On the cardinality of the θ-closed hull of sets

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Introduction

Let X be a topological space and $A \subseteq X$.
The concepts of the θ-*derivative* $\theta(A)$ (or $cl_\theta(A)$) and the θ-*closed hull* $\overline{A}^{\theta}$ (or $[A]_\theta$) are known in literature ([BeCa, CaKo, Vel]).
In the past years, a major goal concerning the mentioned concepts was to provide upper bounds on the cardinalities of θ-closed hulls of sets, in terms of cardinal functions of the space X. Recently, following this line of research, we define a new topological cardinal function, the θ-*bitightness small number* of a space X, $bts_\theta(X)$, and prove that in every topological space X, $|\overline{A}^{\theta}|$ is at most $|A|^{bts_\theta(X)}$.

Using this cardinal bound, we synthesize all earlier results on bounds on the cardinality of θ-closed hulls.
Moreover, we provide applications to P-spaces and to the almost-Lindelöf number.

Notations and terminologies

Let X be a topological space and $A \subseteq X$.

- A *Hausdorff* (resp. *Urysohn*) space X is a space in which distinct points can be separated by open (resp. closed) neighborhoods.
- $\theta(A) := \{x \in X : \overline{U} \cap A \neq \emptyset$ whenever, $x \in U \in \tau(X)\}$ is the *θ-derivative of A* ([Vel]). $\overline{A}^{\theta} := \bigcap\{C \subseteq X : A \subseteq C$ and $C = \theta(C)\}$ is the *θ-closed hull of A* ([BeCa]).
- $\chi(X, x)$ is the minimal cardinality of a local base at $x \in X$, and the *character* $\chi(X)$ of X is the maximum of $\aleph_0$ and $\sup_{x \in X} \chi(X, x)$ ([Juh]).
- $\chi_\theta(X, x)$ is the minimal cardinality of a family of closed neighborhoods of $x \in X$ such that each closed neighborhood of x contains one from this family, and the *θ-character* $\chi_\theta(X)$ of X is the maximum of $\aleph_0$ and $\sup_{x \in X} \chi_\theta(X, x)$ ([AlKo]).
- [AlKo] For Hausdorff spaces X, $\chi_\theta(X) \leq \chi(X)$.

- [AlKo] It is easy to check that $\chi_\theta(X) = \chi(X_s)$ where X_s denote the semiregularization of the space X (i.e. the space in which the base consits of regular open sets).
- For a Urysohn space X, the θ-*bitighness* $bt_\theta(X)$ is the minimal cardinal κ such that, for each non-θ-closed $A \subseteq X$, there are $x \in \theta(A) \setminus A$ and sets $A_\alpha \in [A]^{\leq\kappa}$, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \theta(A_\alpha) = \{x\}$ ([CaKo]).
- [CaKo] For Urysohn spaces X, $bt_\theta(X) \leq \chi_\theta(X)$.
- In their recent work, Bonanzinga, Cammaroto and Matveev defined the *Urysohn number* $U(X)$ to be the minimal cardinal κ such that, for each set $\{x_\alpha : \alpha < \kappa\} \subseteq X$, there are open neighborhoods U_α of x_α, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \overline{U_\alpha} = \emptyset$ ([BoCaMa]). Thus, X is Urysohn if and only if $U(X) = 2$. A space X is said *finitely-Urysohn* if $U(X)$ is finite.
  This line of research is continued by Bonanzinga and Pansera in [BoPa].

Well-known results

In 1988, Bella and Cammaroto proved the following result:

Theorem 1 ([BeCa])

If X is Urysohn and $A \subseteq X$, then $|\overline{A}^{\theta}| \leq |A|^{\chi(X)}$.

In 2000, Alas and Kočinac improved the previous result showing:

Theorem 2 ([AlKo])

If X is Urysohn and $A \subseteq X$, then $|\overline{A}^{\theta}| \leq |A|^{\chi_\theta(X)}$.

