Algebraically determined topologies on permutation groups

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Permutation groups

For a set X by $S(X)$ we denote the *symmetric group*, i.e., the group all permutations (=bijections) of X.

Symmetric groups are important because of

Theorem (Cayley, 1854)

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The symmetric group $S(X)$ contains the normal subgroup $S_\omega(X)$ consisting of all permutations $f : X \to X$ that have finite support

$$\mathrm{supp}(f) = \{x \in X : f(x) \neq x\}.$$

Fact

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A natural topology on permutation groups

On each permutation group $G \subset S(X) \subset X^X$ we can consider the *topology of pointwise convergence* $\mathcal{T}_p$ inherited from the Tychonoff power X^X of X endowed with the discrete topology.

Fact

The topology $\mathcal{T}_p$ turns G into a Hausdorff topological group. In other words, $\mathcal{T}_p$ is a Hausdorff group topology on G.

A neighborhood base of the topology $\mathcal{T}_p$ at the neutral element 1_G consists of the subgroups

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Theorem (Dierolf-Schwanengel, 1977)

For any group G with $S_\omega(X) \subset G \subset S(X)$ the topology $\mathcal{T}_p$ is a minimal Hausdorff group topology on G.

Theorem (Gaughan, 1967)

For the group $G = S(X)$, the topology $\mathcal{T}_p$ is the smallest Hausdorff group topology on G.

Problem (Dikranjan, 2010)

Let G be a group such that $S_\omega(X) \subset G \subset S(X)$. Is $\mathcal{T}_p$ the smallest Hausdorff group topology on G?

Answer (B-G-P, 2011)

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Various sorts of topologized groups

A group G endowed with a topology $\mathcal{T}$ is called

- a topological group if the binary operation $(x, y) \mapsto xy^{-1}$ is continuous;
- a quasi-topological group if the binary operation $(x, y) \mapsto xy^{-1}$ is separately continuous;
- a semi-topological group if the binary operation $(x, y) \mapsto xy$ is separately continuous;
- a [quasi]-topological group if the binary operations $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ are separately continuous;
- a [semi]-topological group if the binary operations $(x, y) \mapsto xy$ and $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ are separately continuous.

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Interplay between various sorts of topologized groups

$$\begin{tikzcd}
\text{topological} \arrow[r] & \text{[quasi]-topological} \arrow[r] \arrow[d] & \text{[semi]-topological} \arrow[d] \\
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\end{tikzcd}$$

Fact

A group G with topology $\mathcal{T}$ is [semi]-topological if and only if for any $a, b \in G$

- *the shift $s_{a,b} : x \mapsto axb$ and*
- *the conjugator $\gamma_a : x \mapsto xax^{-1}$*

are $\mathcal{T}$-continuous.

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$$\begin{array}{ccccc} \text{topological} & \longrightarrow & \text{[quasi]-topological} & \longrightarrow & \text{[semi]-topological} \\ & & \downarrow & & \downarrow \\ & & \text{quasi-topological} & \longrightarrow & \text{semi-topological} \end{array}$$

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Main result answering the Dikranjan's Problem

Theorem (B-G-P, 2011)

For any group G with $S_\omega(X) \subset G \subset S(X)$, the topology $\mathcal{T}_p$ is the smallest T_1-topology turning G into a [semi]-topological group.

Proof of Theorem

Let $S_\omega(X) \subset G \subset S(X)$ and $\mathcal{T}$ be a T_1-topology on G such that $(G, \mathcal{T})$ is a [semi]-topological group.

Our aim: *To prove that $\mathcal{T}_p \subset \mathcal{T}$.*
This is trivial if X is finite. So, we assume that X is infinite.

Observe that the subgroups

$$G_A = \{g \in G : g|A = \mathrm{id}\}, \quad |A| < \infty$$

form a neighborhood base of the topology $\mathcal{T}_p$ at 1_G,
while the family

$$\{G_A : A \subset X, \ |A| = 3\}$$

is a neighborhood subbase of $\mathcal{T}_p$ at 1_G.
So, to prove the theorem, it suffices to check that
for each 3-element subset $A \subset X$ the subgroup G_A is $\mathcal{T}$-open.

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Proof.

Given any permutation $g \notin G_A$, find a point $a \in A$ with $g(a) \neq a$. Choose any $b \in A \setminus \{a, g(a)\}$ and consider the transposition $t : X \to X$ such that $\mathrm{supp}(t) = \{a, b\}$. Then $t \circ g \neq g \circ t$ as $g \circ t(a) = g(b)$ while $t \circ g(a) = g(a)$.
So,

$$U = \{f \in G : f \circ t \neq t \circ f\} = \{f \in G : f \circ t \circ f^{-1} \neq t\} = \gamma_t^{-1}(G \setminus \{t\})$$

is a $\mathcal{T}$-open neighborhood of g, which is disjoint with G_A. $\square$

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Continuation of the Proof

Lemma

For some 3-element subset $A \subset X$ the subgroup G_A is $\mathcal{T}$-open.

Proof. Assume not. Then for each 3-element subset $A \subset X$ the subgroup G_A is not open and being closed is nowhere dense in $(G, \mathcal{T})$.

Claim

For any 3-element subset $A \subset X$ and any finite set $B \subset X$ the set

$$G(A, B) = \{g \in G : g(A) \subset B\}$$

is closed and nowhere dense in $(G, \mathcal{T})$.

