infty)$.

Furthermore the quasi-uniformity $\mathcal{U}_{u^{-1}}$ *is not weightable.*

Example 2 *The fine quasi-uniformity $\mathcal{F}$ of the topological space of the rationals $\mathbb{Q}$ (equipped with its usual topology) is not weightable.*

Proposition 1 *A topological space has a compatible weightable quasi-pseudometric if and only if it has a compatible weightable quasi-uniformity with a countable base.*

Proposition 2 *Let $(X, \mathcal{U})$ be a T_0-quasi-uniform space. Then its bicompletion $(\widetilde{X}, \widetilde{\mathcal{U}})$ is a weightable T_0-quasi-uniform space, too.*

In the following we discuss some analogues of uniform hyperspaces which yield some important examples of weightable quasi-uniformities.

Let (X, d) be a quasi-pseudometric space and let $\mathcal{P}_0(X)$ be the set of nonempty subsets of X. Moreover let $A, B \in \mathcal{P}_0(X)$. We set

$$H_d^+(A, B) = \sup_{b \in B} d(A, b)$$

and

$$H_d^-(A, B) = \sup_{a \in A} d(a, B).$$

(As usual, here for instance $d(A, b) = \inf\{d(a, b) : a \in A\}$.)

Furthermore

$$H_d(A, B) = H_d^+(A, B) \vee H_d^-(A, B).$$

Then H_d^+ is the extended *upper Hausdorff quasi-pseudometric*, H_d^- is the extended *lower Hausdorff quasi-pseudometric* and H_d is the extended *Hausdorff quasi-pseudometric* on $\mathcal{P}_0(X)$.

Similarly for each $x \in X$, set

$$(W_d^+)_x(A, B) = d(A, x) \dot{-} d(B, x)$$

and

$$(W_d^-)_x(A, B) = d(x, B) \dot{-} d(x, A).$$

Moreover for each $x \in X$, let

$$(W_d)_x(A, B) = (W_d^+)_x(A, B) \vee (W_d^-)_x(A, B).$$

Then for each $x \in X$, $(W_d^+)_x$ is the *upper Wijsman quasi-pseudometric at* x, $(W_d^-)_x$ is the *lower Wijsman quasi-pseudometric at* x, and $(W_d)_x$ is the *Wijsman quasi-pseudometric at* x.

For each $x \in X$, $(W_d^-)_x$ is a weightable quasi-pseudometric with weight $(w_d^-)_x(A) = d(x, A)$ whenever $A \in \mathcal{P}_0(X)$.

Let $A, B \in \mathcal{P}_0(X)$. It is known that

$$H_d^+(A, B) = \sup_{x \in X} (W_d^+)_x(A, B) = \sup_{x \in X} (d(A, x) - d(B, x)).$$

Similarly we have

$$H_d^-(A, B) = \sup_{x \in X} (W_d^-)_x(A, B) = \sup_{x \in X} (d(x, B) - d(x, A)).$$

Hence also $H_d(A, B) = \sup_{x \in X} (W_d)_x(A, B)$.

Proposition 3 *Each totally bounded quasi-uniformity $\mathcal{U}$ on a set X is weightable.*

Given any quasi-pseudometric space (X, d), we can define quasi-uniformities

$$\mathcal{U}_{W_d^+} = \bigvee_{x \in X} \mathcal{U}_{(W_d^+)_x}$$

and similarly

$$\mathcal{U}_{W_d^-} = \bigvee_{x \in X} \mathcal{U}_{(W_d^-)_x}$$

and

$$\mathcal{U}_{W_d} = \bigvee_{x \in X} \mathcal{U}_{(W_d)_x}$$

on $\mathcal{P}_0(X)$.

Corollary 1 *Let (X, d) be a quasi-pseudometric space. When restricting the Wijsman quasi-uniformities to X (where we identify the points with singletons), we have*

$$\mathcal{U}_{W_d^+} \subseteq \mathcal{U}_d$$

and

$$\mathcal{U}_{W_d^-} \subseteq \mathcal{U}_d;$$

consequently

$$\mathcal{U}_{W_d} \subseteq \mathcal{U}_d.$$

Equality holds in the three inclusions provided that $\mathcal{U}_d$ is totally bounded.

