On theory of $\mathcal{C}$-spaces and $\mathcal{C}$-groups

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Weak Topology

A topological space (X, τ) is determined by a cover $\mathcal{P}$, or $\mathcal{P}$ determines (X, τ), if $U \subseteq X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$.

Examples

(i) $\mathcal{S}$ - the class of all compact countable spaces with at most one limit point (convergent sequences).

X is sequential iff $\mathcal{S} \cap X$ is a determining cover of X.

(ii) $\mathcal{K}$ - the class of all compact spaces.

X is k-space iff $\mathcal{K} \cap X$ is a determining cover of X.

(iii) $\mathcal{CK}$ - the class of all countably compact spaces.

X is quasi-k-space iff $\mathcal{CK} \cap X$ is a determining cover of X.

(iv) $\mathcal{CT}_{\aleph_0}$ - the class of all countable topological spaces.

X has countable tightness iff $\mathcal{CT}_{\aleph_0} \cap X$ is a determining cover of X.

The Main Goal

The main goal is to give a general construction of some natural classes of topological spaces and groups including the above mentioned the most important ones and to obtain their complete characterizations.

Part I. Theory of $\mathcal{C}$-spaces

Definition of $\mathcal{T}$-determining classes

Definition: A subclass $\mathcal{C}$ of the category $\mathbf{Top}$ containing a non-empty space is called topology determining ($\mathcal{T}$-determining) in $\mathbf{Top}$ if it satisfies the following two conditions:

(i) $\mathcal{C}$ is closed under taking of closed subspaces;

(ii) $\mathcal{C}$ is closed under taking of homeomorphic images.

The elements of $\mathcal{C}$ are called $\mathcal{C}$-basic or simply basic spaces.

Example: All the classes $\mathcal{S}$, $\mathcal{K}$, $\mathcal{CK}$ and $\mathcal{CT}_{\aleph_0}$ are $\mathcal{T}$-determining.

In what follows $\mathcal{C}$ denotes a $\mathcal{T}$-determining class of topological spaces.

$\mathcal{C}$-modification of topological spaces

Definition: Let X be a topological space. A subset E of X is called $\mathcal{C}$-open (resp. $\mathcal{C}$-closed) if $E \cap A$ is relatively open (resp. closed) in A for every $A \in \mathcal{C} \cap X$.

The set $\tau_{\mathcal{C}}$ of all $\mathcal{C}$-open sets of a topological space (X, τ) is a topology on X which is finer than the original one, i.e., $\tau \leq \tau_{\mathcal{C}}$. The space $(X, \tau_{\mathcal{C}})$ is called the $\mathcal{C}$-modification of (X, τ) and $\tau_{\mathcal{C}}$ is called the $\mathcal{C}$-extension of τ.

Definition of $\mathcal{C}$-spaces

Definition: A topological space X is called a $\mathcal{C}$-space if every $\mathcal{C}$-open subset is open.

Proposition: The $\mathcal{C}$-modification $(X, \tau_{\mathcal{C}})$ of a topological space (X, τ) is a $\mathcal{C}$-space.

Mappings

Definition: Let $\mathcal{C}$ be a $\mathcal{T}$-determining class and f be a mapping from a topological space (X, τ) to a topological space (Y, ν).

(i) f is called $\mathcal{C}$-continuous if for every $A \in \mathcal{C}(X)$ the restriction $f|_A$ of f onto A is continuous.

(ii) f is called $\mathcal{C}$-covering if for every $B \in \mathcal{C}(Y)$ there is $A \in \mathcal{C}(X)$ such that $f(A) = B$.

Example: If $\mathcal{C} = \mathcal{K}$ (resp. $\mathcal{C} = \mathcal{S}$) we obtain the definitions of k-continuous and compact-covering maps (resp. sequentially continuous and sequence-covering mappings).

$\mathcal{C}$-continuous maps and $\mathcal{C}$-spaces

Theorem: For a topological space (X, τ) the following assertions are equivalent:

(i) (X, τ) is a $\mathcal{C}$-space.

(ii) Every $\mathcal{C}$-continuous mapping from (X, τ) to a topological space is continuous.

Coreflectivity of $\mathbf{C}$

Definition: Let $\mathcal{C}$ be a $\mathcal{T}$-determining class in $\mathbf{Top}$.

(i) $\mathcal{C}$ is called stable if for every $(X, \tau) \in \mathcal{C}$ and every continuous mapping f from (X, τ) to a topological space (Y, ν) the image $f(X)$ (with the induced topology $\nu|_{f(X)}$) is a $\mathcal{C}$-space.

(ii) $\mathcal{C}$ is called closed under taking of continuous images if for every $(X, \tau) \in \mathcal{C}$ and every continuous mapping f from (X, τ) to a topological space (Y, ν) the image $f(X)$ belongs to $\mathcal{C}$.

Example: The $\mathcal{T}$-determining classes $\mathcal{S}$, $\mathcal{K}$, $\mathcal{CK}$ and $\mathcal{CT}_{\aleph_0}$ are closed under taking of continuous images.

Theorem: Let a $\mathcal{T}$-determining class $\mathcal{C}$ be stable. Then the category $\mathbf{C}$ of all $\mathcal{C}$-spaces is a coreflective subcategory of $\mathbf{Top}$ and the assignment $(X,\tau) \mapsto \mathbf{C}(X,\tau)$ is a coreflector for $\mathbf{C}$ from $\mathbf{Top}$ to $\mathbf{C}$.

$\mathcal{LC}$-spaces

In analogy to the local compactness:

Definition: Let $\mathcal{C}$ be a $\mathcal{T}$-determining class in $\mathbf{Top}$. A topological space (X, τ) is called an $\mathcal{LC}$-space if for every point x of X and an open neighborhood U of x there is an open neighborhood V of x such that the closure $\overline{V}$ is a $\mathcal{C}$-basic space and $\overline{V} \subseteq U$.

