Uniform continuity of a product of real functions

Joint work with Som Naimpally

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1. Introduction

While our results are all valid for complex valued functions, they will be stated about functions defined on metric spaces with values in the set of real numbers $\mathbb{R}$.

Notation for a metric space

- $\langle X, d\rangle$ will denote a set X equipped with a metric d
- $\mathbb{R}^X$ = the vector space of real-valued functions defined on X
- $B(x, \varepsilon)$ = the open ball with center $x \in X$ and radius $\varepsilon > 0$
- For $A \subseteq X$, $\operatorname{diam}(A) = \sup\{d(a_1, a_2) : a_1 \in A, a_2 \in A\}$.

Both continuity and uniform continuity of real-valued functions on metric spaces can be explained in terms of the classical notion of oscillation.

For each $n \in \mathbb{N}$, $f : X \to \mathbb{R}$, and $x \in X$, put $\omega_n(f, x) := \operatorname{diam} f(B(x, \frac{1}{n}))$, where of course the diameter of the image is taken with respect to the usual metric for $\mathbb{R}$. Then the *oscillation* of f at x is defined by the formula

$$\omega(f, x) := \lim_{n\to\infty} \omega_n(f, x) = \inf_{n\in\mathbb{N}} \omega_n(f, x).$$

While f takes values in $\mathbb{R}$, the associated oscillation function $x \mapsto \omega(f, x)$ takes values in $[0, \infty]$.

Example 1.1. Suppose E and $X\setminus E$ are both dense in X; if $f : X \to \mathbb{R}$ is defined by $f(x) = 1$ if $x \in E$ and $f(x) = 0$ if $x \notin E$ satisfies $\omega(f, x) \equiv 1$.

Example 1.2. Suppose $f : [0, \infty) \to \mathbb{R}$ is defined by $f(x) = 1/x$ for $x \in (0, 1)$ and $f(x) = 0$ otherwise, the associated oscillation function is defined by

$$\omega(f, x) = \begin{cases} \infty & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Given $f \in \mathbb{R}^X$,

- $\omega(f, \cdot)$ is upper semicontinuous;
- f is continuous at $p \in X$ if and only if $\omega(f, p) = 0$;
- f is uniformly continuous on X if and only if $\langle \omega_n(f, \cdot) \rangle$ converges uniformly to the zero function.

While the pointwise product of two continuous real functions remains continuous, uniformly continuity is not preserved under taking products: in $\mathbb{R}^{\mathbb{R}}$, take $f(x) = g(x) = x$.

It is standard exercise to show that the product of two bounded real uniformly continuous functions remains uniformly continuous, but uniform continuity of the product may fail if just one of the factors is bounded.

For counterexamples we will **always** work in discrete metric subspaces of the real line.

Put $\widehat{X} := \{1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{3},\ 3,\ 3\frac{1}{4},\ 4,\ 4\frac{1}{5}, \ldots\}$, equipped with the usual metric from the line.

Example 1.3. Let f be the identity function on $\widehat{X}$, i.e, $f(x) = x$, and let g be defined by $g(x) = \frac{1}{x}$ if $x \in \mathbb{N}$ and $g(x) = 0$ otherwise. Then f and g are in fact both Lipschitz on $\widehat{X}$ and g is bounded, while

$$(fg)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

fails to be uniformly continuous.

2. Necessary and Sufficient Conditions for Uniform Continuity of a Product

We intend to give necessary and sufficient conditions on a pair of uniformly continuous real functions such that their pointwise product is uniformly continuous.

Our conditions prove sufficient without assuming anything whatsoever about the factors. On the other hand, necessity holds for a class of pairs Δ that is much broader than the class of all pairs of functions that are both uniformly continuous. Seemingly, one must put some restriction on the class of pairs of functions under consideration for uniform continuity of their product, in that given any strictly positive function f no matter how pathological it may be, $f \cdot \frac{1}{f}$ will be uniformly continuous.

Our distinguished class of function pairs Δ is characterized by a uniform joint oscillation condition. For $\{f,g\} \subseteq \mathbb{R}^X$ and $\delta > 0$ put

$$\lambda(f,g,\delta) := \sup\{|(f(x)-f(p))(g(x)-g(p))| : \{x,p\} \subseteq X \text{ and } d(x,p) < \delta\}.$$

We now define Δ by

$$\Delta := \{(f,g) \in \mathbb{R}^X \times \mathbb{R}^X : \lim_{n\to\infty} \lambda(f,g,\frac{1}{n}) = \inf_{n\in\mathbb{N}} \lambda(f,g,\frac{1}{n}) = 0\}.$$

Evidently, Δ contains all pairs (f,g) where both functions are uniformly continuous. It also contains all (f,g) where one function is uniformly continuous and the other function is bounded. But how is membership to Δ related to the classical notion of oscillation?

Theorem 2.1. *Let $\langle X,d\rangle$ be a metric space and let $\{f,g\} \subseteq \mathbb{R}^X$. Then $(f,g) \in \Delta$ if and only if $lim_{n\to\infty} sup_{x\in X}\ \omega_n(f,x)\omega_n(g,x) = 0$.*

Remark: For $(f,g) \in \Delta$, at each point of X, at least one of the functions must be continuous!

Remark: Proof of sufficiency is trivial whereas proof of necessity is delicate.

