# Eberlein-Grothendieck scattered spaces that are $\sigma$-discrete

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## Definitions

Unless otherwise stated, every topological space in this presentation is assumed to be Tychonoff. The set of real numbers with the natural topology is denoted by $\mathbb{R}$. For a space $X$ the family of all open subsets of $X$ is denoted by $\tau(X)$ and the family of all compact subspaces of $X$ is denoted by $\mathcal{K}(X)$. A space $X$ is scattered if for every non void $Y \subset X$ there is $y \in Y$ such that $\{y\} \in \tau(Y)$. For a space $X$ denote by $X^{(0)}$ the set of the isolated points of $X$. If $X^{(\alpha)}$ is defined for any $\alpha < \gamma$, let $X^{(\gamma)}$ be the set of isolated points of $X \setminus \bigcup_{\alpha<\gamma} X^{(\alpha)}$. The set $X^{(\gamma)}$ is called the $\gamma$-th scattering level of the space $X$.

## Definitions

It is clear that a scattered space is the union of its scattering levels. The height of a scattered space is the first ordinal $\kappa$ for which $X^{(\kappa)} = \varnothing$. A transfinite sequence $\{x_\alpha : \alpha < \lambda\}$ of elements of a space $X$ is right-separated if for every $\mu < \lambda$ there is $U \in \tau(X)$ such that $U \cap \{x_\alpha : \alpha < \lambda\} = \{x_\alpha : \alpha < \mu\}$. A space $X$ is scattered if and only if it can be written as a right separated sequence $X = \{x_\alpha : \alpha < \lambda\}$.

## Definitions

The space of all continuous functions from a space $X$ into a space $Y$, endowed with the topology inherited from the product space $Y^X$, is denoted by $C_p(X, Y)$. On the other hand, $C_u(X)$ is the space of all continuous real-valued functions on a space $X$, with the topology of uniform convergence.

## Definitions

Arhangel'skii defined Eberlein-Grothendieck spaces as those homeomorphic to a subspace of $C_p(K)$ for some compact space $K$. Notice that if $X$ is a subset of a Banach space $E$ with the weak topology then $X$ embeds in $C_p(\mathbf{B}X^*)$ hence $X$ is Eberlein-Grothendieck.

# Answer to Benyamini, Rudin, Simon & Wage

**Theorem (Alster)**

*For any compact space $K$ TFAE:*

- *$K$ is $\sigma$-discrete and Corson;*
- *$K$ is scattered and Corson;*
- *$K$ is strong Eberlein.*

# What if $X$ is not compact? (Arhangel'skii?)

**Particular case (Haydon)**

Assuming $K$ is compact, is $C_p(K, \{0, 1\})$ $\sigma$-discrete whenever it is scattered?

**Problem 1**

*Are Eberlein-Grothendieck scattered spaces $\sigma$-discrete?*

# The case when $w(K) = \omega_1$ and $X \subset C_p(K)$

**Theorem**

*If $X$ is an Eberlein-Grothendieck locally compact scattered space of height lower than $\omega_1 \cdot \omega_1$, then $X$ is $\sigma$-discrete.*

**Theorem**

*If $X$ is an Eberlein-Grothendieck locally countable scattered space of cardinality $\omega_1$, then $X$ is $\sigma$-discrete.*

**Corollary**

*If $X = \{x_\alpha : \alpha < \omega_1\} \subset C_p(K)$ is a right-separated $\omega_1$-sequence, then $X$ is $\sigma$-discrete.*

# Furthermore

## Corollary

*Suppose* $w(K) = \omega_1$, *if* $X \subset C_p(K)$ *is locally countable separable and scattered then* $X$ *is countable.*

## Corollary

*If* $K$ *is a scattered compact space of weight* $\omega_1$ *then* $C_p(K)$ *is SLD if and only if for every* $\varepsilon > 0$ *there exists a family* $\{X_n : n \in \omega\}$ *that covers* $C_p(K, \{0, 1\})$ *and for each* $n \in \omega$ *the set* $X_n$ *has an* $\varepsilon$*-open partitioning of length* $\omega_1$.

# Proven by means of topological games

**Theorem**

*Every Lindelöf Čech-complete scattered space is $\sigma$-compact.*

**Corollary**

*Every Eberlein-Grothendieck Lindelöf Čech-complete scattered space is $\sigma$-discrete.*

# A bad space

We applied the hereditary metalindelöfness of certain scattered spaces to prove they are $\sigma$-discrete. It is not very clear that this property implies $\sigma$-discreteness of scattered Tychonoff spaces. This is not true for general spaces as we can deduce from the following example.

**Example**

*There exists a scattered space of class $T_1$ which is hereditarily meta-Lindelöf but is not $\sigma$-discrete.*

# Open problems, the Lindelöf case

- Is every Eberlein-Grothendick Lindelöf scattered space $\sigma$-discrete?
- Suppose that $L$ is a Lindelöf scattered space and $K$ is a compact subspace of $C_p(L)$. Is $C_p(K)$ hereditarily weakly $\theta$ refinable?

# The Čech complete case

We showed that Problem 1 has a positive partial answer for the case of Lindelöf Čech-complete scattered spaces. What if we remove the hypothesis that the space is Lindelöf?

- Is every Eberlein-Grothendieck Čech-complete scattered space $\sigma$-discrete?

# Meta-Lindelöf spaces

- Is every Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space $\sigma$-discrete?
- Let $X$ be an Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space of height and cardinality equal to $\omega_1$. For every point $x \in X$ there is an open set $U_x$ that isolates $x$ in its scattering level. There is a point countable open refinement $\mathcal{V}$ of the cover $\{U_x : x \in X\}$. Let $V_x$ be the intersection of $U_x$ and the union of all the elements of $\mathcal{V}$ that contain $x$. Define the partially ordered set $\mathbb{P} = \{p \subset X : p \text{ is finite and } V_x \cap p = \{x\} \text{ for every } x \in p\}$ and $q < p$ if $p \subset q$. Has $\mathbb{P}$ got the countable chain condition?

# Right separated sequences

For an Eberlein-Grothendieck right-separated transfinite sequence $X = \{x_\alpha : \alpha < \lambda\}$ we showed that $X$ is hereditarily metalindelöf for $\lambda < \omega_2$. Moreover we proved that hereditarily metalindelöfness implies $\sigma$-discreteness of $X$ for $\lambda < \omega_1 \cdot \omega_1$. It is not yet clear if hereditarily metalindelöfness implies $\sigma$-discreteness of $X$ for $\lambda = \omega_1 \cdot \omega_1$.

- Suppose that $\omega_1 \cdot \omega_1 \leq \lambda < \omega_2$ and $X = \{x_\alpha : \alpha < \lambda\}$ is an Eberlein-Grothendieck right-separated transfinite sequence. Is $X$ $\sigma$-discrete?

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