Proof. Since the set of maps $A \to B$ is finite, we can choose a finite subset $F \subset G(A, B)$ such that for each $g \in G(A, B)$ there is $f \in F$ with $f|A = g|A$. Then $f^{-1} \circ g \in G_A$ and hence $g \in f \circ G_A$. So, $G(A, B) = \bigcup_{f \in F} f \circ G_A$ is closed and nowhere dense as a finite union of closed nowhere dense subspaces.

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Choose two disjoint 3-element subsets $A, B \subset X$ and consider the nowhere dense subset $G(A, A \cup B) \cup G(B, A \cup B)$ in $(G, \mathcal{T})$.

For any distinct points $a, b \in A \cup B$ let $t_{a,b} \in S_\omega(X) \subset G$ be the transposition with $\mathrm{supp}(t_{a,b}) = \{a, b\}$.

Put $T = \{t_{a,b} : a, b \in A \cup B\}$.

For every $t \in T$ the set

$$V_t = \{u \in G : u \circ t \neq t \circ u\} = \gamma_t^{-1}(G \setminus \{t\})$$

is $\mathcal{T}$-open and contains each transposition $s \in T$ with $s \circ t \neq t \circ s$.

Then the set

$$U_{s,t} = \gamma_s^{-1}(V_t) = \{u \in G : (usu^{-1})t \neq t(usu^{-1})\}$$

is a $\mathcal{T}$-open neighborhood of 1_G and so is the intersection

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Choose two disjoint 3-element subsets $A, B \subset X$ and consider the nowhere dense subset $G(A, A \cup B) \cup G(B, A \cup B)$ in $(G, \mathcal{T})$.
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$$V_t = \{u \in G : u \circ t \neq t \circ u\} = \gamma_t^{-1}(G \setminus \{t\})$$

is $\mathcal{T}$-open and contains each transposition $s \in T$ with $s \circ t \neq t \circ s$.
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Claim

For each 3-element subset $B \subset X$ the subgroup G_B is $\mathcal{T}$-open.

Proof. Choose any permutation $f \in S_\omega(X) \subset G$ with $f(A) = B$ and observe that $G_B = f \circ G_A \circ f^{-1}$ is $\mathcal{T}$-open, being a two-sided shift of the $\mathcal{T}$-open subgroup G_A.

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What we have just proved

Theorem (B-G-P, 2011)

For any group G with $S_\omega(X) \subset G \subset S(X)$, the topology $\mathcal{T}_p$ is the smallest T_1-topology turning G into a [semi]-topological group.

Remark

The [semi]-topological cannot be replaced by semi-topological as the group $G = S_\omega(\mathbb{Z})$ admits a shift-invariant Hausdorff topology $\mathcal{T}$ which is incomparable with $\mathcal{T}_p$.

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Topologizable groups

Definition

A group G is *topologizable* if G admits a non-discrete Hausdorff group topology.

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Each infinite abelian group G is topologizable as G embeds in $\mathbb{T}^{|G|}$.

Problem (Markov, 1946)

Is each infinite group topologizable?

Answer

There exist:

- an uncountable non-topologizable group (Hesse, 1979);
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- *$\mathfrak{Z}_G \subset \mathfrak{M}_G \subset \mathcal{T}$ for each group T_2-topology $\mathcal{T}$ on G.*
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Is $\mathfrak{Z}_G = \mathfrak{M}_G$ for each symmetric group $G = S(X)$?

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Coincidence of Zariski and Markov topologies

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Proof.

Take Hesse's non-topologizable group H with $\mathfrak{Z}_H \neq \mathfrak{M}_H$ and using Cayley theorem, embed H into the permutation group $G = S(H)$. Then G is a group with $\mathfrak{Z}_G = \mathfrak{M}_G$ containing the subgroup $H \subset G$ with $\mathfrak{Z}_H \neq \mathfrak{M}_H$. □

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Topologizing the quotient group $S(X)/S_\omega(X)$

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Two natural topologies on $S(X)$

Each discrete space X has two natural compactifications:

- αX, the *Aleksandrov* one-point compactifications;
- βX, the *Čech-Stone* compactification.

Fact

Each bijection $f : X \to X$ can be uniquely extended to homeomorphisms $\alpha f : \alpha X \to \alpha X$ and $\beta f : \beta X \to \beta X$.

Consequently, the group $S(X)$ can be identified with the homeomorphisms groups $\mathcal{H}(\alpha X)$ and $\mathcal{H}(\beta X)$ of the compactifications αX and βX.
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$\mathcal{T}_\alpha = \mathcal{T}_{\mathbf{p}}$. *Consequently,* $S_\omega(X)$ *is a dense subgroup of the topological group* $(S_\omega(X), \mathcal{T}_\alpha) = \mathcal{H}(\alpha X)$.

Theorem (B-G-P, 2011)

The subgroup $S_\omega(X)$ *is closed and nowhere dense in the topological group* $(S(X), \mathcal{T}_\beta) = \mathcal{H}(\beta X)$.
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Consequently, the quotient topological group $S(X)/S_\omega(X)$ is not discrete and thus is topologizable.

An Open Problem

Definition

A group G is *quasi-topologizable* if G admits a Hausdorff topology turning G into a quasi-topologizable group.

$$\begin{array}{ccccc} \text{topologizable} & \longrightarrow & \text{[quasi]-topologizable} & \longrightarrow & \text{[semi]-topologizable} \\ & & \downarrow & & \downarrow \\ & & \text{quasi-topologizable} & \longrightarrow & \text{semi-topologizable} \end{array}$$

Theorem (Zelenyuk, 2000)

Each infinite group is quasi-topologizable.

Open Problem

Is each infinite group [quasi]-topologizable? [semi]-topologizable?

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References

T.Banakh, I.Guran, I.Protasov,
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Thanks!

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