Actually, in 1993, Cammaroto and Kočinac have already improved the first result in this way:

Theorem 3 ([CaKo])

If X is Urysohn and $A \subseteq X$, then $|\overline{A}^{\theta}| \leq |A|^{bt_\theta(X)}$.

In 2010, Bonanzinga, Cammaroto and Matveev extended the *Theorem 1* from Urysohn spaces to finitely-Urysohn spaces:

Theorem 4 ([BoCaMa])

If X is finitely-Urysohn and $A \subseteq X$, then $|\overline{A}^{\theta}| \leq |A|^{\chi(X)}$.

Analogously, in 2011, Bonanzinga and Pansera extended the *Theorem 2* from Urysohn spaces to finitely-Urysohn spaces:

Theorem 5 ([BoPa])

If X is finitely-Urysohn and $A \subseteq X$, then $|\overline{A}^{\theta}| \leq |A|^{\chi_\theta(X)}$.

Now, our goal was to find a similar result using the cardinal function $bt_\theta(X)$ or something like that.
Acually, using a variation of $bt_\theta(X)$, we found a very interesting result that hold in every topological space.

The situation is summarized in the following diagram:

$$
\begin{array}{ccc}
\forall\,\text{fin-Ury } X,\ |\overline{A}^{\theta}| \leq |A|^{\chi(X)} & \longrightarrow & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \leq |A|^{\chi(X)} \\
\uparrow & & \uparrow \\
\forall\,\text{fin-Ury } X,\ |\overline{A}^{\theta}| \leq |A|^{\chi_\theta(X)} & \longrightarrow & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \leq |A|^{\chi_\theta(X)} \\
\uparrow & & \uparrow \\
?????????????????????????? & \longrightarrow & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \leq |A|^{bt_\theta(X)}
\end{array}
$$

Finite θ-bitightness & θ-bitightness small number

Definition 1 (Cam-Cat-Pan-Tsa, 2012)

The *finite θ-bitightness* of a space X, $fbt_\theta(X)$, is the minimal cardinal κ such that, for each non-θ-closed $A \subseteq X$, there are sets $A_\alpha \in [A]^{\leq\kappa}$, $\alpha < \kappa$, such that $\bigcap_{\alpha<\kappa} \theta(A) \setminus A$ is finite and nonempty.
The invariant $fbt_\theta(X)$ is defined for all finitely-Urysohn spaces.
Also, when $bt_\theta(X)$ is defined so is $fbt_\theta(X)$, and $fbt_\theta(X) \leq bt_\theta(X)$.

We have the following result:

Proposition 1 (Cam-Cat-Pan-Tsa, 2012)

For each finitely-Urysohn space X, $fbt_\theta(X) \leq \chi_\theta(X)$.

Definition 2 (Cam-Cat-Pan-Tsa, 2012)

The *θ-bitightness small number* of a space X, $bts_\theta(X)$, is the minimal cardinal κ such that, for each non-θ-closed $A \subseteq X$ that is not a singleton, there are $A_\alpha \in [A]^{\leq\kappa}$, $\alpha < \kappa$, such that

$$\bigcap_{\alpha<\kappa} \theta(A_\alpha) \setminus A \neq \emptyset \text{ and } \left|\bigcap_{\alpha<\kappa} \theta(A_\alpha)\right| \leq |A|^\kappa.$$

The invariant $bts_\theta(X)$ is defined for all spaces.
Obviously, $bts_\theta(X) \leq fbt_\theta(X)$ whenever the latter is defined.

We have the following result:

Proposition 2 (Cam-Cat-Pan-Tsa, 2012)

For Urysohn space X, $bts_\theta(X) \leq fbt_\theta(X) \leq bt_\theta(X) \leq \chi_\theta(X) \leq \chi(X)$.

Now, we obtained the following results:

Proposition 3 (Cam-Cat-Pan-Por, 2012)

If X is Urysohn, then $fbt_\theta(X) \leq bt_\theta(X) \leq 2^{fbt_\theta(X)}$.

Theorem 1 (Cam-Cat-Pan-Por, 2012)

If X is Urysohn and $A \subseteq X$, then $|\overline{A}^\theta| \leq |A|^{fbt_\theta(X)}$.

