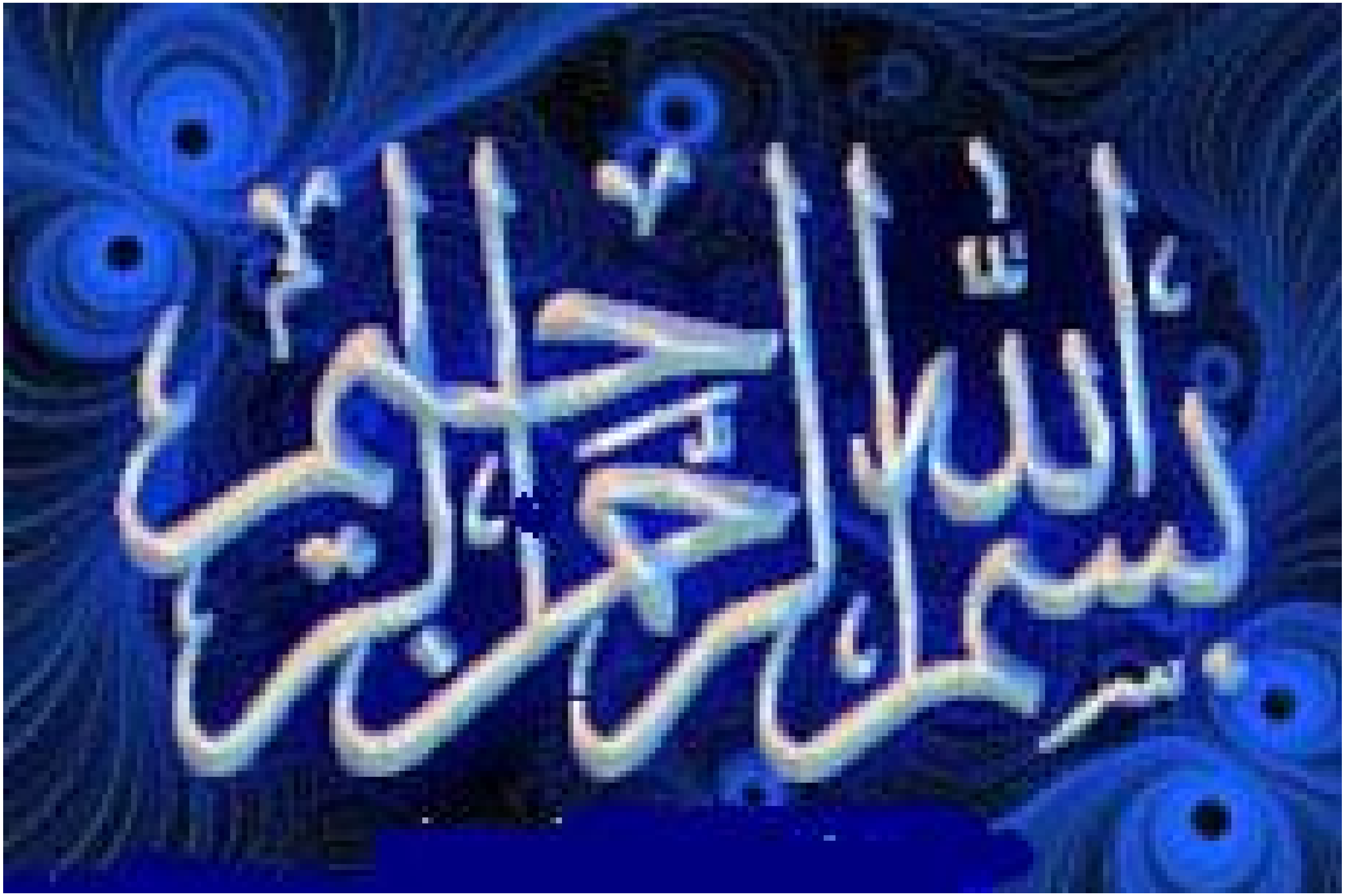
بسم الله الرحمن الرحيم

Semi-Topological Groups

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- Elwood Bohn [El] was the first who studied the notion of semi-topological Groups in 1965.
- Definition and results were published in The American Mathematical Monthly, vol. 72, No. 9 (1965), 996-998.
- El wood defined semi-topological Groups by using semi-open sets defined by Levine in 1963.

