Metrization theory and the Kadec property

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Introduction

A classical definition in Banach spaces theory is the following:

Definition (Kadec)

A normed space $(X, \|\cdot\|)$ is called Kadec if the weak and the norm topologies coincide on $\mathcal{S}_X$, i.e. $Id : (\mathcal{S}_X, w) \to (X, \|\cdot\|)$ is continuous.

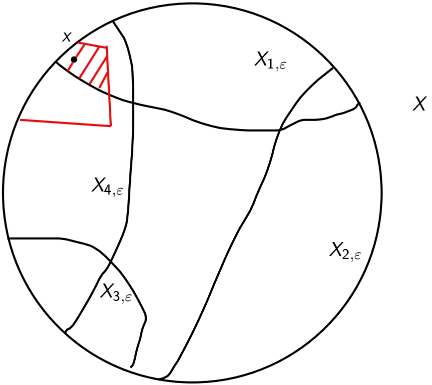
- The natural norms of $\ell_p(\Gamma)$ for every set Γ and $1 < p < +\infty$ are Kadec norms.
- ℓ_∞ does not admits an equivalent Kadec renorming (M. Talagrand, 1977).
- All separable Banach spaces admits an equivalent Kadec renorming (M. I. Kadec, 1958).

Introduction

The studies in this area have brought to two definitions: let $(X, \|\cdot\|)$ a Banach space

Definition (SLD (Jayne, Namioka and Rogers, 1992))

If for every $\varepsilon > 0$ we can write $X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon}$ such that for all $n \in \mathbb{N}$ and every $x \in X_{n,\varepsilon}$ exists a w-open set V such that $\|\cdot\|$-diam$(V \cap X_{n,\varepsilon}) < \varepsilon$.



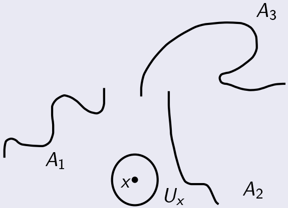
Definition (Descriptive (R. W. Hansell, 1989))

If the norm topology has a network $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$, where every one of the subfamilies $\mathcal{N}_n$ is a w-isolated family.

It is known that Kadec⇒SLD⇔Descriptive, but there are no known examples of a SLD spaces without an equivalent Kadec renorming.

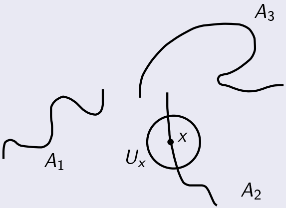
Definition (Discrete family)

For every $x \in X$ exists U_x such that at most one element of $\mathcal{A}$ has non-empty intersection with U_x.



Definition (Isolated family)

For every $x \in \bigcup_{A \in \mathcal{A}} A$ exists U_x such that at most one element of $\mathcal{A}$ has non-empty intersection with U_x.



Introduction

An interresting result of Raja says:

Theorem (M. Raja, 1999)

X is descriptive if and only if exists a function $\varphi : X \to \mathbb{R}$ positively homogeneous, w-lsc,

$$\|x\| \leq \varphi(x) \leq (1+\varepsilon)\|\cdot\|$$

such that the weak and the strong topologies coincide on the "sphere" $\{x \in X | \varphi(x) = 1\}$.

It is possible to construct a norm-continuous function φ?

The main result

In this result we work with topologies generated by particular subset of the dual space (which generate the same norm). As an example we can take $\mathscr{C}(K)$ and $\mathscr{C}_p(K)$,

Theorem (Kadec quasi-renorming)

Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*, the following conditions are equivalent:

1. *X is $\sigma(X, Z)$-descriptive.*
2. *There is an equivalent $\sigma(X, Z)$-lsc and $\sigma(X, Z)$-Kadec quasinorm $q(\cdot)$.*
3. *The norm topology admits a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that every one of the families $\mathcal{B}_n$ is $\sigma(X, Z)$-isolated and norm discrete.*

Remeber that a quasi-norm $q : X \to \mathbb{R}^+$ has the followings properties:

- $x = 0$ if $q(x) = 0$;
- $q(\alpha x) = |\alpha| q(x)$;
- $q(x + y) \leq C(q(x) + q(y))$.