Theorem 2 (Cam-Cat-Pan-Tsa, 2012)

Let X be a space with $A \subseteq X$, then $|\overline{A}^\theta| \leq |A|^{bts_\theta(X)}$.

Now, the new situation is summarized in the following diagram:

$$\begin{tikzcd}
\forall\,\text{fin-Ury } X,\ |\overline{A}^{\theta}| \le |A|^{\chi(X)} \arrow[r] & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \le |A|^{\chi(X)} \\
\forall\,\text{fin-Ury } X,\ |\overline{A}^{\theta}| \le |A|^{\chi_\theta(X)} \arrow[r] \arrow[u] & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \le |A|^{\chi_\theta(X)} \arrow[u] \\
 & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \le |A|^{bt_\theta(X)} \arrow[u] \\
\forall\,\text{space } X,\ |\overline{A}^{\theta}| \le |A|^{bts_\theta(X)} \arrow[r] \arrow[uu] & \forall\,\text{Ury } X,\ |\overline{A}^{\theta}| \le |A|^{fbt_\theta(X)} \arrow[u]
\end{tikzcd}$$

Some examples

The following spaces are presented to show the existence of spaces where $bt_\theta(X)$ is not defined, $fbt_\theta(X) = \omega$, and $bts_\theta(X) > \chi_\theta(X)$ (*Example 1*) and where $bt_\theta(X)$ and $fbt_\theta(X)$ are defined with $fbt_\theta(X) = bt_\theta(X) = \omega$ (*Example 2*). *Example 3* gives a negative answer to a question present in [BoCaMa] and in [BoPa]. What remains open is the existence of a space X where $bt_\theta(X)$ and $fbt_\theta(X)$ are defined and $fbt_\theta(X) < bt_\theta(X)$.

All these examples are contained in [Cam-Cat-Pan-Por].

Example 1 [A first countable, Hausdorff (not Urysohn) space X for which $bt_\theta(X)$ is not defined, $fbt_\theta(X) = \omega$, and $bts_\theta(X) > \chi_\theta(X)$]

Let $\mathbb{Q} = \{r_n : n \in \omega\}$ denote the space of rational numbers with the usual topology and $\mathbb{D} = \mathbb{Q} + \sqrt{2}$ denote the dense subspace of irrational numbers. Let Λ be nonempty set and $X(\Lambda) = \mathbb{Q} \cup (\mathbb{D} \times \Lambda)$.
A set $U \subseteq X(\Lambda)$ is defined to be open if

- ★ $p \in U \cap \mathbb{Q}$ implies there is $\epsilon > 0$ such that $((p - \epsilon, p + \epsilon)\cap) \cup ((p - \epsilon, p + \epsilon) \cap \mathbb{D}) \times \Lambda) \subseteq U$ and
- ★ $(p, \alpha) \in U \cap (\mathbb{D} \times \{\alpha\})$ for some $\alpha \in \Lambda$ implies there is $\epsilon > 0$ such that $((p - \epsilon, p + \epsilon) \cap \mathbb{D}) \times \{\alpha\} \subseteq U$.

Now, we have that:

- For $|\Lambda| \geq 2$, the space $X(\Lambda)$ is Hausdorff, semiregular, and first countable but not Urysohn.
- If $|\Lambda| < \omega$, $U(X(\Lambda)) = |\Lambda| + 1$. Otherwise, if $|\Lambda| \geq \omega$, $U(X(\Lambda)) = \omega$.