[El] Elwood Bohn, Semi-Topological Groups, The American Math. Month., vol.72(9) (1965), 996-998.

- In this talk we will follow the definition of Elwood and by using semi-open sets, deduce some results.

Preliminaries

- (X,τ) denotes a topological space with no separation properties assumed.
- cl(A) and Int(A) denote the closure and interior of a set A in X.
- A subset A of a topological space X is called semi-open [NL] if there exists an open set U in X such that $U \subseteq A \subseteq cl(U)$.

[NL] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1) (1963), 36-41.

- Complement of a semi-open set is called semi-closed set.
- Collection of all semi-open (respectively, semi-closed sets) in X is denoted by SO(X) (respectively, SC(X)).
- sCl(A) represents the semi-closure of A and is the intersection of all semi-closed sets containing A.

- Theorem 1. If $A_\alpha \in SO(X)$ for each $\alpha \in \nabla$, then $\bigcup_{\alpha\in\nabla} A_\alpha \in SO(X)$

- Theorem 2. A set $M_x \subseteq X$ is semi-neighbourhood of a point $x \in X$ if and only if there exists $A \in SO(X)$ such that $x \in A \subseteq M_x$

- If $A \subseteq X$, then necessary and sufficient condition that $A \in SO(X)$ is that A be a semi-neighbourhood of each $x \in A$.
- A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is semi-continuous at $x \in X$ if and only if for each neighbourhood $N_{f(x)}$ of $f(x)$, $f^{-1}(N_{f(x)})$ is semi-neighbourhood of x. f is semi-continuous on X if and only if f is semi-continuous at each point of X.

- Theorem 3. A function f : (X, τ) $\rightarrow$ (Y,σ) is semi-continuous on X if and only if for each $x \in X$ and each neighbourhood $N_{f(x)}$ there exists a semi-neighbourhood M_x such that $f(M_x) \subseteq N_{f(x)}$

- Theorem 4. $A \in SO(X)$ if and only if $A^{-1} \in SO(X)$

- Theorem 5.
- Let (X, τ) and (Y, σ) be topological spaces and let $(X \times Y, \tau \times \sigma)$ be their product space. If $A \in SO(X)$ and $B \in SO(Y)$, then $A \times B \in SO(X \times Y)$.
- Note that converse of this theorem is not true in general.

Semi-Topological Geoup (By Elwood)

- A triple (G, $\circ$, τ) is termed a semi-topological group [El] if and only if (G, $\circ$) is a group, (G, τ) is a topological space, and for each $x, y \in G$ and each nbd $N_{x\circ y^{-1}}$ there exist semi-nbds M_x and M_y such that $M_x \circ M_y^{-1} \subseteq N_{x \circ y^{-1}}$

[El] Elwood Bohn, Semi-Topological Groups, The American Math. Month., vol.72(9) (1965), 996-998.

- Theorem 6. If (G, o, τ) is a semi-topological group then the function $f : G \times G \rightarrow G$, where $f((x \circ y)) = x \circ y^{-1}$ is semi-continuous relative to the product topology for $G \times G$.

- Theorem 7. If (G, o, τ) is a semi-topological group then the function $i : G \rightarrow G$ defined by $i(x) = x^{-1}$ and $m : G \times G \rightarrow G$ defined by $m((x \circ y)) = x \circ y$ are semi-continuous functions.

- Theorem 8. Let $(G, \circ, \tau)$ be a semi-topological group. If $A \in SO(G)$ and $B \subseteq G$, then $A \circ B$, $B \circ A \in SO(G)$.

- Remark 1. The converse of Theorem 6 and 7 are not true.

- Example 1. Let (G, +) be the group of integers modulo 2, with the usual operation of addition, and let $\tau = \{\phi, \{0\}, G\}$. $i : G \longrightarrow G$ is continuous on G. Similarly $m : G \times G \longrightarrow G$ is continuous at (0, 0), (1, 0), and (0, 1) and semi-continuous at (1,1). However, since {1} is not semi-open, $(G, \circ, \tau)$ is not a semi-topological group.

