Around Malykhin's problem on metrizablity of the separable Fréchet groups

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joint work with Dmitri Shakhmatov

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Dedicated to Ljubiša Kočinac in occasion of his 65^{th} birthday

- Finding the right setting
  - the scale of convergence properties
  - Malykhin's problem
  - minimal groups and locally minimal groups
- The main results and their corollaries
- The non-abelian case
- How it works (a look into the kitchen)
- Open problems

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For Tychonov topological space X one has the following implications

$$w(X) \leq \omega \xRightarrow{(1)} \text{metrizable} \xRightarrow{(2)} \text{Fréchet} \xRightarrow{(3)} \text{sequential} \xRightarrow{(4)} t(X) \leq \omega$$

For compact spaces the implication (1) is invertible, while (2) and (3) are not invertible. Invertibility of the implication (4) (known as Moore-Mròwka problem) is independent on the axioms of ZFC ([$\Diamond$] Ostaszewski,Fedorchuk 1976, Balough 1989, PFA).
For compact topological groups all implications are invertible, as $t(K) = w(K)$ holds for every compact group Ismail [1981] ($\{0,1\}^{w(K)} \hookrightarrow K$ and $\{0,1\}^{w(K)} = w(K)$).
Actually, he showed the more precise equality $t(K) = \chi(K)$ for locally compact groups, so in particular countable tightness implies metrizability for locally compact groups.
This motivates the question of whether countable tightness in the presence of a weaker compactness-like property can still imply some of the stronger properties like sequential, Fréchet-Urysohn or metrizable.

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Malykhin's problem

By the classical metrization theorem of Birkhoff and Kakutani every first countable group is metrizable. This justified:

Malykhin's problem [1978]

Are separable Fréchet groups metrizable ?

Here separable can be replaced by countable since a topological group with a dense metrizable subgroup is metrizable.
Malykhin was aware that $\{0,1\}^{\omega_1}$ is a separable non-metrizable Fréchet group under the assumption $\mathfrak{p} > \omega_1$ (so this is consistency problem). The next example shows why "separable" is relevant:

Example

For every uncountable cardinal κ the Σ-product G of κ-many copies of the circle group $\mathbb{T}$ is abelian, ω-bounded (closed separable subsets are compact) and Fréchet (hence $t(G) = \omega$), while G is non-metrizable (as $\chi(G) = w(G) = \kappa$). The group G is connected.

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Minimal topological groups

Definition (Choquet, Doïtchinov, Stephenson 1970/71)

A Hausdorff group topology τ on a group G is minimal if every Hausdorff group topology τ' on G such that $\tau' \subseteq \tau$ satisfies $\tau' = \tau$.

Theorem [Arhangel'skiĭ 1981]

$nw(G) = w(G)$ for every minimal group G.

Since $t(X) \leq nw(X) \leq w(X)$ for every space X, this shows that minimality can be used as the desired compactness like property. Minimal abelian groups of countable pseudocharcater are metrizable. It was proved by Guran, by Pestov and by Shakhmatov that this fails to be true in the non-abelian case. This confirms

- the appropriateness of the minimal groups for this kind of problems;
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Locally minimal topological groups

Theorem [R.M. Stephenson, Jr. 1971]

Every locally compact minimal abelian group is compact.

So the class of the minimal abelian groups does not contain the locally compact abelian groups (where the equality $t(K) = \chi(K)$ holds). This suggests to replace the class of minimal groups by a larger class that contains also all locally compact groups.

Definition (Morris and Pestov 1998 and Banakh 2000)

A Hausdorff topological group (G, τ) is locally minimal if there exists a τ-neighborhood U of e_G such that U fails to be a nbd of e_G in any Hausdorff group topology on G strictly coarser than τ.

Examples of locally minimal groups are the minimal groups, the locally compact groups and the subgroups of Banach-Lie groups (so abelian locally minimal groups need not be even locally precompact, i.e., subgroups of locally compact groups).