- If $|\Lambda| < \omega$, $bts_\theta(X(\Lambda)) = \omega$. On the other hand, if $|\Lambda| \geq \omega$, $bts_\theta(X(\Lambda)) = \log_2(|\Lambda|)$. In particular, if $|\Lambda| = 2^{\mathfrak{c}}$, then $bts_\theta(X(\Lambda)) = \mathfrak{c} > \omega = \chi_\theta(X(\Lambda))$.
- For $|\Lambda| = 2$ (i.e. $\Lambda = \{0, 1\}$), $U(X) = 3$ and the set $\mathbb{Q}$ is not θ-closed and $\theta(\mathbb{Q}) = X$. In fact, the points $\{(\sqrt{2}, 0), (\sqrt{2}, 1)\}$ can not be separated by disjoint closed neighborhoods. Again, let $B = \{r_n : n \in \omega\}$ a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ and $C \subseteq B$ be an infinite subset. As $\theta(C) = C \cup \{(\sqrt{2}, 0), (\sqrt{2}, 1)\}$, $bt_\theta(X)$ is not defined. Moreover, it is easy to show that $fbt_\theta(X) = \omega$. It is straightforward to show that if Y is the irrational slope topological space (see *Example 75* in [StSe]), then $U(Y) = 3$, $fbt_\theta(Y) = \omega$, and $bt_\theta(Y)$ is not defined.
- For each $n \in \mathbb{N}$, let Λ_n be a set with n elements and $X_n = X(\Lambda_n)$. The topological sum space $Y = \bigsqcup_{n \in \mathbb{N}} X_n$ is Hausdorff but not n-Urysohn for any $n \in \mathbb{N}$ even though $U(Y) = \omega$. However, $fbt_\theta(Y) = \omega$ and $bt_\theta(Y)$ is not defined.

Example 2 [**CH** - An Urysohn space X for which $fbt_\theta(X) = bt_\theta(X) = \omega$]

This example is like *Example 2.3* in [CaKo]. Let $\tau(\mathbb{R})$ be the usual topology on $\mathbb{R}$ and let the underlying set of X be $\mathbb{R}$ with this finer topology:

$$\tau(X) \text{ is generated by } \{U \setminus C : U \in \tau(\mathbb{R}), C \in [\mathbb{R}]^{\leq\omega_1}\}.$$

Now, we have $C \in [\mathbb{R}]^{\leq\omega_1}$ in the above definition whereas, in the example in [CaKo], it is $C \in [\mathbb{R}]^{\leq\omega}$. So, we need that $\mathfrak{c} > \omega_1$ (i.e., $\neg$**CH**). Anyway, let's look at the example where $\kappa < \mathfrak{c}$. That is, X is $\mathbb{R}$ with this finer topology:

$$\tau(X) \text{ is generated by } \{U \setminus C : U \in \tau(\mathbb{R}), C \in [\mathbb{R}]^{\leq\kappa}\}.$$

Now, we have that:

- X is Urysohn.
- $fbt_\theta(X) = bt_\theta(X) = \omega$.

Example 3 [A Hausdorff space X with $U(X)=\chi(X)=\omega$ for which $|\overline{A}^{\theta}|>|A|^{\chi(X)}\cdot U(X)$]

In this example, we provide a Hausdorff space X such that $U(X)=\omega$ and $|\overline{A}^{\theta}|>|A|^{\chi(X)}\cdot U(X)$; however, we know that $|\overline{A}^{\theta}|\leq|A|^{\chi_\theta(X)}$ when X is Hausdorff and finitely-Urysohn. The future research goal is to identify those spaces X for which $U(X)$ is infinite and $|\overline{A}^{\theta}|\leq|A|^{\chi_\theta(X)}$. This research project is simplified by using that $U(X)=U(X_s)$ for any space X and then applying the equality $|\overline{A}^{\theta}_X|=|\overline{A}^{\theta}_{X_s}|$ for $A\subseteq X$. So, to obtain that $|\overline{A}^{\theta}|\leq|A|^{\chi_\theta(X)}$ is reduced to verifying $|\overline{A}^{\theta}|\leq|A|^{\chi_\theta(X)}$ for a semiregular Hausdorff space X for which $U(X)$ is infinite. The question asked in both [BoCaMa, BoPa] is whether $|[A]_\theta|\leq|A|^{\chi_\theta(X)}\cdot U(X)$ is true for all Hausdorff spaces X, i.e., when $U(X)$ is infinite. A negative answer is presented here following *Example 1*.
Let Λ be a set such that $|\Lambda|>\mathfrak{c}$ and $X(\Lambda)$ be defined as in *Example 1*. As noted in *Example 1*, $X(\Lambda)$ is a first countable Hausdorff space with $U(X)=\omega$. As $\theta(\mathbb{Q})=X(\Lambda)$, $|\theta(\mathbb{Q})|=|\Lambda|>\mathfrak{c}$. However, $|\mathbb{Q}|^{\chi(X(\Lambda))}\cdot U(X(\Lambda))=\omega^\omega\cdot\omega=2^\omega=\mathfrak{c}$. Thus, $|\overline{\mathbb{Q}}^{\theta}|>|\mathbb{Q}|^{\chi(X(\Lambda))}\cdot U(X(\Lambda))$.