- Theorem 9. Let (G, ∘, τ) be a semi-topological group. Then the map $L_g : G \longrightarrow G$ defined by $L_g(h) : g \circ h$ is semi-continuous.
- The result for right multiplication is similar.

- Theorem 10. Let (G, $\circ$, τ) be a semi-topological group and (H, $\circ$) is a semi-open subgroup of G. Then any coset $x \circ H$ is semi-open.

- Theorem 11. Let (G, $\circ$, τ) be a semi-topological group then every semi-open subgroup of G is also semi-closed.

- Theorem 12. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a semi-continuous function and let A be an open subspace of X, then the restriction map $f/_A : A \longrightarrow Y$ defined by $f/_A(y) = f(y)$ is semi-continuous.

- Theorem 12. Every open subgroup (H, $\circ$) of a semi-topological group (G, $\circ$, τ) is a semi topological group and is called semi topological subgroup of G.

- Theorem 13. Let $(G, \circ, \tau)$ be a semi-topological group and $(H, \circ)$ be a subgroup of G. If H contains a non-empty semi-open set, then H is semi-open in G.

- Theorem 14. Let (G, $\circ$, τ) be a semi-topological group and β_e be a base at the identity e of G. Then we have following properties:

a) For every $O \in \beta_e$, there is an element $V \in SO(G,e)$ such that $V^2 \subseteq O$.

b) For every $O \in \beta_e$, there is an element $V \in SO(G,e)$ such that $V^{-1} \subseteq O$.

c) For every $O \in \beta_e$, and for every $x \in O$, there is an element $V \in SO(G, e)$ such that $V x \subseteq O$.

- Definition. Let (G, $\circ$, τ) be a semi-topological group. Then a subset U of G is called symmetric if $U = U^{-1}$.

- Definition [DC]. Topological space (X, τ) is called s-regular if for each closed set F and any point $x \in X - F$, there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $x \in V$.
- Theorem [DC]. Let U be an open subset of an s-regular space X and $x \in U$, then there exists a semi-open set V in X such that $x \in V \subseteq sCl(V) \subseteq U$.

[DC] D. A. Carnahan, Some properties related to compactness in topological spaces, Ph. D Thesis, Univ. Arkansas, 1973

- Theorem 15. If $(G, \circ, \tau)$ is a semi-topological group with base at identity e consisting of symmetric semi-nbd then G satisfies the axiom of s-regularity at e .

- Lemma [DC]. Let (G, $\circ$, τ) be a semi-topological group and V be a semi-nbd of e in G. Then $V \subseteq sCl(V) \subseteq V^2$.

- Lemma 2. If (G, $\circ$, τ) is a semi-topological group, then (G, τ) is semi-T_2 and s-regular.

- Lemma 3. Let (G, $\circ$, τ) be a semi-topological group. Then for any subset A of G and any open nbd U of e ,

  sCl(A) $\subseteq$ AU.

- Theorem . Let (G, $\circ$, τ) be a semi-topological group, and β_e a base of the space (G, τ) at e . Then, for every subset A of G, $sCl(A) = \cap\{AU : U \in \beta_e\}$.

References

1. D. A. Carnahan, Some properties related to compactness in topological spaces, Ph. D Thesis, Univ. Arkansas, 1973.
2. C. Dorsett, Semi Compactness, semi separation axioms, and product spaces, Bull. Malaysian Math. Soc. (2) 4 (1981), 21-28.
3. C. Dorsett, Semicontinuity and semicompactness, J. Inst. Math. Comput. Sci. Math. Ser., 3 (3) (1990), 317-323.
4. M. Ganster, Some Remarks on Strongly Compact Spaces and Semi Compact Spaces, Bull. Malaysian Math. Soc. (10) 2 (1987), 67-81.

5. M. Ganster, D.S. Jankovic, I.L. Reilly, On compactness with respect to semi-open sets, Comment. Math. Univ. Carolinse 31,1(1990), 37-39.
6. K. Kuratowski, Topology I, Warszawa, 1933.
7. N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1) (1963), 36-41.
8. O. Nja˚stad, Remarks on topologies defined by local properties, Avh. Norske Vid.-Akad. Oslo I(N.S), 8 (1966) 1-16.
9. T. Noiri, A Note on Semi Continuous Mappings.
10. R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, 1960.

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