In particular, if $\mathcal{C} = \mathcal{K}$ then $\mathcal{LC}$-spaces are locally compact ones.

Characterization of $\mathcal{C}$-spaces

Theorem: Let a $\mathcal{T}$-determining class of topological spaces $\mathcal{C}$ be stable. Then a topological space (Y, ν) is a $\mathcal{C}$-space if and only if it is a quotient space of an $\mathcal{LC}$-space of the form

$$X = \bigoplus_{A \in \mathcal{C}_0} A, \text{ where } \mathcal{C}_0 \subseteq \mathcal{C}.$$

Examples

Corollary: Let (X, τ) be a topological space. Then:

(i) (Franklin) X is sequential iff it is a quotient image of a metric space.

(ii) (Cohen) X is a k-space iff it is a quotient image of a locally compact space.

(iii) (Nagata) X is a quasi-k-space iff it is a quotient image of a locally countably compact space.

Part II. Theory of $\mathcal{C}$-groups

Definition of $\mathcal{C}$-groups

Definition: A topological group (G, τ) is called a $\mathcal{C}$-group if every $\mathcal{C}$-continuous homomorphism from (G, τ) to a topological group (H, ν) is continuous.

If $\mathcal{C} = \mathcal{K}$ or $\mathcal{C} = \mathcal{S}$ we obtain the definitions of k-groups and s-groups respectively given by Noble.

Characterization of $\mathcal{C}$-groups

Theorem: For a topological group (G, τ) the following are equivalent:

(i) (G, τ) is a $\mathcal{C}$-group;

(ii) τ is the finest group topology on G which is weaker than the $\mathcal{C}$-extension $\tau_{\mathcal{C}}$ of τ.

$\mathcal{C}$-modification of topological groups

Denote by $\tau_{\mathcal{C}}^h$ the finest group topology on G which is weaker than the $\mathcal{C}$-extension $\tau_{\mathcal{C}}$ of τ. We call the group $(G, \tau_{\mathcal{C}}^h)$ the group $\mathcal{C}$-modification of (G, τ) and the topology $\tau_{\mathcal{C}}^h$ the group $\mathcal{C}$-extension of τ, we write $\mathbf{C}^h(G, \tau) := (G, \tau_{\mathcal{C}}^h)$.

Proposition: The $\mathcal{C}$-modification $(G, \tau_{\mathcal{C}}^h)$ of a topological group (G, τ) is a $\mathcal{C}$-group.

Coreflectivity of $\mathbf{CGr}$

Theorem: Let a $\mathcal{T}$-determining class $\mathcal{C}$ be stable. Then the category $\mathbf{CGr}$ of all $\mathcal{C}$-groups is a coreflective subcategory of $\mathbf{TopGr}$ and the assignment $(G, \tau) \mapsto \mathbf{C}^h(G, \tau)$ is a coreflector for $\mathbf{CGr}$ from $\mathbf{TopGr}$ to $\mathbf{CGr}$.

Characterization of $\mathcal{C}$-groups

Theorem: Let a $\mathcal{T}$-determining class of topological spaces $\mathcal{C}$ be stable. Then for a topological group (G, τ) the following assertions are equivalent:

(i) (G, τ) is a (resp. abelian) $\mathcal{C}$-group;

(ii) (G, τ) is a quotient group of the Graev free (resp. abelian) topological group over a Tychonoff $\mathcal{C}$-space;

(iii) (G, τ) is a quotient group of the Graev free (resp. abelian) topological group over a Tychonoff $\mathcal{LC}$-space of the form

$$\bigoplus_{A \in \mathcal{C}_0} A, \text{ where } \mathcal{C}_0 \subseteq \mathcal{C}.$$

Examples

Corollary: For a topological group (G, τ) TFAE:

(i) (G, τ) is a (resp. abelian) k-group;

(ii) (G, τ) is a quotient group of the Graev free (resp. abelian) topological group over a k-space;

(iii) (G, τ) is a quotient group of the Graev free (resp. abelian) topological group over a locally compact space.

Corollary: For a topological group (G, τ) TFAE:

(i) (G, τ) is an (resp. abelian) s-group;

(ii) (G, τ) is a quotient group of the Graev free (resp. abelian) topological group over a sequential space;

(iii) (G, τ) is a quotient group of the Graev free (resp. abelian) topological group over a metrizable space.

$\mathcal{C}$-order of $\mathcal{C}$-spaces

Definition: Let E be a subset of a topological space (X, τ). We shall write $x \in [E]^c$ if and only if $x \in A \cap \overline{A \cap E} = \mathrm{cl}_A(A \cap E)$ for some $A \in \mathcal{C}(X)$. Set $[E]_0 = E$, $[E]_{\alpha+1} = [[E]_\alpha]^c$ and $[E]_\alpha = \cup_{\beta<\alpha}[E]_\beta$ for a limit ordinal α.

Theorem: A topological space (X, τ) is a $\mathcal{C}$-space if and only if for every subset E of X there exists an ordinal α such that $[E]_\alpha = \overline{E}$.

Definition: The $\mathcal{C}$-order $\mathrm{Co}(X)$ of X is the least ordinal α such that $[E]_\alpha = \overline{E}$ for every subset $E \subseteq X$. A $\mathcal{C}$-space (X, τ) is called $\mathcal{C}$-Fréchet-Urysohn if $\mathrm{Co}(X) \leq 1$.

Theorem: Every hereditary $\mathcal{C}$-space is a $\mathcal{C}$-Fréchet-Urysohn space.