Locally minimal topological groups

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So the class of the minimal abelian groups does not contain the locally compact abelian groups (where the equality $t(K) = \chi(K)$ holds). This suggests to replace the class of minimal groups by a larger class that contains also all locally compact groups.

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A Hausdorff topological group (G, τ) is locally minimal if there exists a τ-neighborhood U of e_G such that U fails to be a nbd of e_G in any Hausdorff group topology on G strictly coarser than τ.

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Tightness and character coincide in locally minimal abelian groups

$t(X) \leq \chi(X)$ for every topological space X, here we discuss the coincidence of these invariants for locally minimal abelian groups.

Theorem (D. Shakhmatov, DD)

A locally minimal and locally precompact abelian group G satisfies $t(G) = \chi(G)$.

The equality $t(G) = \chi(G)$ should be compared with

Theorem [L. Ausenhofer, M. J. Chasco, X. Dominguez, DD 2009]

A locally minimal group G satisfies $w(G) = nw(G)$.

Corollary

A locally minimal and locally precompact abelian group of countable tightness is metrizable.

So, sequential locally minimal and locally precompact abelian groups are metrizable and we get a connection to Malykhin's problem:

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For a topological group G let

$$b(G) = \min\{\kappa : (\forall \text{ open } \emptyset \neq U \subseteq G)(\exists A \in [G]^{\leq\kappa}) \text{ with } G = AU\}$$

Then from $w(G) = \chi(G)b(G)$ and the above corollary we get:

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A locally minimal and locally precompact abelian group G satisfies $w(G) = t(G)b(G)$.

Since $t(X) \leq nw(X)$ for every space X, our next corollary strengthens the above mentioned result of Arhangel'skiĭ for abelian groups.

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$t(G) = w(G)$ for every locally minimal precompact abelian group G.

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Since minimal abelian groups are precompact we obtain:

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The next corollary answers a question of Oleg Okunev [2007]:

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Theorem (D. Shakhmatov, DD)

Every radial locally minimal and locally precompact abelian group is metrizable. So radial minimal abelian groups are metrizable.

(a space X is radial if, for any subset A of X and $x \in \overline{A}$ there exists a transfinite sequence (a_α) in A with $x = \lim_\alpha a_\alpha$).

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Minimal groups of countable tightness in the non-abelian case

The hypothesis "abelian" cannot be removed in the above results.

Example (Shakhmatov, DD - 1992)

Let κ be an uncountable cardinal.

(a) Let G be the Σ-product of κ-many copies of a finite simple non-abelian group F. Then G is both ω-bounded and totally minimal (every Hausdorff quotient is minimal), since every closed normal subgroup of the completion F^κ of G has the form F^A for some subset $A \subseteq \kappa$.
On the other hand, G is Fréchet, hence of countable tightness, while G is non-metrizable. The group G is zero-dimensional.

(b) Replacing F by a connected compact simple Lie group L one obtains a similar example of a non-metrizable Fréchet ω-bounded and totally minimal connected group.

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Do resolvable groups with similar properties exist ?

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Example

(c) Replace the finite simple group F by any center-free resolvable finite group (e.g., $F = S_3$), then a similar argument shows that the (resolvable) group G will still be minimal. Hence a resolvable minimal ω-bounded (hence, precompact) group of countable tightness (actually, Fréchet) need not be metrizable.

The group G from (c) is not totally minimal due to the following

Fact.

If G is a totally minimal countably compact group, then $G \cdot \overline{K'} = K$, where K is the completion of G.

This follows from the fact that the image of G in the compact abelian quotient $K/\overline{K'}$ is totally minimal and countably compact, so must coincide with $K/\overline{K'}$ (as the abelian totally minimal and countably compact groups are compact (Shakhmatov, DD 1992). In the above example (c), $G \cdot K' \neq K = F^{\kappa}$, while G is ω-bounded, hence countably compact. Therefore, G is not totally minimal.