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p-convexity

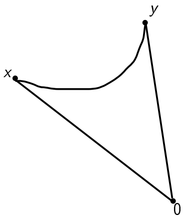
We need two lemmata in order to prove our theorem.

Definition

Let $p \in (0,1)$. A set A of a vector space is called p-convex if for every $x, y \in A$ and $\tau, \mu \in [0,1]$ such that $\tau^p + \mu^p = 1$

$$\tau x + \mu y \in A.$$

A real function $\phi : X \to \mathbb{R}$ is said to be p-convex if its epigraph is p-convex.



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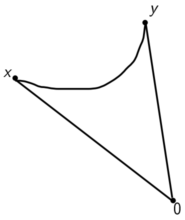


A usefull example

Proposition (p-distance)

Let X be a normed space and Z a norming subspace in the dual space X^*. If C is a w^*-compact and p-convex subset of X^{**}, $0 < p \leq 1$, and we define

$$\varphi(x) := \inf \{ \|x - c^{**}\|_Z \mid c^{**} \in C \}$$

Then φ is p-convex, $\sigma(X, Z)$-lower semicontinuous and 1-Lipschitz map from X to $\mathbb{R}^+$.

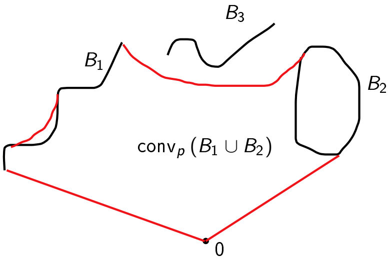
p-isolated family

Definition

Let $(X, \|\cdot\|)$ be a normed space, Z be a norming subspace in X^* and $0 < p \leq 1$. A family $\mathcal{B} := \{B_i \mid i \in I\}$ of subsets in the normed space X is said to be p-isolated for the $\sigma(X, Z)$-topology when

$$B_i \cap \overline{\operatorname{conv}_p \{B_j \mid j \neq i,\ j \in I\}}^{\sigma(X,Z)} = \emptyset$$

for every $i \in I$.



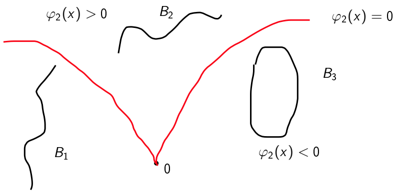
Interplay lemma

Lemma

Let $(X, \|\cdot\|)$ be a normed space and Z be a norming subspace in X^*. Let $\mathcal{B} := \{B_i \mid i \in I\}$ be an uniformly bounded family of subsets of X. The following are equivalent:

1. *The family $\mathcal{B}$ is p-isolated for the $\sigma(X, Z)$-topology;*
2. *There is a family $\mathcal{L} := \{\varphi_i : X \to \mathbb{R}^+ \mid i \in I\}$ of p-convex and $\sigma(X, Z)$-lsc functions such that*

$$\{x \in X \mid \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i \qquad \forall i \in I.$$



