TOPOLOGIES INDUCED BY $(3,1,\rho)$-METRICES AND $(3,2,\rho)$-METRICES

- The geometric problems in metric spaces, and their axiomatic classification and generalization have been considered in many papers such as:
- [Ne1] V. Nemytzki: On the “third axiom of metric spaces”, Trans. Amer. Math. Soc. 29, 1927, 507-513
- [Me] K. Menger; Untersuchungen über allge-meine Metrik, Math. Ann. 100, 1928, 75-163 .
- [Ne2] V. Nemytzki: Uber die Axiome des metrichen Raumes, Math. Ann. 104, No 5, 1931, 666-671
- [AN] P.S. Aleksandrov, V.V. Nемыскii: Uslovija metrizuemosti topologiceskih prostranstv i aksioma simetrii, Mat. sb. 3 : 3, 1938, 663-672

- [M] Z.Mamuzić; Uvod u opstu topologiju, Matematicka biblioteka-17, Beograd, 1960;
- [G1] S.Gähler; 2-metrische Räume und ihre topologische Struktur, Math. Nachr. 26, 1963, 115-148
- [A] A.V. Arhangelskii: O povedenii metrizuemosti pri faktornых tobrazenijah, DAN 164, No 2, 1965, 247-250
- [N1] S. Nedev: Ob obobçeno metrizuemыih prostranstvah, Dokl. Bolg. Akad. Nauk, 20, No 6, 1967, 513-516
- [NC] S. Nedev, M. Coban: O metrizuemosti topologiceskih grupp, Vestn. Mosk. Un-ta, ser. Matem., meh, No 6, 1968, 18-20

- [G2] S. Gähler, Untersuchungen über verallgemenerte m-metrische Räume. I and II, Math. Nach. 40, 1969, 165-189 and 229-264.
- [N2] S. Nedev; O-metrizuemie prostranstva, Trud. Mosk. Mat. Ob. Tom 24, 1971, 201-236
- [Dh] Dhage,B.C: Generalized metric spaces and topological structure I, An. stiint. Univ. Al.I. Cuza Iasi. Mat(N.S) 46, 2000, 3-24
- [Mu1] Z. Mustafa, B. Sims: Some Remarks Concerning D-Metric Spaces, Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia (Span), July 2003, 189-198
- [Mu2] Z. Mustafa, B. Sims: A new approach to generalized metric spaces, Jurnal of Nonlinear and Convex Analysis, Vol. 7, Number 2, 2006, 289-297

- I introduced the notion of (n,m,ρ)-metrices in: Mat. Bilten, 16, Skopje, 1992, 73-76. Here I will discuss only the $(3,1,\rho)$-metrices and $(3,2,\rho)$-metrices,i.e.the cases $n=3$, and $m=1$ or $m=2$.
- Let M be a nonempty set and let $\rho\subseteq M^3$. For a map $d:M^3\to\mathbb{R}$ we state the following conditions:

  1) **$d(x,y,z)\geq 0$,**

  2) **$d(x,y,z)=0$ if and only if $(x,y,z)\in\rho$;**

  3) **$d(x,y,z)=d(x,z,y)=d(y,x,z)$,**

  4) **$d(x,y,z)\leq d(x,y,u)+d(x,u,z)+d(u,y,z)$,**

  5) **$d(x,y,z)\leq d(x,u,v)+d(y,u,v)+d(z,u,v)$,**

  6) **$d(x,x,y)=d(x,y,y)$,**

  for $(x,y,z)\in M^3$ and $(u,v)\in M^2$.

A map d satisfying 1), 2), 3) and 4) is called a **$(3,1,\rho)$-metric;**

1) $\mathbf{d(x,y,z)\geq 0}$,

2) **$d(x,y,z)=0$ if and only if (x,y,z)**$\in\rho$;

3) $\mathbf{d(x,y,z)=d(x,z,y)=d(y,x,z)}$,

4) $\mathbf{d(x,y,z)\leq d(x,y,u)+d(x,u,z)+d(u,y,z)}$,

A map d satisfying 1), 2), 3) and 5) is called a **$(3,2,\rho)$-metric**; and

1) $\mathbf{d(x,y,z)\geq 0}$,

2) **$d(x,y,z)=0$ if and only if (x,y,z)**$\in\rho$;

3) $\mathbf{d(x,y,z)=d(x,z,y)=d(y,x,z)}$,

5) $\mathbf{d(x,y,z)\leq d(x,u,v)+d(y,u,v)+d(z,u,v)}$,

A $(3,1,\rho)$-metric satisfying 6) is called a **$(3,1,\rho)$-symmetric**.

