

Algebraically determined topologies on permutation groups

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SPM 2012, Caserta

Permutation groups

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Symmetric groups are important because of

Theorem (Cayley, 1854)

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The group $S_\omega(X)$ of finitely supported permutation

The symmetric group $S(X)$ contains the normal subgroup $S_\omega(X)$ consisting of all permutations $f : X \rightarrow X$ that have **finite support**

$$\text{supp}(f) = \{x \in X : f(x) \neq x\}.$$

Fact

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A natural topology on permutation groups

On each permutation group $G \subset S(X) \subset X^X$ we can consider the *topology of pointwise convergence* \mathcal{T}_p inherited from the Tychonoff power X^X of X endowed with the discrete topology.

Fact

*The topology \mathcal{T}_p turns G into a Hausdorff topological group.
In other words, \mathcal{T}_p is a *Hausdorff group topology* on G .*

A neighborhood base of the topology \mathcal{T}_p at the neutral element 1_G consists of the subgroups

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Extremal properties of the topology \mathcal{T}_p

Theorem (Dierolf-Schwanengel, 1977)

*For any group G with $S_\omega(X) \subset G \subset S(X)$
the topology \mathcal{T}_p is a minimal Hausdorff group topology on G .*

Theorem (Gaughan, 1967)

*For the group $G = S(X)$,
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Problem (Dikranjan, 2010)

*Let G be a group such that $S_\omega(X) \subset G \subset S(X)$.
Is \mathcal{T}_p the smallest Hausdorff group topology on G ?*

Answer (B-G-P, 2011)

Yes!

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Various sorts of topologized groups

A group G endowed with a topology \mathcal{T} is called

- a **topological group** if the binary operation $(x, y) \mapsto xy^{-1}$ is continuous;
- a **quasi-topological group** if the binary operation $(x, y) \mapsto xy^{-1}$ is separately continuous;
- a **semi-topological group** if the binary operation $(x, y) \mapsto xy$ is separately continuous;
- a **[quasi]-topological group** if the binary operations $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ are separately continuous;
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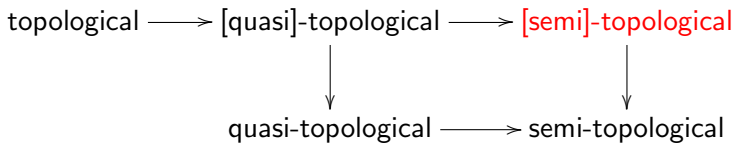
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Interplay between various sorts of topologized groups



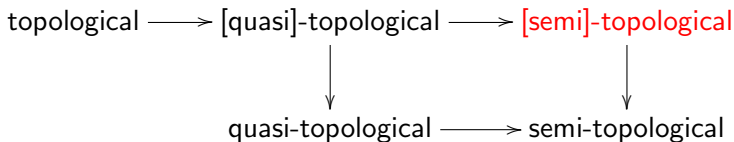
Fact

A group G with topology \mathcal{T} is [semi]-topological if and only if for any $a, b \in G$

- the shift $s_{a,b} : x \mapsto axb$ and
- the **conjugator** $\gamma_a : x \mapsto xax^{-1}$

are \mathcal{T} -continuous.

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Main result answering the Dikranjan's Problem

Theorem (B-G-P, 2011)

For any group G with $S_\omega(X) \subset G \subset S(X)$, the topology \mathcal{T}_p is the smallest T_1 -topology turning G into a [semi]-topological group.

Proof of Theorem

Let $S_\omega(X) \subset G \subset S(X)$ and \mathcal{T} be a T_1 -topology on G such that (G, \mathcal{T}) is a [semi]-topological group.

Our aim: *To prove that $\mathcal{T}_p \subset \mathcal{T}$.*

This is trivial if X is finite. So, we assume that X is infinite.

Observe that the subgroups

$$G_A = \{g \in G : g|_A = \text{id}\}, \quad |A| < \infty$$

form a neighborhood **base** of the topology \mathcal{T}_p at 1_G , while the family

$$\{G_A : A \subset X, |A| = 3\}$$

is a neighborhood **subbase** of \mathcal{T}_p at 1_G .