Remark 2 *There are obvious connections between the upper resp. lower constructions considered above and the operation of conjugation.*

For instance, for any quasi-pseudometric d *on a set* X *we have that for any* $x \in X$,

$$((W^{+}_{d^{-1}})_x)^{-1} = (W^{-}_{d})_x,$$

and thus

$$(\mathcal{U}_{W^{+}_{d^{-1}}})^{-1} = \mathcal{U}_{W^{-}_{d}}$$

on $\mathcal{P}_0(X)$.

Proposition 4 *Let d be a totally bounded quasi-pseudometric on a set X. Then*

$$\mathcal{U}_{W_d^+} = \mathcal{U}_{H_d^+},$$

$$\mathcal{U}_{W_d^-} = \mathcal{U}_{H_d^-}$$

and

$$\mathcal{U}_{W_d} = \mathcal{U}_{H_d}$$

on $\mathcal{P}_0(X)$.

Given a quasi-pseudometric space (X, d), for each $x \in X$ set

$$r_x(a) = d(a, x)$$

and

$$l_x(a) = d(x, a)$$

whenever $a \in X$.

Similarly let us define for each $x \in X$,

$$R_x(A) = d(A, x)$$

and

$$L_x(A) = d(x, A)$$

whenever $A \in \mathcal{P}_0(X)$.

The following definition is well known.

A quasi-pseudometric d on a set X is called *bounded* provided that there is a constant $M > 0$ such that $d(x, y) \leq M$ whenever $x, y \in X$.

Let us observe that for any $x \in X$, r_x is bounded on X if and only if R_x is bounded on $\mathcal{P}_0(X)$.

Analogously for any $x \in X$, l_x is bounded on X if and only if L_x is bounded on $\mathcal{P}_0(X)$.

Lemma 1 *A quasi-pseudometric d on a set X is bounded if and only if there is an $x \in X$ such that both r_x and l_x are bounded.*

Boundedness conditions imply hereditary precompactness properties of Wijsman type quasi-uniformities:

Remark 3 *The quasi-uniformity $\mathcal{U}_{u^{-1}}$ is hereditarily precompact on $[0, \infty)$. Hence given any set X, for any function $f : X \to [0, \infty)$, the initial quasi-uniformity $(f \times f)^{-1}\mathcal{U}_{u^{-1}}$ is hereditarily precompact.*

Suppose now that d *is a quasi-pseudometric on* X. *Observe that on* $\mathcal{P}_0(X)$ *we have*

$$\mathcal{U}_{(W_d^+)_x} = (R_x \times R_x)^{-1}\mathcal{U}_{u^{-1}}$$

and

$$\mathcal{U}_{(W_d^-)_x} = (L_x \times L_x)^{-1}\mathcal{U}_u$$

whenever $x \in X$. *Therefore for any quasi-pseudometric space* (X, d) *and* $x \in X$, $\mathcal{U}_{(W_d^+)_x}$ *is hereditarily precompact on* $\mathcal{P}_0(X)$ *and* $(\mathcal{U}_{(W_d^-)_x})^{-1}$ *is hereditarily precompact on* $\mathcal{P}_0(X)$.

Consequently $\mathcal{U}_{W_d^+}$ *and* $(\mathcal{U}_{W_d^-})^{-1}$ *are hereditarily precompact on* $\mathcal{P}_0(X)$, *since hereditary precompactness is preserved under arbitrary suprema of quasi-uniformities.*

Given $x \in X$, r_x is bounded on X if and only if $\mathcal{U}_{(W_d^+)_x}$ *is totally bounded on* $\mathcal{P}_0(X)$.

Similarly, given $x \in X$, l_x is bounded on X if and only if $\mathcal{U}_{(W_d^-)_x}$ *is totally bounded on* $\mathcal{P}_0(X)$.

Given $x \in X$, $\mathcal{U}_{(W_d)_x}$ *is totally bounded on* $\mathcal{P}_0(X)$ *if and only if* d *is bounded on X.*

Corollary 2 *Let* (X, d) *be a quasi-pseudometric space.*

Then $\mathcal{U}_{W_d^+}$ *is totally bounded on* $\mathcal{P}_0(X)$ *if and only if for each* $x \in X$*,* r_x *is bounded on* X*.*

Similarly $\mathcal{U}_{W_d^-}$ *is totally bounded on* $\mathcal{P}_0(X)$ *if and only if for each* $x \in X$*,* l_x *is bounded on* X*.*

Finally $\mathcal{U}_{W_d}$ *is totally bounded on* $\mathcal{P}_0(X)$ *if and only if* d *is bounded on* X*.*

Proposition 5 *Let q be a weightable quasi-pseudometric with weight function f on a set X. Then*

$$[(f \times f)^{-1}\mathcal{U}_u] \subseteq \mathcal{U}_q$$

and

$$\mathcal{U}_q \subseteq [(f \times f)^{-1}\mathcal{U}_u] \vee \mathcal{U}_{q^{-1}}.$$

In the following we replace the quasi-metric theory of weightability by a quasi-uniform approach to weightability.

Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{U}$ contains a finest symmetric quasi-uniformity coarser than $\mathcal{U}$, namely $\mathcal{U} \wedge \mathcal{U}^{-1}$, and $\mathcal{U}$ is contained in a coarsest symmetric quasi-uniformity finer than $\mathcal{U}$, namely $\mathcal{U}^s$.

A quasi-uniformity $\mathcal{A}$ on X will be called a *symmetrizer* of $\mathcal{U}$ provided that

$$\mathcal{U} \vee \mathcal{A}$$

is symmetric, that is, $\mathcal{U} \vee \mathcal{A}$ is a uniformity.

A symmetrizer $\mathcal{A}$ of $\mathcal{U}$ will be called *adequate* provided that $\mathcal{A} \subseteq \mathcal{U}^{-1}$.

Example 3 *Let* $\mathcal{U}$ *and* $\mathcal{Z}$ *be quasi-uniformities on a set* X. *Then* $\mathcal{Z}$ *satisfies both* $\mathcal{Z} \subseteq \mathcal{U}^{-1}$ *and* $\mathcal{U}^{-1} \subseteq \mathcal{U} \vee \mathcal{Z}$ *if and only if* $\mathcal{Z}$ *on* X *is an adequate symmetrizer of* $\mathcal{U}$.

Remark 4 *Let d be a weightable quasi-pseudometric with weight function f on a set X. We set $\mathcal{A} = (f \times f)^{-1}(\mathcal{U}_{u^{-1}})$.*

We have $\mathcal{A} \subseteq \mathcal{U}_{d^{-1}}$ and $\mathcal{U}_{d^{-1}} \subseteq \mathcal{U}_d \vee \mathcal{A}$.

Hence $\mathcal{A}$ is an adequate symmetrizer for $\mathcal{U}_d$. It follows that

$$\mathcal{U}_{d^s} = \mathcal{U}_d \vee \mathcal{A}.$$

For any weight f, $\mathcal{A}$ is hereditarily precompact, since f is bounded below by 0.

For any weight f, $\mathcal{A}^s$ is preLindelöf.

For a bounded weight f, $\mathcal{A}$ is totally bounded.

If $\mathcal{A}$ is a(n adequate) symmetrizer for a quasi-uniformity $\mathcal{U}$, then $\mathcal{A}^{-1}$ is a(n adequate) symmetrizer for $\mathcal{U}^{-1}$.

For each quasi-uniformity $\mathcal{U}$, the conjugate quasi-uniformity $\mathcal{U}^{-1}$ is an adequate symmetrizer of $\mathcal{U}$.

Consider a quasi-uniformity $\mathcal{U}_T$ generated by a partial order T on a set X (that is, $\mathcal{U}_T$ has the base $\{T\}$) and let L by a linear extension of T on X.

Then in general the quasi-uniformity $\mathcal{U}_{L^{-1}}$ generated by L^{-1} is strictly coarser than the quasi-uniformity $\mathcal{U}_{T^{-1}}$ generated by T^{-1} on X, but obviously both are adequate symmetrizers of $\mathcal{U}_T$.

For each uniformity $\mathcal{U}$, any quasi-uniformity coarser than $\mathcal{U}$ is an adequate symmetrizer of $\mathcal{U}$.

Trivially, each totally bounded quasi-uniformity $\mathcal{U}$ on a set X can be made transitive by taking the supremum with the finest possible (transitive) totally bounded quasi-uniformity $\mathcal{D}_\omega$ on X.

On the other hand there are quasi-uniformities that cannot be made transitive by taking the supremum with any totally bounded quasi-uniformity.

Similarly there are quasi-uniformities that cannot by made symmetric by taking the supremum with any totally bounded quasi-uniformity.

Obviously each quasi-uniformity on a set X can be made symmetric by taking the supremum with the discrete uniformity $\mathcal{D}$ on X.

Lemma 2 *Suppose that* $\mathcal{U}$ *is a quasi-uniformity on a set* X. *Let* $\{A_i : i = 1, \ldots, n\}$ *be a finite cover of* X *and let* $V \in \mathcal{U}$.