1) $d(x,y,z) \geq 0$,
2) $d(x,y,z)=0$ if and only if $(x,y,z) \in \rho$;
3) $d(x,y,z)=d(x,z,y)=d(y,x,z)$,
4) $d(x,y,z) \leq d(x,y,u)+d(x,u,z)+d(u,y,z)$,
6) $d(x,x,y)=d(x,y,y)$,

A $(3,2,\rho)$-metric satisfying 6) is called a **$(3,2,\rho)$-symmetric**.

1) $d(x,y,z) \geq 0$,
2) $d(x,y,z)=0$ if and only if $(x,y,z) \in \rho$;
3) $d(x,y,z)=d(x,z,y)=d(y,x,z)$,
5) $d(x,y,z) \leq d(x,u,v)+d(y,u,v)+d(z,u,v)$,
6) $d(x,x,y)=d(x,y,y)$,

Remark: In the notion of (n,m,ρ)-metrices, the case $n=2$ allows only one possibility for m, $m=1$. Then, for a nonempty set M and $\rho\subseteq M^2$, a map $d:M^2\to\mathbb{R}$ is a $(2,1,\rho)$-metric if:

1) $d(x,y)\geq 0$,

2) $d(x,y)=0$ if and only if $(x,y)\in\rho$,

3) $d(x,y)=d(y,x)$, and

4) $d(x,y)\leq d(x,u)+d(u,y)$,

for $(x,y)\in M^2$ and $u\in M$.

It follows that ρ has to be an equivalence relation on M. If $\rho=\{(x,x) \mid x\in M\}$, then the notion of a $(2,1,\rho)$-metrics t is the notion of a metrics on M.

If d is a $(3,j,\rho)$-metric, then ρ has to be a **(3,j)-eqivalence**, for j=1 or 2, i.e.:

1) $(x,x,x)\in\rho$;

2) if $(x,y,z)\in\rho$, then (x,z,y), $(y,x,z)\in\rho$;

3.1) if $(x,y,u),(x,u,z),(u,y,z)\in\rho$, then $(x,y,z) \in\rho$;

3.2) if $(x,u,v),(y,u,v),(z,u,v)\in\rho$, then $(x,y,z) \in\rho$.

The set $\Delta=\{ (x,x,x) \mid x \in M \}$ is a (3,j)-eqivalence for both j=1 and j=2.

A $(3,j,\Delta)$-metric (symmetric) d is called a **(3,j)-metric**, (**(3,j)-symmetric**), j=1or 2.

Any $(3,j,\rho)$-metric d on M induces a map $D: M^2 \to \mathbb{R}$ defined by $D(x,y)=d(x,x,y)$. The map D satisfies: $D(x,y) \geq 0$; and $D(x,x)=0$, and is called a **distance** in [M] and **pseudo o-metric** in [N2].

If d is a $(3,j)$-metric, D is a o-metric as in [N2].

If d is a $(3,j)$-symmetric, D is a **symmetric** as in [N2].

If d is a $(3,2)$-symmetric, then D satisfies:

$D(x,y) \geq 0$; $D(x,y)=0$ iff $x=y$; $D(x,y)=D(y,x)$; and

$D(x,y) \leq 3/2(D(x,z)+D(z,y))$, and is called **quasimetric, nearmetrics** or **inframetrics**.

Let d be a $(3,j,\rho)$-metric on M, j=1 or 2.

For $\varepsilon>0$, we define three ε-balls, as follows:

$B(x,y,\varepsilon)=\{z \mid d(x,y,z)<\varepsilon\}$; $K(x,\varepsilon)=\{y \mid d(x,y,y)<\varepsilon\}$;
$B(x,\varepsilon)=\{y \mid \text{there is a } z, \text{ such that } d(x,y,z)<\varepsilon\}$.

