Jointly Continuous Utility Functions defined on submetrizable k_ω-spaces.

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preference relations

Definition

A preference relation $\preceq$ *on a set (of alternatives)* X *is a preorder, that is a reflexive and transitive binary relation.*
The preference relation $\preceq$ *is complete or total if every pair of elements of* X *is comparable.*

In Economics, *preference relations* are often described by means of *utility functions*.

utility functions

Definition

A function $u : X \longrightarrow \mathbb{R}$ *is a utility function representing a preference relation* $\preceq$ *if:*

(i) $\forall x, y \in X$ *t. c.* $x \preceq y \Rightarrow u(x) \leq u(y)$;

(ii) $\forall x, y \in X$ *t. c.* $x \prec y \Rightarrow u(x) < u(y)$.

utility functions

X a commodity set

$\preceq$ a customer preference relation

$x \preceq y, \quad x, y \in X$
means that the commodity x is weakly preferred to y

to represent $\preceq$ by a utility function $u : X \longrightarrow \mathbb{R}$ means to numerically measure the ranking of a customer preference by associating to each possible consumption bundle a real number that measures its utility: the greater the utility, the more preferred is the bundle, and conversely.

Continuous utility representation problem

It has been interesting to introduce some structures (topological, linear, algebrical. . . ..) on $(X, \preceq)$ and to require that the utility function has properties connected with the introduced structure.

We are interested in ***continuous*** utility functions

Definition

A preference relation on a topological space X is continuous if for every $x \in X$ the sets $(-\infty, x]$ and $[x, +\infty)$ are closed in X.

Continuous utility representation problem

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Definition

A preference relation on a topological space X is continuous if for every $x \in X$ the sets $(-\infty, x]$ and $[x, +\infty)$ are closed in X.

Another frequently taken assumption is $\preceq$ to be closed (cf. Nachbin (1965) and Levin (1983)).

Definition

A preference relation $\preceq$ on a topological space X is said to be closed if its graph $\{(x, y) \in X \times X : x \preceq y\}$ is a closed subset of the topological product $X \times X$.

Continuity and *closedness* properties are equivalent in the total case.
In general a closed preorder is always continuous (Nachbin (1965)).

Peleg (*) was the first who presented sufficient conditions for the existence of a continuous utility function for a partial order on a topological space.
Peleg solved a problem which was posed by Aumann in the context of expected utility.

Aumann observed that a rational decision-maker may express *indecisiveness* (or equivalently *incomparability*) between two alternatives, so that he is not forced to express *indifference*.

(*) B.PELEG, Utility functions for partially ordered topological spaces, Econometrica 38 (1970), 93–96.

Existence of jointly continuous utility functions

Let X be a topological space and Γ a set of closed preorders on X.

The *Problem of the Existence of Jointly Continuous Utility Functions* is
to find topological conditions on Γ and X in order to exist a continuous function

$$u : \Gamma \times X \to \mathbb{R}$$

such that $u(\preceq, \cdot)$ is a utility function for every $\preceq \in \Gamma$.

Clearly if $\Gamma = \{\preceq\}$ we have the classic continuous representation problem of a continuous preference relation.

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Levin's Theorem

Theorem (Levin, 1983)

Let Γ be metrizable and let X be locally compact and second countable. Moreover, assume that the set

$$G = \{(\preceq, x, y) : x \preceq y\}$$

is closed in $\Gamma \times X \times X$.
Then there exists a continuous function $u : \Gamma \times X \to [0,1]$ such that, for each $\preceq \in \Gamma$, $u(\preceq, \cdot)$ is a continuous utility function.

(*) V.L. LEVIN, A continuous utility theorem for closed preorders on a σ-compact metrizable space, Soviet Math. Dokl. 28 (1983), 715–718.

A natural topology on the set Γ of preorders of X should satisfy the following condition:

$$x_n \to x\,,\; y_n \to y\,,\; \preceq_n \to \preceq\,,\; x_n \preceq_n y_n \Longrightarrow x \preceq y\,.$$

If the spaces Γ and X are metrizable, the former condition is equivalent to require the set $G = \{(\preceq, x, y) : x \preceq y\}$ to be closed in $\Gamma \times X \times X$.

Levin's Theorem

Let $\mathcal{P}$ be a space of closed preorders defined on closed subsets $D \subset X$ (preorders with moving domain $D(\preceq)$) and $\Phi = \{(\preceq, x) : \preceq \in \mathcal{P},\ x \in D(\preceq)\}$

Theorem (Levin, 1983)

If $\mathcal{P}$ is metrizable and X is locally compact second countable and

$$M = \{(\preceq, x, y) : \preceq \in \mathcal{P},\ x, y \in D(\preceq),\ x \preceq y\}$$

is closed in $\mathcal{P} \times X \times X$, there exists a continuous function $u : \Phi \to \mathbb{R}$ such that $u(\preceq, \cdot)$ is a utility function for every $\preceq \in \mathcal{P}$.

Theorem (Back, 1986)

Let X be a locally compact and second countable space. There exists a continuous map $\nu : \mathcal{P} \to \mathcal{U}_\tau$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$. Any such map ν is actually a homeomorphism of $\mathcal{P}_{lns}$ onto $\nu(\mathcal{P}_{lns})$, where $\mathcal{P}_{lns}$ is the family of total locally non-satied preorders.

$\mathcal{P}$ is the space of total closed preorders defined on closed subsets of X, endowed with the Fell topology

$\mathcal{U}_\tau$ is the space of all continuous utility functions defined on closed subsets of X with the τ_c topology, a generalized compact-open topology.