So, to prove the theorem, it suffices to check that

for each 3-element subset $A \subset X$ the subgroup G_A is \mathcal{T} -open.

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Lemma

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Proof.

Given any permutation $g \notin G_A$, find a point $a \in A$ with $g(a) \neq a$. Choose any $b \in A \setminus \{a, g(a)\}$ and consider the transposition $t : X \rightarrow X$ such that $\text{supp}(t) = \{a, b\}$. Then $t \circ g \neq g \circ t$ as $g \circ t(a) = g(b)$ while $t \circ g(a) = g(a)$.

So,

$$U = \{f \in G : f \circ t \neq t \circ f\} = \{f \in G : f \circ t \circ f^{-1} \neq t\} = \gamma_t^{-1}(G \setminus \{t\})$$

is a \mathcal{T} -open neighborhood of g , which is disjoint with G_A . \square

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For *some* 3-element subset $A \subset X$ the subgroup G_A is \mathcal{T} -open.

Proof. Assume not. Then for each 3-element subset $A \subset X$ the subgroup G_A is not open and being closed is nowhere dense in (G, \mathcal{T}) .

Claim

For any 3-element subset $A \subset X$ and any finite set $B \subset X$ the set

$$G(A, B) = \{g \in G : g(A) \subset B\}$$

is closed and nowhere dense in (G, \mathcal{T}) .

Proof. Since the set of maps $A \rightarrow B$ is finite, we can choose a finite subset $F \subset G(A, B)$ such that for each $g \in G(A, B)$ there is $f \in F$ with $f|_A = g|_A$. Then $f^{-1} \circ g \in G_A$ and hence $g \in f \circ G_A$. So, $G(A, B) = \bigcup_{f \in F} f \circ G_A$ is closed and nowhere dense as a finite union of closed nowhere dense subspaces.

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For any distinct points $a, b \in A \cup B$ let $t_{a,b} \in S_\omega(X) \subset G$ be the transposition with $\text{supp}(t_{a,b}) = \{a, b\}$.

Put $T = \{t_{a,b} : a, b \in A \cup B\}$.

For every $t \in T$ the set

$$V_t = \{u \in G : u \circ t \neq t \circ u\} = \gamma_t^{-1}(G \setminus \{t\})$$

is \mathcal{T} -open and contains each transposition $s \in T$ with $s \circ t \neq t \circ s$.

Then the set

$$U_{s,t} = \gamma_s^{-1}(V_t) = \{u \in G : (usu^{-1})t \neq t(usu^{-1})\}$$

is a \mathcal{T} -open neighborhood of 1_G and so is the intersection

$$U = \bigcap \{U_{s,t} : s, t \in T, t \circ s \neq s \circ t\}.$$

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For every $t \in T$ the set

$$V_t = \{u \in G : u \circ t \neq t \circ u\} = \gamma_t^{-1}(G \setminus \{t\})$$

is \mathcal{T} -open and contains each transposition $s \in T$ with $s \circ t \neq t \circ s$.

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is a \mathcal{T} -open neighborhood of 1_G and so is the intersection

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Continuation of the Proof

Choose two disjoint 3-element subsets $A, B \subset X$ and consider the nowhere dense subset $G(A, A \cup B) \cup G(B, A \cup B)$ in (G, \mathcal{T}) .

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For each 3-element subset $B \subset X$ the subgroup G_B is \mathcal{T} -open.

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What we have just proved

Theorem (B-G-P, 2011)

For any group G with $S_\omega(X) \subset G \subset S(X)$, the topology \mathcal{T}_ρ is the smallest T_1 -topology turning G into a **[semi]-topological** group.

Remark

The **[semi]-topological** cannot be replaced by **semi-topological** as the group $G = S_\omega(\mathbb{Z})$ admits a shift-invariant Hausdorff topology \mathcal{T} which is incomparable with \mathcal{T}_ρ .

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Topologizable groups

Definition

A group G is *topologizable* if G admits a non-discrete Hausdorff group topology.

Remark

Each infinite abelian group G is topologizable as G embeds in $\mathbb{T}^{|G|}$.

Problem (Markov, 1946)

Is each infinite group topologizable?