Next we define several topologies on M:

a) $\tau(G,d)$ generated by the ε-balls $B(x,y,\varepsilon)$;

b) $\tau(H,d)$ generated by the ε-balls $B(x,\varepsilon)$;

c) $\tau(D,d)$ generated by the ε-balls $B(x,x,\varepsilon)$;

d) $\tau(N,d)$: $U\in\tau(N,d)$ iff $\forall x\in U$, $\exists\varepsilon>0$, $B(x,x,\varepsilon)\subseteq U$;

e) $\tau(W,d)$: $U\in\tau(W,d)$ iff $\forall x\in U$, $\exists\varepsilon>0$,$B(x,\varepsilon)\subseteq U$;

f) $\tau(S,d)$ generated by the ε-balls $K(x,\varepsilon)$;

e) $\tau(K,d)$: $U\in\tau(K,d)$ iff $\forall x\in U$, $\exists\varepsilon>0$,$K(x,\varepsilon)\subseteq U$;

For any $(3,j,\rho)$-metric d on M, $j=1$ or 2, these topologies satisfy the following inclusions:

$\tau(W,d)\subseteq\tau(N,d)\subseteq\tau(D,d)\subseteq\tau(G,d)$,

$\tau(W,d) \subseteq\tau(H,d)\subseteq\tau(G,d)$, and

$\tau(W,d) \subseteq\tau(K,d)\subseteq\tau(S,d)$.

For a $(3,2,\rho)$-metric d on M, we have the following additional inclusions:

$\tau(W,d)\subseteq\tau(N,d)=\tau(K,d)\subseteq\tau(S,d)\subseteq\tau(D,d)\subseteq\tau(G,d)$.

In general, these inclusions are strict.

For each $X, Y \in \{W, N, D, G, H\}$ we denote by:

a) **Co(X–Y)** b) **C(X–Y)**

c) **Cos(X–Y)** d) **Cs(X–Y)**

the class of all topological spaces (M,τ), such that there exist a:

a) $(3,1,\rho)$-metric b) $(3,1)$-metric

c) $(3,1,\rho)$-symmetric d) $(3,1)$-symmetric

d, such that $\tau=\tau(X,d)=\tau(Y,d)$.

For X=Y, we write only Co(X), C(X), Cos(X) and Cs(X).

The class Cs(N) is the class of symmetrizable spaces, the class Co(N) is the class of pseudo o-metrizable spaces, and the class C(N) is the class of o-metrizable spaces, as in [N2].

The 2-metrizable spaces investigated in [Me] and [G1] are from the class Co(G), with some additional requirements on the set ρ, i.e. with the requirement that $d(x,x,y)=0$.

In [Mu], it is shown that under some additional conditions on a (3,1)-metric d, the topological space (M, $\tau(N,d)$) is a metrizable space.

The class of metrizable spaces is a subclass of each of the above mentioned classes Cs(X–Y).

The class Co(N–D) is the class of all the topological spaces satisfying the I axiom of countability.

The classes Cos(N–D) and Cos(W–G) coincide. Similarly, Cs(N–D)=Cs(W–G).

Question:

What are the classes: Cs(W–G), C(W–G)???

For a $(3,1,\rho)$-metric d on a set M, the map

$Fab : M \to \mathbb{R}$, defined by $Fab(x)=d(a,b,x)$, in general, is not continuous for $a,b \in M$ and any induced topology on M by d.

But, for any $a \in M$ the multvalued map

$Fa : M \to \mathbb{R}$, defined by $Fa(x)=\{d(a,x,y) \mid y \in M\}$

is l.s.c for the induced topologies $\tau(G,d)$ and $\tau(H,d)$.

Questions:

What conditions on d would imply the existence of a continuous selection for Fa ??

What is the subclass of C(W-G), (Cs(W-G)) satisfying the property that each Fa has a continuous selection ???

Example: Let $M=\mathbb{R}$, and let $d:M^3\to\mathbb{R}$ be defined by:

$d(x,x,x)=0$

$d(x,y,y)=1$ for $x<y$

$d(x,y,y)=y-x$ for $y\leq x$

$d(x,y,z)=\max\{z-y, |z-y-1|\}$ for $x<y<z$.

This d is a (3,1)-metric,

$\tau(W,d)=\mathcal{I}$, $\tau(H,d)=\tau(G,d)=\mathcal{D}$,

$\tau(N,d)=\tau(D,d)=\tau$ is generated by the half open intervals $[x,y)$, $((\mathbb{R},\tau)$ is called Sorgenfrey line),

$\tau(K,d)=\tau(S,d)$ is generated by the half open intervals $(x,y]$.