Then

$$\bigcup_{i=1}^{n} (V^{-1}(A_i) \times V(A_i))$$

belongs to $\mathcal{U}_\omega$.

Theorem 1 *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be quasi-uniformities on a set* X *such that there is a totally bounded quasi-uniformity* $\mathcal{Z}$ *on* X *with*

$$\mathcal{U} \subseteq \mathcal{V} \vee \mathcal{Z}.$$

Then

$$\mathcal{U} \subseteq \mathcal{V} \vee (\mathcal{U} \vee \mathcal{V}^{-1})_\omega.$$

Let us call a quasi-uniformity $\mathcal{U}$ on a set X *t-symmetrizable* provided that there is a totally bounded quasi-uniformity $\mathcal{Z}$ on X such that $\mathcal{U} \vee \mathcal{Z}$ is a uniformity, that is, $\mathcal{U}$ possesses a totally bounded symmetrizer $\mathcal{Z}$.

Remark 5 *(a) The supremum of any family of* t*-symmetrizable quasi-uniformities on a set* X *is* t*-symmetrizable.*

(b) The conjugate of a t*-symmetrizable quasi-uniformity is* t*-symmetrizable.*

(c) Each totally bounded quasi-uniformity is t*-symmetrizable.*

(d) The restriction to a subspace of a t*-symmetrizable quasi-uniformity is* t*-symmetrizable.*

(e) The product quasi-uniformity of any family of t*-symmetrizable quasi-uniformities is* t*-symmetrizable.*

Corollary 3 *(a) Let* $(q_i)_{i\in I}$ *be a family of quasi-pseudometrics on a set* X *where* q_i *is weightable by a bounded weight* f_i $(i \in I)$*. Then* $\bigvee_{i\in I}\mathcal{U}_{q_i}$ *is* t*-symmetrizable.*

(b) Each weightable quasi-uniformity inducing a countably compact topology is t*-symmetrizable.*

Proposition 6 *Let* $\mathcal{U}$ *be a* t*-symmetrizable quasi-uniformity on a set* X*. Then* $\mathcal{U}$ *can be adequately symmetrized by* $(\mathcal{U}^{-1})_\omega$ *to* $\mathcal{U}^s$*.*

Corollary 4 *The following statements for a quasi-uniformity $\mathcal{U}$ on a set X are equivalent:*

(a) $\mathcal{U}$ is t-symmetrizable.

(b) $\mathcal{U}$ has a totally bounded adequate symmetrizer.

Remark 6 *Let $\mathcal{U}$ be a quasi-uniformity on a set X. Note that the existence of a quasi-uniformity $\mathcal{Z}$ on X such that $\mathcal{U} = \mathcal{U}^{-1} \vee \mathcal{Z}$ implies that $\mathcal{U}^s = \mathcal{U}^{-1} \vee \mathcal{Z} = \mathcal{U}$, hence that $\mathcal{U}$ is a uniformity.*

Let $\mathcal{U}$ be a t-symmetrizable quasi-uniformity.

Then according to the preceding results $\mathcal{U}^s$ is clearly the smallest uniformity that we can obtain by t-symmetrization of $\mathcal{U}$.

$$\mathcal{U}^s \vee \mathcal{D}_\omega = \mathcal{U} \vee (\mathcal{U}^{-1})_\omega \vee \mathcal{D}_\omega = \mathcal{U} \vee \mathcal{D}_\omega$$

is obviously the finest uniformity that we can reach by t-symmetrization of $\mathcal{U}$.

Indeed each uniformity in between these two extreme cases can be obtained by t-symmetrization of $\mathcal{U}$:

Proposition 7 *Let* $\mathcal{U}$ *be a* t*-symmetrizable quasi-uniformity on a set* X*. Then any uniformity* $\mathcal{V}$ *on* X *such that* $\mathcal{U}^s \subseteq \mathcal{V} \subseteq \mathcal{U}^s \vee \mathcal{D}_\omega$ *satisfies* $\mathcal{V} = \mathcal{U} \vee \mathcal{V}_\omega$.