Answer

There exist:

- an uncountable non-topologizable group (Hesse, 1979);
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Zariski and Markov topologies on groups

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For a group G

- the *Markov topology* \mathfrak{M}_G is the intersection of all Hausdorff groups topologies on G ;
- the *Zariski topology* \mathfrak{Z}_G is generated by the subbase consisting of algebraically open sets

$$\{x \in G : a_1 x^{k_1} a_2 x^{k_2} \dots a_n x^{k_n} \neq 1_G\}$$

where $a_1, \dots, a_n \in G$ and $k_1, \dots, k_n \in \mathbb{Z}$.

Fact

- $\mathfrak{Z}_G \subset \mathfrak{M}_G \subset \mathcal{T}$ for each group T_2 -topology \mathcal{T} on G .
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Coincidence of Zariski and Markov topologies

Theorem

$\mathfrak{Z}_G = \mathfrak{M}_G$ if the group G is:

- countable (Markov, 1946);
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Theorem (Hesse, 1979)

There is an uncountable non-topologizable group G with $\mathfrak{M}_G \neq \mathfrak{Z}_G$ (so, \mathfrak{M}_G is discrete while \mathfrak{Z}_G is not).

Problem (Dikranjan-Shakhmatov, 2007 (OPIT2))

Is $\mathfrak{Z}_G = \mathfrak{M}_G$ for each symmetric group $G = S(X)$?

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Proof.

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Topologizing the quotient group $S(X)/S_\omega(X)$

Since the subgroup $S_\omega(X)$ is normal in $S(X)$, we can consider the quotient group $S(X)/S_\omega(X)$.

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Two natural topologies on $S(X)$

Each discrete space X has two natural compactifications:

- αX , the *Aleksandrov* one-point compactifications;
- βX , the *Čech-Stone* compactification.

Fact

Each bijection $f : X \rightarrow X$ can be uniquely extended to homeomorphisms $\alpha f : \alpha X \rightarrow \alpha X$ and $\beta f : \beta X \rightarrow \beta X$.

Consequently, the group $S(X)$ can be identified with the **homeomorphisms groups** $\mathcal{H}(\alpha X)$ and $\mathcal{H}(\beta X)$ of the compactifications αX and βX .

This identification allows us to introduce the compact-open topologies \mathcal{T}_α and \mathcal{T}_β on $S(X)$.

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Topologies \mathcal{T}_α and \mathcal{T}_β on $S(X)$

Fact

$\mathcal{T}_\alpha = \mathcal{T}_p$. Consequently, $S_\omega(X)$ is a dense subgroup of the topological group $(S_\omega(X), \mathcal{T}_\alpha) = \mathcal{H}(\alpha X)$.

Theorem (B-G-P, 2011)

The subgroup $S_\omega(X)$ is closed and nowhere dense in the topological group $(S(X), \mathcal{T}_\beta) = \mathcal{H}(\beta X)$.

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Each infinite group is quasi-topologizable.

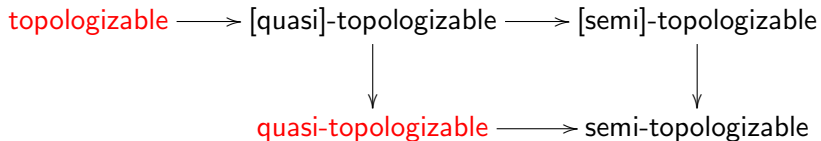
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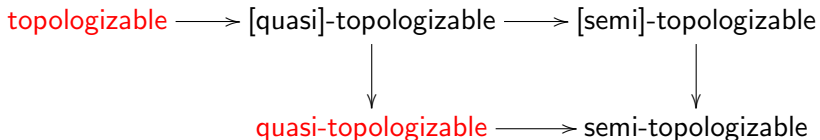
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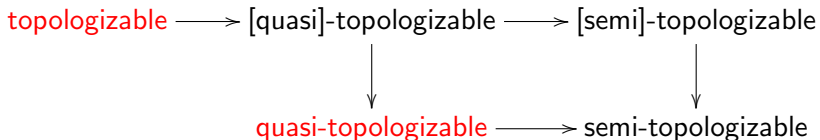
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* * *

Thanks!



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