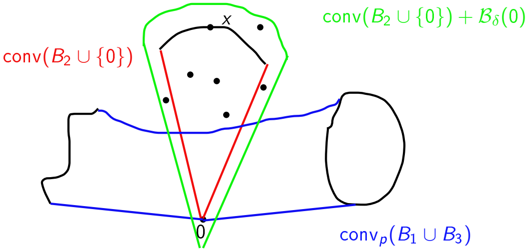
Connection lemma

Lemma

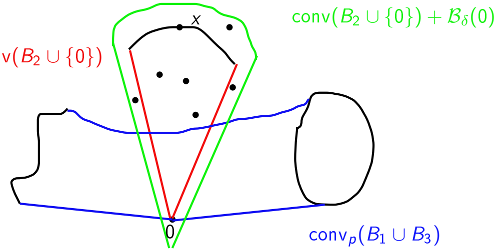
Let $(X, \|\cdot\|)$ *be a normed space and* Z *be a norming subspace in* X^*. *Let* $\mathcal{B} := \{B_i \mid i \in I\}$ *be an uniformly bounded and p-isolated family of subsets of* X *for the* $\sigma(X, Z)$ *topology and some* $p \in (0, 1]$. *Then there is an equivalent quasinorm* $q_{\mathcal{B}}(\cdot)$ *on* X *such that: for all net* $\{x_\alpha \mid \alpha \in A\}$ *and* x *in* X *with* $x \in B_{i_0}$ *for* $i_0 \in I$, *the conditions* $\sigma(X, Z)\text{-}\lim_\alpha x_\alpha = x$ *and* $\lim_\alpha q_{\mathcal{B}}(x_\alpha) = q_{\mathcal{B}}(x)$ *imply that*

1. *exists* α_0 *such that* $x_\alpha \notin \overline{\operatorname{conv}_p \{B_i \mid i \neq i_0,\ i \in I\}}^{\sigma(X,Z)}$ *for* $\alpha \geq \alpha_0$;
2. *for every* $\delta > 0$ *there is* $\alpha_\delta \in A$ *such that*

$$x_\alpha \in \overline{(\operatorname{conv}(B_{i_0} \cup \{0\}) + \mathcal{B}(0, \delta))}^{\sigma(X,Z)} \quad \forall \alpha \succ \alpha_\delta.$$



- Put φ_i the p-distance from $\overline{\text{conv}_p \{B_j | j \neq i,\ j \in I\}}^{\sigma(X^{**},X^*)}$;
- p_i^δ the Minkowski functional of $\overline{\text{conv}\,(B_i \cup \{0\}) + \mathcal{B}_\delta(0)}^{\sigma(X,Z)}$;
- $\psi_i(x) = \|x\|_Z + \sum_{n=1}^{+\infty} \frac{1}{n2^n} p_i^{\frac{1}{n}}(x)$;
- Apply the Deville lemma with the families $\{\varphi_i\}$ and $\{\psi_i\}$;
- The Minkowski functional of $\{x \in X | \theta(x) + \theta(-x) \leq 1\}$ is the quasi-norm that we were seeking.



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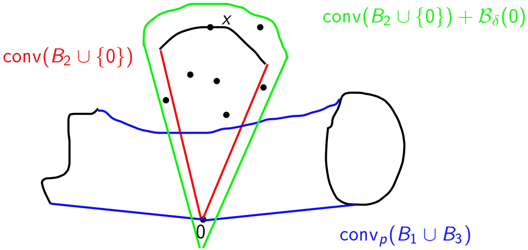

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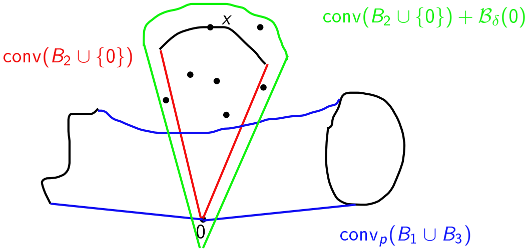

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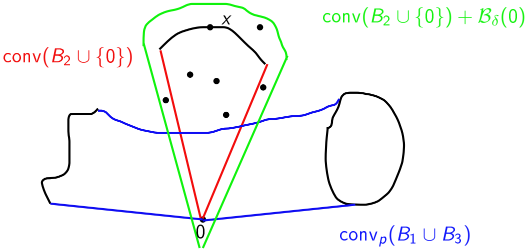

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Observations

It is important to know that modifying the proof, we can fix $\varepsilon \in (0,1)$ such that the quasi-norm $q(\cdot)$ verify

$$q(x+y) \leq \frac{1+\varepsilon}{1-\varepsilon}(q(x)+q(y))$$

and

$$(1-\varepsilon)\|x\|_Z \leq q(x) \leq (1+\varepsilon)\|x\|_Z.$$

Decomposition lemma

Lemma

Let $(X, \|\cdot\|)$ *be a normed space,* Z *be a norming subspace in* X^* *and* $q_n \searrow 0$ *with in* $(q_n) \subseteq (0,1]$*. Let* $\mathcal{B}$ *an isolated family of sets, for the* $\sigma(X, Z)$ *topology. Then there is a decomposition of every* $B \in \mathcal{B}$ *as*

$$B = \bigcup_{n=1}^{+\infty} B_n$$

with $\{B_n \mid B \in \mathcal{B}\}$ *a* $\sigma(X, Z)$*-*q_m*-isolated for* m *big enough and every* $n \in \mathbb{N}$*.*

