Jointly Continuous Utility Functions defined on submetrizable k_{ω} -spaces.

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preference relations

Definition

A preference relation \leq on a set (of alternatives) X is a preorder, that is a reflexive and transitive binary relation. The preference relation \leq is complete or total if every pair of elements of X is comparable.

In Economics, *preference relations* are often described by means of *utility functions*.

utility functions

Definition

A function $u : X \longrightarrow \mathbb{R}$ is a utility function representing a preference relation \leq if: (i) $\forall x, y \in X$ t. c. $x \leq y \Rightarrow u(x) \leq u(y)$; (ii) $\forall x, y \in X$ t. c. $x \prec y \Rightarrow u(x) < u(y)$.

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utility functions

- X a commodity set
- \leq a customer preference relation

 $x \preceq y, \ x, y \in X$ means that the commodity x is weakly preferred to y

to represent \leq by a utility function $u: X \longrightarrow \mathbb{R}$ means to numerically measure the ranking of a customer preference by associating to each possible consumption bundle a real number that measures its utility: the greater the utility, the more preferred is the bundle, and conversely.

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Continuous utility representation problem

It has been interesting to introduce some structures (topological, linear, algebrical....) on (X, \preceq) and to require that the utility function has properties connected with the introduced structure.

We are interested in *continuous* utility functions

Definition

A preference relation on a topological space X is continuous if for every $x \in X$ the sets $(-\infty, x]$ and $[x, +\infty)$ are closed in X.

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Definition

A preference relation on a topological space X is continuous if for every $x \in X$ the sets $(-\infty, x]$ and $[x, +\infty)$ are closed in X.

Another frequently taken assumption is \leq to be closed (cf. Nachbin (1965) and Levin (1983)).

Definition

A preference relation \leq on a topological space X is said to be closed if its graph $\{(x, y) \in X \times X : x \leq y\}$ is a closed subset of the topological product $X \times X$.

Continuity and *closedness* properties are equivalent in the total case.

In general a closed preorder is always continuous (Nachbin (1965)).

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Peleg (*) was the first who presented sufficient conditions for the existence of a continuous utility function for a partial order on a topological space.

Peleg solved a problem which was posed by Aumann in the context of expected utility.

Aumann observed that a rational decision-maker may express *indecisiveness* (or equivalently *incomparability*) between two alternatives, so that he is not forced to express *indifference*.

(*) B.PELEG, Utility functions for partially ordered topological spaces, Econometrica 38 (1970), 93–96.

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Existence of jointly continuous utility functions

Let X be a topological space and Γ a set of closed preorders on X.

The Problem of the Existence of Jointly Continuous Utility Functions is to find topological conditions on Γ and X in order to exist a continuous function

 $u:\Gamma\times X\to\mathbb{R}$

such that $u(\leq, \cdot)$ is a utility function for every $\leq \in \Gamma$.

Clearly if $\Gamma = \{ \leq \}$ we have the classic continuous representation problem of a continuous preference relation.

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Levin's Theorem

Theorem (Levin, 1983)

Let Γ be metrizable and let X be locally compact and second countable. Moreover, assume that the set

$$G = \{(\preceq, x, y) : x \preceq y\}$$

is closed in $\Gamma \times X \times X$. Then there exists a continuous function $u: \Gamma \times X \rightarrow [0,1]$ such that, for each $\leq \in \Gamma$, $u(\leq, \cdot)$ is a continuous utility function.

(*) V.L. LEVIN, A continuous utility theorem for closed preorders on a σ -compact metrizable space, Soviet Math. Dokl. 28 (1983), 715–718.

A natural topology on the set Γ of preorders of X should satisfy the following condition:

$$x_n \to x, y_n \to y, \preceq_n \to \preceq, x_n \preceq_n y_n \Longrightarrow x \preceq y.$$

If the spaces Γ and X are metrizable, the former condition is equivalent to require the set $G = \{(\preceq, x, y) : x \preceq y\}$ to be closed in $\Gamma \times X \times X$.

Levin's Theorem

Let \mathcal{P} be a space of closed preorders defined on closed subsets $D \subset X$ (preorders with moving domain $D(\preceq)$) and $\Phi = \{(\preceq, x) : \preceq \in \mathcal{P}, x \in D(\preceq)\}$

Theorem (Levin, 1983)

If \mathcal{P} is metrizable and X is locally compact second countable and

$$M = \{ (\preceq, x, y) : \preceq \in \mathcal{P}, x, y \in D(\preceq), x \preceq y \}$$

is closed in $\mathcal{P} \times X \times X$, there exists a continuous function $u : \Phi \to \mathbb{R}$ such that $u(\preceq, \cdot)$ is a utility function for every $\preceq \in \mathcal{P}$.

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Theorem (Back, 1986)

Let X be a locally compact and second countable space. There exists a continuous map $\nu : \mathcal{P} \to \mathcal{U}_{\tau}$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$. Any such map ν is actually a homeomorphism of \mathcal{P}_{Ins} onto $\nu(\mathcal{P}_{Ins})$, where \mathcal{P}_{Ins} is the family of total locally non-satied preorders.

 ${\cal P}$ is the space of total closed preorders defined on closed subsets of X, endowed with the Fell topology

 U_{τ} is the space of all continuous utility functions defined on closed subsets of X with the τ_c topology, a generalized compact-open topology.