Remark 7 *Suppose that* $\mathcal{U}$ *is a quasi-uniformity on a set* X *that has a countable base and is* t*-symmetrizable. Then* $\mathcal{U}$ *can be symmetrized by some totally bounded quasi-uniformity on* X *having a countable base.*

Theorem 2 *Let* $\mathcal{U}$ *be a* t*-symmetrizable quasi-uniformity on a set* X*. Then* $\mathcal{U}$ *can be written as the supremum of a family of quasi-pseudometric quasi-uniformities* $\mathcal{U}_q$ *where* q *is a quasi-pseudometric on* X *such that the quasi-uniformity* $\mathcal{U}_q$ *is* t*-symmetrizable.*

A quasi-uniformity $\mathcal{U}$ on a set X is called *proximally symmetric* provided that the finest totally bounded quasi-uniformity $\mathcal{U}_\omega$ coarser than $\mathcal{U}$ is a uniformity on X.

Proposition 8 *Each* t*-symmetrizable proximally symmetric quasi-uniformity* $\mathcal{U}$ *on a set* X *is a uniformity.*

Proposition 9 *Each* t*-symmetrizable quasi-uniformity* $\mathcal{U}$ *on a set* X *satisfies* $(\mathcal{U}^s)_\omega = (\mathcal{U}^{-1})_\omega \vee \mathcal{U}_\omega$.

Several results that were originally proved for weightable quasi-pseudometrics (with bounded weights) indeed hold for t-symmetrizable quasi-uniformities: In our next result $|X|$ will denote the (infinite) cardinality of a set X.

Proposition 10 *For each t-symmetrizable quasi-uniformity $\mathcal{U}$ on a set X, if D is an infinite $\mathcal{U}^s$-discrete subset of X, then there is $B \subseteq D$ such that $|B| = |D|$ and B is $\mathcal{U}$-discrete.*

A base $\mathcal{B}$ is a θ-*base* for a topological space X if $\mathcal{B}$ can be written as $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ in such a way that given any open set U of X and any point $x \in U$ there is $n_x \in \mathbb{N}$ such that $\mathrm{ord}(x, \mathcal{B}_{n_x})$ (that is, the order of $\mathcal{B}_{n_x}$ at x) is finite and some member B of $\mathcal{B}_{n_x}$ has $x \in B \subseteq U$.

It is known that a topological space has a θ-base if and only if it is quasi-developable.

The following result generalizes the fact that for each weightable quasi-pseudometric d on a set X the topology $\tau_{\mathcal{U}_d}$ has a θ-base.

Proposition 11 *Let $\mathcal{U}$ be a quasi-uniformity with a countable base on a set X possessing the property that $\mathcal{U}^s = \mathcal{U} \vee \mathcal{A}$ where $\mathcal{A}$ is a quasi-uniformity on X such that $\mathcal{A}^s$ is preLindelöf.*

Then $\tau_{\mathcal{U}}$ has a θ-base.

A quasi-uniformity $\mathcal{U}$ on a set X is called *Smyth completable* provided that each left K-Cauchy filter on $(X, \mathcal{U})$ is a $\mathcal{U}^s$-Cauchy filter.

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called a *left K-Cauchy filter* (resp. *right K-Cauchy filter*) provided that for each $U \in \mathcal{U}$ there is $\mathcal{F}_U \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ (resp. $U^{-1}(x) \in \mathcal{F}$) whenever $x \in \mathcal{F}_U$.

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called *$\mathcal{U}$-stable* provided that for each $U \in \mathcal{U}$ we have that

$$\bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{F}.$$

Remark 8 *Let* $(X,\mathcal{U})$ *be a quasi-uniform space such that each left* K*-Cauchy filter on* $(X,\mathcal{U})$ *is contained in a* $\mathcal{U}^s$*-Cauchy filter. Then each left* K*-Cauchy filter* $\mathcal{F}$ *on* X *is in fact a* $\mathcal{U}^s$*-Cauchy filter.*

The following result generalizes the fact that for each weightable quasi-pseudometric d on a set X with a weight $|\cdot|$ the quasi-uniformity $\mathcal{U}_d$ is Smyth completable.

Proposition 12 *Each quasi-uniformity* $\mathcal{U}$ *on a set* X *possessing an adequate symmetrizer* $\mathcal{A}$ *where* $\mathcal{A}$ *is hereditarily precompact is Smyth completable.*

Problem 1 *Let* q *be a quasi-pseudometric on a set* X *such that the quasi-uniformity* $\mathcal{U}_q$ *is t-symmetrizable.*

Is $\mathcal{U}_q = \mathcal{U}_d$ *where* d *is a quasi-pseudometric on* X *that is weightable by a bounded weight?*

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