The proof

Theorem

In a normed space $(X, \|\cdot\|)$ *with a norming subspace* Z *in* X^* *we have an equivalent* $\sigma(X,Z)$*-lower semicontinuous and* $\sigma(X,Z)$*-Kadec quasinorm if and only if there are isolated families for the* $\sigma(X,Z)$ *topology*

$$\{\mathcal{B}_n \mid n = 1, 2, \dots\}$$

in the unit sphere $\mathcal{S}_X$ *such that for every* x *in* $\mathcal{S}_X$ *and every* $\varepsilon > 0$ *there is some positive integer* n *and a set* $B \in \mathcal{B}_n$ *with the property that* $x \in B$ *and that* $\|\cdot\|\text{-diam}(B) < \varepsilon$.

The use of the decomposition lemma

- Fix a sequence $q_n \searrow 0$ in $(0, 1]$;
- Apply the decomposition lemma to the sets of the families $\mathcal{B}_n$;
- We obtain families $\mathcal{B}_n^m$ that are q_{s_m}-isolated;
- Let's renumber the sequence and assume that the families $\mathcal{B}_n$ are already p_n-isolated.

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The use of the connection lemma

- For every family $\mathcal{B}_n$ we apply the connection lemma and obtain an equivalent quasi-norm $Q_{\mathcal{B}_n}(\cdot)$;
- Consider $Q(x) := \sum_{n=1}^{+\infty} c_n Q_{\mathcal{B}_n}(x)$;
- It is possible to chose (c_n) accordingly for the convergence of the series, since we can assume the inequality

$$(1-\delta)\|x\|_Z \leq Q_{\mathcal{B}_n}(x) \leq (1+\delta)\|x\|_Z$$

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Another application of the connection lemma

- Let's prove the Kadec property;
- Take a net $\{x_\alpha \,|\, \alpha \in (A, \succ)\}$ and x with $\|x\|_Z = 1$, $\lim_\alpha Qx_\alpha = Qx$ and (x_α) being $\sigma(X, Z)$-convergent to x;
- By a standard lower semicontinuous argument we have $\lim_\alpha Q_{\mathcal{B}_q}(x_\alpha) = Q_{\mathcal{B}_q}(x)$ for every $q \in \mathbb{N}$;
- Fix $\varepsilon > 0$ and consider $q \in \mathbb{N}$ such that for some $B \in \mathcal{B}_q$ we have $x \in B$ and $\|\cdot\|_Z$-diam$(B) < \frac{\varepsilon}{2}$;
- Use the connection lemma to obtain $\alpha_{\frac{\varepsilon}{2}}$ such that

$$x_\alpha \in \overline{\mathrm{conv}(B \cup \{0\}) + \mathcal{B}(0, \varepsilon/2)}^{\sigma(X,Z)}$$

whenever $\alpha \succ \alpha_{\frac{\varepsilon}{2}}$.

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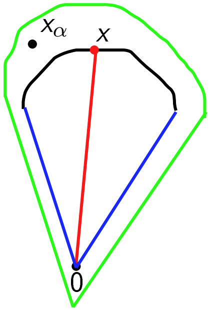
So we are in this condition, where everything has small diameter