K. BACK, Concepts of similarity for utility functions, Journ. of Math. Econ., 15 (1986), 129-142.

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The Fell topology on $CL((X, \tau))$, has as a subbase

$$U^{-} = \{B \in CL((X,\tau)) : B \cap U \neq \emptyset\}, \ U \in \tau \text{ and}$$
$$(K^{c})^{+} = \{B \in CL((X,\tau)) : B \cap K = \emptyset\}, K \text{ compact in } (X,\tau)$$

 au_c -topology on $\mathcal{U}_{ au}$ has as a subbase

 $[G] = \{ (D, u) \in \mathcal{U}_{\tau} : D \cap G \neq \emptyset \}$

 $[K:I] = \{(D, u) \in \mathcal{U}_{\tau}: u(D \cap K) \subset I\}$

where G is an open subset of X, $K \subset X$ is compact and $I \subset \mathbb{R}$ is open (possibly empty).

Definition

A preference relation \leq on a topological space X is said to be locally non satied if for every $x \in X$ and for every neighbourhood U of x there is $y \in U$ such that $x \prec y$.

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Theorem (CCH, 2010-11)

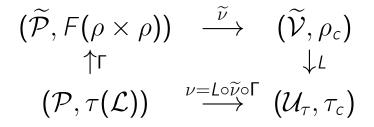
Let (X, τ) be a regular space submetrizable by a boundedly compact metric ρ . There exists a continuous map

$$\nu: (\mathcal{P}, \tau(\mathcal{L})) \to (\mathcal{U}_{\tau}, \tau_{c})$$

such that $\nu(\preceq)$ is a utility function for \preceq , for every $\preceq \in \mathcal{P}$.

A. CATERINO, R. CEPPITELLI, L. HOLÀ, Some generalizations of Back's theorem, to appear.

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The space S' of tempered distributions (Example 3.3)(*) is an example of a submetrizable k_{ω} -space, not submetrizable by a boundedly compact metric.

(*) C. Castaing, P. Raynaud de Fitte and A. Salvadori, Some variational convergence results with applications to evolution inclusions, Adv. Math. Econ. 8 (2006), 33–73.

David Carfi, *S-Linear Algebra in Economics and Physics*, Applied Sciences, (APPS) ISSN 1454-5101 Vol. **9** (2007)

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Definition

A Hausdorff topological space is a submetrizable k_{ω} -space if it is the inductive limit of a nondecreasing sequences of metrizable compact subspaces.

 $X = \bigcup_n K_n$ We will say that $(K_n)_n$ determines the topology of X.

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Theorem

Every submetrizable k_{ω} -space X is a quotient space of a locally compact second countable space.

$$\pi: (\hat{X}, \eta) \to (X, \tau)$$

if
$$X = \bigcup_n K_n$$

then $\hat{X} = \bigoplus_n \{n\} \times K_n$
 $\pi = \nabla_n i|_{K_n}$

Image: A matrix

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We put

$$\mathcal{P} = \{ \preceq : \ \preceq \ \text{is a preorder on } D(\preceq) \subset X \text{ and } \ \preceq \in \mathit{CL}((X, \tau) imes (X, \tau)) \}.$$

For every
$$\leq \in \mathcal{P}$$
 let $\stackrel{\sim}{\leq}$ be the preorder so defined:
- $D(\stackrel{\sim}{\leq}) = \pi^{-1}(D(\leq))$
- for every $a, b \in D(\stackrel{\sim}{\leq}), a \stackrel{\sim}{\leq} b$ if and only if $\pi(a) \leq \pi(b)$.

$$\widetilde{\mathcal{P}} = \{ \widetilde{\preceq} = p^{-1}(\preceq) : \preceq \in \mathcal{P} \} \subset \mathcal{CL}(\hat{X} \times \hat{X})$$

where

$$p = \pi \times \pi : \hat{X} \times \hat{X} \to X \times X.$$

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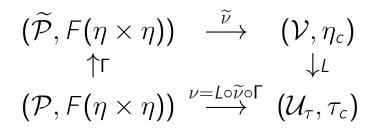
Theorem

Let (X, τ) be a submetrizable k_{ω} -space. There exists a continuous map

$$\nu: (\mathcal{P}, F(\eta \times \eta)) \to (\mathcal{U}_{\tau}, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for \preceq , for every $\preceq \in \mathcal{P}$.

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 Γ is a homeomorphism L is continuous $L(\widetilde{u}) = u : D(\preceq) \rightarrow \mathbb{R}$ with $u(x) = \widetilde{u}(\pi^{-1}(x))$ is a utility function for \preceq .

Theorem

There exists a continuous map

$$\nu_0: (\mathcal{P}, F(\tau \times \tau)) \to (\mathcal{U}_{\tau}, \tau_c)$$

such that $\nu_0(\preceq)$ is a utility function for \preceq , for every $\preceq \in \mathcal{P}$.

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 $(\mathcal{P}, F(\eta \times \eta)) \xrightarrow{\nu} (\mathcal{U}_{\tau}, \tau_c)$ $\downarrow^{i} \nearrow_{\nu_{0}}$ $(\mathcal{P}, F(\tau \times \tau))$

$u_0(\preceq) = \nu(\preceq)$ for every $\preceq \in \mathcal{P}$.

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