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If G is a totally minimal countably compact group, then $G \cdot \overline{K'} = K$, where K is the completion of G.

This follows from the fact that the image of G in the compact abelian quotient $K/\overline{K'}$ is totally minimal and countably compact, so must coincide with $K/\overline{K'}$ (as the abelian totally minimal and countably compact groups are compact (Shakhmatov, DD 1992). In the above example (c), $G \cdot K' \neq K = F^\kappa$, while G is ω-bounded, hence countably compact. Therefore, G is not totally minimal.

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(c) Replace the finite simple group F by any center-free resolvable finite group (e.g., $F = S_3$), then a similar argument shows that the (resolvable) group G will still be minimal. Hence a resolvable minimal ω-bounded (hence, precompact) group of countable tightness (actually, Fréchet) need not be metrizable.

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Example (Megrelishvili, DD 2012)

For every integer $m > 1$ there exists locally countably compact (so, locally precompact) non-metrizable Fréchet minimal groups G of arbitrarily large character with some additional options:

(a) the center $Z(G)$ is either the circle group, or a finite cyclic group of size m;

(b) the quotient $G/Z(G)$ is abelian;

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Recognizing dense locally minimal subgroups

Definition

A subgroup G of a topological group H is called:

(a) *essential* if $G \cap N = 0$ for a closed normal subgroup N of H implies $N = 0$.

(b) [ACDD] *locally essential* if there exists a neighborhood V of 0 in G such that $H \cap N = \{0\}$ implies $N = \{0\}$ for all closed normal subgroups N of G contained in V.

The next criterion for local minimality was inspired by Banaschewski-Prodanov-Stephenon's minimality criterion for dense subgroups, using (in a similar way) essential subgroups:

Theorem [Aussenhofer, Chasco, Dominguez, DD]

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Corollary

A locally precompact group G is locally minimal iff G is a locally essential subgroup of its completion.

Main Theorem [D. Shakhmatov – DD]

If K is a locally compact abelian group, then $\chi(K) \leq t(G)$ for every locally essential subgroup G of K.

The next lemma is relevant for the proof ($\mathbb{Z}_p$ - the p-adic integers).

Lemma

If κ is an infinite cardinal, $p \in \mathbb{P}$ and G is a locally essential subgroup of $\mathbb{Z}_p^\kappa$, then $t(G) \geq \kappa$.

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Corollary

If K is a compact abelian group, then $w(K) \leq t(G)$ for every locally essential subgroup G of K.

Obviously, it is equivalent to ask $\chi(K) = t(G)$ for all locally essential subgroups G of K.

Theorem

If K is a compact abelian group containing a locally essential radial subgroup, then K is metrizable.

Lemma

Let p be a prime and let N be a non-trivial abelian pro-p-group. Then no locally essential subgroup of N^{ω_1} can be radial.

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Does the tightness of an arbitrary locally minimal abelian group coincide with its character?

Question

Is every locally minimal abelian group of countable tightness necessarily metrizable? Is it necessarily Fréchet or sequential ?

Question

(a) Is every totally minimal (precompact) resolvable group of countable tightness metrizable.

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(a space X is pseudoradial if, for any subset A of X, A is closed if along with any transfinite sequence it contains all its limits).

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Malykhin's problem: some news results

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Theorem [Alan Dow and Doyel Barman 2012]

In the Cohen model, every countable Fréchet space has π-weight at most $\aleph_1$.

Here comes one of the main recent results of M. Hrusak and his student U. Ramos Garcia

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It is consistent with continuum arbitrarily large that every countable Fréchet space of weight $< \mathfrak{c}$ has countable π-weight.

Moreover, in the forcing model they build

- there is a countable Fréchet space of weight and π-weight equal to $\mathfrak{c}$ (noticed by Alan Dow);
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