The construction of a converging subnet

- So we have that $\|\cdot\|_Z$- $\text{dist}(x_\alpha, I_x) \leq \varepsilon$ for $\alpha \succ \alpha_{\frac{\varepsilon}{2}}$ where I_x is the segment joining x with the origin;
- So there are numbers $r_{\alpha,\varepsilon} \in [0,1]$ such that

$$\left\| x_\alpha - r_{(\alpha,\varepsilon)} x \right\| \leq \varepsilon$$

  for every $\alpha \succ \overline{\alpha}_\varepsilon$. Choose $\gamma_\varepsilon \succ \alpha_{\frac{\varepsilon}{2}}$ and $\gamma_\varepsilon \succ \overline{\alpha}_\varepsilon$.
- Consider the direct set $D = \{(\alpha, \varepsilon) \in A \times (0,1] | \alpha \succ \gamma_\varepsilon\}$.
- By compactness, extract a converging subnet from $\{r_{\alpha,\varepsilon}\}_{(\alpha,\varepsilon)\in D}$ then we have $\sigma : B \to D$ and $r = \lim_{\beta \in B} r_{\sigma(\beta)}$.
- Let $\overline{\sigma}$ the composition of σ with the projection onto A and we have

$$\|\cdot\|_Z\text{-} \lim_{\beta \in B} x_{\overline{\sigma}(\beta)} = rx.$$

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$$\|\cdot\|_Z\text{-}\lim_{\beta\in B} x_{\overline{\sigma}(\beta)} = rx.$$

The construction of a converging subnet

- So we have that $\|\cdot\|_Z$- $\text{dist}(x_\alpha, I_x) \leq \varepsilon$ for $\alpha \succ \alpha_{\frac{\varepsilon}{2}}$ where I_x is the segment joining x with the origin;
- So there are numbers $r_{\alpha,\varepsilon} \in [0,1]$ such that

  $$\left\| x_\alpha - r_{(\alpha,\varepsilon)} x \right\| \leq \varepsilon$$

  for every $\alpha \succ \overline{\alpha}_\varepsilon$. Choose $\gamma_\varepsilon \succ \alpha_{\frac{\varepsilon}{2}}$ and $\gamma_\varepsilon \succ \overline{\alpha}_\varepsilon$.
- Consider the direct set $D = \{(\alpha, \varepsilon) \in A \times (0,1] | \alpha \succ \gamma_\varepsilon\}$.
- By compactness, extract a converging subnet from $\{r_{\alpha,\varepsilon}\}_{(\alpha,\varepsilon)\in D}$ then we have $\sigma : B \to D$ and $r = \lim_{\beta \in B} r_{\sigma(\beta)}$.
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End of the proof

- The hypothesis $\lim_{\alpha \in A} Q(x_\alpha) = Q(x)$ tell us that $Q(rx) = Q(x) \neq 0$ and so $r = 1$.
- The same proof is valid for every subnets of the given net, and this end the proof.

Furthermore we obtain

$$Q(x+y) \leq \frac{1+\delta}{1-\delta}(Q(x)+Q(y)).$$

Last, but not least

Theorem

In a normed space $(X, \|\cdot\|)$ with a norming subspace Z in X^*, if there are isolated families for the $\sigma(X, Z)$ topology

$$\{\mathcal{B}_n \mid n = 1, 2, \ldots\}$$

in the unit sphere $\mathcal{S}_X$ such that for every x in $\mathcal{S}_X$ and every $\varepsilon > 0$ there is some positive integer n and a set $B \in \mathcal{B}_n$ with the property that $x \in B$ and that $\|\cdot\|$-diam$(B) < \varepsilon$, then there exists a LUR an Kadec F-norm.

Remember that an F-norm is a function that satisfies the followings:

$$F(x) = 0 \Leftrightarrow x = 0;$$
$$F(\lambda x) \leq F(x) \text{ if } |\lambda| \leq 1;$$
$$F(x+y) \leq F(x) + F(y);$$
$$F(\lambda x_n) \to 0 \text{ if } F(x_n) \to 0;$$
$$F(\lambda_n x) \to 0 \text{ if } \lambda_n \to 0.$$

This type of function determine the uniform structure of a topological metrizable vector space. The LUR condition is a sort of weak parallelogram law which state that $x_n \to x$ in the strong topology when

$$2F^2(x) + 2F^2(x_n) - F^2(x + x_n) \to 0.$$

Thanks you