Applications to P-spaces

Bonanzinga-Cammaroto-Matveev ([BoCaMa]) and Bonanzinga-Pansera ([BoPa]) asked whether, in all Hausdorff spaces X, $|\overline{A}^{\theta}| \leq |A|^{\chi_\theta} \cdot U(X)$. Here, we give an interesting partial answer.

Definition 3

The *θ-P-point number* of a space X is the minimal cardinal κ such that some $x \in X$ has closed neighborhoods V_α, $\alpha < \kappa$, with $\bigcap_{\alpha<\kappa} V_\alpha$ not a neighborhood of x.

As the *θ-P-point number* of any space is at leat $\aleph_0$, the following theorem generalizes the Bonanzinga-Pansera Theorem, and thus also the earlier three theorems discussed in the introduction.

Theorem 3 (Cam-Cat-Pan-Tsa, 2012)

Let X be a space whose Urysohn number is smaller than its θ-P-point number. Then, for each $A \subseteq X$, $|\overline{A}^{\theta}| \leq |A|^{\chi_\theta} \cdot U(X)$.

Definition 4

A space X is a *P-space* if each countable intersection of neighborhoods is a neighborhood. Thus, the θ-P-point number of a P-space is $\geq \aleph_1$.

Corollary 1 (Cam-Cat-Pan-Tsa, 2012)

Let X be a P-space with $U(X) = \aleph_0$.
Then, for each $A \subseteq X$, $|\overline{A}^{\theta}| \leq |A|^{\chi_\theta}$.

Applications to the almost-Lindelöf number

Definition 5

The *almost-Lindelöf number* $aL(A, X)$ of a set $A \subseteq X$ is the minimal cardinal κ such that, for each open cover $\mathcal{U}$ of A, there is $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that $A \subseteq \bigcup_{V \in \mathcal{V}} \overline{V}$.

Now, we have this interesting result:

Theorem 4 (Cam-Cat-Pan-Tsa, 2012)

Let X be a Hausdorff space. For each $A \subseteq X$, $|A| \leq 2^{aL(A,X)\chi_\theta(X) bts_\theta(X)}$.

The following corollary improves upon a result of Bonanzinga, Cammaroto and Matveev [BoCaMa], asserting that for Hausdorff, finitely-Urysohn spaces X, $|X| \leq 2^{aL(X,X)\chi(X)}$.

Corollary 2 (Cam-Cat-Pan-Tsa, 2012)

Let X be a Hausdorff, finitely-Urysohn space. For each $A \subseteq X$, $|A| \leq 2^{aL(A,X)\chi_\theta(X)}$. In particular, $|X| \leq 2^{aL(X,X)\chi_\theta(X)}$.

Open problems

Question 1 (Cam-Cat-Pan-Por-Tsa, 2012)

An unsolved problem is to characterize those Hausdorff spaces X for which $bt_\theta(X)$ and $fbt_\theta(X)$ are defined?

Question 2 (Cam-Cat-Pan-Por-Tsa, 2012)

Does there exist a Hausdorff (or Urysohn) space X for which $bt_\theta(X)$ and $fbt_\theta(X)$ are defined and $fbt_\theta(X) < bt_\theta(X)$?

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Thank You and... *Happy Birthday Prof. Kočinac...!!!* :-) :-) :-)