Example 2.2. As an example of (f, g) in Δ where neither function is uniformly continuous nor bounded, let $\widehat{X}$ be the discrete metric space introduced before and let $f : \widehat{X} \to \mathbb{R}$ and $g : \widehat{X} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} n & \text{if } x = n \text{ and } n \text{ is even} \\ 2n & \text{if } x = n + \frac{1}{n+1} \text{ and } n \text{ is even} \\ 0 & \text{otherwise} \end{cases},$$

$$g(x) = \begin{cases} n & \text{if } x = n \text{ and } n \text{ is odd} \\ 2n & \text{if } x = n + \frac{1}{n+1} \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}.$$

Note $\forall x \in X,\ (fg)(x) = 0.$ This example shows that we can have $(f, g) \in \Delta$ while for each $n \in \mathbb{N}$,

$$\sup \{|f(x) - f(p)| : d(x, p) < \frac{1}{n}\} = \sup \{|g(x) - g(p)| : d(x, p) < \frac{1}{n}\} = \infty,$$

because large local variability in one function is corrected by small local variability in the other.

Our necessary and sufficient conditions amount to a continuity notion for a product fg that is properly stronger than uniform continuity of the product, and which for uniformly continuous factors, reduces to uniform continuity. We call a product fg that satisfies this condition *emphatically uniformly continuous*.

Definition 2.3. Let f, g be real-valued functions on a metric space $\langle X, d\rangle$. We call their product fg *emphatically uniformly continuous* provided
$\forall\varepsilon > 0,\ \exists\delta > 0$ such that $\forall x \in X,\ \forall p \in X,\ d(x, p) < \delta$ implies

$$|\frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)| < \varepsilon.$$

Proposition 2.4. *Let $\langle X, d\rangle$ be a metric space and let $\{f, g\} \subseteq \mathbb{R}^X$. Then the product fg is emphatically uniformly continuous if and only if fg is uniformly continuous and $(f, g) \in \Delta$*

Proof. For necessity, let $\varepsilon > 0$ and choose $\delta > 0$ such $\forall x \in X, \forall p \in X$, $d(x, p) < \delta \Rightarrow |\frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)| < \frac{\varepsilon}{2}$. By symmetry, it is clear that also $|f(p)g(p) - \frac{1}{2}(f(x)g(p) + f(p)g(x))| < \frac{\varepsilon}{2}$.

Let $\{x, p\} \subseteq X$ satisfying $d(x, p) < \delta$ be arbitrary, and put

$$\alpha = \frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)$$

and

$$\beta = f(p)g(p) - \frac{1}{2}(f(x)g(p) + f(p)g(x)).$$

Then the inequality $|\alpha + \beta| \leq |\alpha| + |\beta|$ gives

$$|f(p)g(p) - f(x)g(x)| < \varepsilon,$$

while the inequality $|\beta - \alpha| \leq |\beta| + |\alpha|$ gives

$$|(f(x) - f(p))(g(x) - g(p))| < \varepsilon,$$

establishing at once uniform continuity of fg and $(f, g) \in \Delta$. $\square$

The next example presents a pair of continuous bounded real functions whose product is uniformly continuous but not emphatically uniformly continuous.

Example 2.5. Let $\widehat{X}$ be our usual discrete metric space. Define $f : \widehat{X} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ -1 & \text{if } x = n + \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \end{cases},$$

and let $g(x) = -f(x)$ for all x. The product fg being constant is uniformly continuous. We show that emphatic uniform continuity of the product fails. For each $n \in \mathbb{N}$ put $x_n = n$ and $p_n = n + \frac{1}{n+1}$. For each n we have

$$\frac{1}{2}(f(x_n)g(p_n) + f(p_n)g(x_n)) = \frac{1}{2}(1 \cdot 1 + (-1) \cdot (-1)) = 1,$$

while $f(x_n)g(x_n) = -1$.

We can also show that the uniform joint oscillation condition fails. We compute

$$(f(x_n) - f(p_n))(g(p_n) - g(x_n)) = 4$$

while $\lim_{n\to\infty} d(x_n, p_n) = 0$. This shows that $(f, g) \notin \Delta$.

Our main result is an immediate consequence of Proposition 2.4

Theorem 2.6. *Let $\langle X, d\rangle$ be a metric space and let $(f, g) \in \Delta$. Then the product fg is emphatically uniformly continuous if and only if fg is uniformly continuous.*

Corollary 2.7. *Let $\langle X, d\rangle$ be a metric space and let f and g be uniformly continuous real functions on X. Then the product fg is emphatically uniformly continuous if and only if fg is uniformly continuous.*

Corollary 2.8. *Let $\langle X, d\rangle$ be a metric space and let $f : X \to \mathbb{R}$ be uniformly continuous and let $g : X \to \mathbb{R}$ be bounded. Then the product fg is emphatically uniformly continuous if and only if fg is uniformly continuous.*

Corollary 2.9. *Let $\langle X, d\rangle$ be a metric space and let $f \in \mathbb{R}^X$ be uniformly continuous. Then f^2 is uniformly continuous if and only if for each $\varepsilon > 0, \exists\delta > 0$ such that whenever $d(x,p) < \delta$, we have $|f(x)(f(p) - f(x))| < \varepsilon$.*

Proof. This follows from $\frac{1}{2}(2f(x)f(p)) - (f(x))^2 = f(x)(f(p) - f(x))$. $\square$