K. BACK, Concepts of similarity for utility functions, Journ. of Math. Econ., 15 (1986), 129-142.

The Fell topology on $CL((X,\tau))$, has as a subbase

$$U^- = \{B \in CL((X,\tau)) : B \cap U \neq \emptyset\}, \;\; U \in \tau \text{ and}$$

$$(K^c)^+ = \{B \in CL((X,\tau)) : B \cap K = \emptyset\}, K \text{ compact in } (X,\tau).$$

τ_C-topology on $\mathcal{U}_\tau$ has as a subbase

$$[G] = \{(D,u) \in \mathcal{U}_\tau : \;\; D \cap G \neq \emptyset\}$$

$$[K : I] = \{(D,u) \in \mathcal{U}_\tau : \;\; u(D \cap K) \subset I\}$$

where G is an open subset of X, $K \subset X$ is compact and $I \subset \mathbb{R}$ is open (possibly empty).

Definition

A preference relation $\preceq$ on a topological space X is said to be locally non satied if for every $x \in X$ and for every neighbourhood U of x there is $y \in U$ such that $x \prec y$.

Theorem (CCH, 2010-11)

Let (X,τ) be a regular space submetrizable by a boundedly compact metric ρ. There exists a continuous map

$$\nu : (\mathcal{P}, \tau(\mathcal{L})) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for $\preceq$, for every $\preceq \in \mathcal{P}$.

A. CATERINO, R. CEPPITELLI, L. HOLÀ, Some generalizations of Back's theorem, to appear.

$$\begin{tikzcd}[column sep=huge]
(\widetilde{\mathcal{P}}, F(\rho \times \rho)) \arrow[r, "\widetilde{\nu}"] & (\widetilde{\mathcal{V}}, \rho_c) \arrow[d, "L"] \\
(\mathcal{P}, \tau(\mathcal{L})) \arrow[u, "\Gamma"'] \arrow[r, "\nu = L \circ \widetilde{\nu} \circ \Gamma"] & (\mathcal{U}_\tau, \tau_c)
\end{tikzcd}$$

The space S' of tempered distributions (Example 3.3)(*) is an example of a submetrizable k_ω-space, not submetrizable by a boundedly compact metric.

(*) C. Castaing, P. Raynaud de Fitte and A. Salvadori, Some variational convergence results with applications to evolution inclusions, Adv. Math. Econ. 8 (2006), 33–73.

David Carfì, *S-Linear Algebra in Economics and Physics*, Applied Sciences, (APPS) ISSN 1454-5101 Vol. **9** (2007)

Definition

A Hausdorff topological space is a submetrizable k_ω-space if it is the inductive limit of a nondecreasing sequences of metrizable compact subspaces.

$X = \bigcup_n K_n$

We will say that $(K_n)_n$ determines the topology of X.

Theorem

Every submetrizable k_ω-space X is a quotient space of a locally compact second countable space.

$$\pi : (\hat{X}, \eta) \to (X, \tau)$$

if $X = \bigcup_n K_n$

then $\hat{X} = \bigoplus_n \{n\} \times K_n$

$\pi = \nabla_n \, i|_{K_n}$

We put

$$\mathcal{P} = \{\preceq : \preceq \text{ is a preorder on } D(\preceq) \subset X \text{ and } \preceq \in CL((X,\tau)\times(X,\tau))\}.$$

For every $\preceq \in \mathcal{P}$ let $\widetilde{\preceq}$ be the preorder so defined:
- $D(\widetilde{\preceq}) = \pi^{-1}(D(\preceq))$
- for every $a, b \in D(\widetilde{\preceq})$, $a \widetilde{\preceq} b$ if and only if $\pi(a) \preceq \pi(b)$.

$$\widetilde{\mathcal{P}} = \{\widetilde{\preceq} = p^{-1}(\preceq) : \preceq \in \mathcal{P}\} \subset CL(\hat{X} \times \hat{X})$$

where

$$p = \pi \times \pi : \hat{X} \times \hat{X} \to X \times X.$$

Theorem

Let (X,τ) be a submetrizable k_ω-space. There exists a continuous map

$$\nu : (\mathcal{P}, F(\eta \times \eta)) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for $\preceq$, for every $\preceq \in \mathcal{P}$.

$$
\begin{array}{ccc}
(\widetilde{\mathcal{P}}, F(\eta \times \eta)) & \xrightarrow{\widetilde{\nu}} & (\mathcal{V}, \eta_c) \\
\uparrow \Gamma & & \downarrow L \\
(\mathcal{P}, F(\eta \times \eta)) & \xrightarrow{\nu = L \circ \widetilde{\nu} \circ \Gamma} & (\mathcal{U}_\tau, \tau_c)
\end{array}
$$

Γ is a homeomorphism
L is continuous
$L(\widetilde{u}) = u : D(\preceq) \to \mathbb{R}$ with $u(x) = \widetilde{u}(\pi^{-1}(x))$
is a utility function for $\preceq$.

Theorem

There exists a continuous map

$$\nu_0 : (\mathcal{P}, F(\tau \times \tau)) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu_0(\preceq)$ *is a utility function for* $\preceq$*, for every* $\preceq \in \mathcal{P}$.

$$
\begin{array}{ccc}
(\mathcal{P}, F(\eta \times \eta)) & \xrightarrow{\nu} & (\mathcal{U}_\tau, \tau_c) \\
\downarrow i & \nearrow \nu_0 & \\
(\mathcal{P}, F(\tau \times \tau)) & &
\end{array}
$$

$\nu_0(\preceq) = \nu(\preceq)$ for every $\preceq \in \mathcal{P}$.