Various aspects of countable dense homogeneity

Jan van Mill

VU University Amsterdam

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Happy birthday Ljubiša!

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- There are many CDH-spaces: Cantor set, manifolds, Hilbert cube, etc. etc.
- 'Nice' spaces tend to be CDH.
- But there are also 'exotic' CDH-spaces.

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Problem

Is there a locally compact separable metrizable CDH*-space with a dense open subset that is not* CDH*?*

- Let X be a space, and let $\alpha \geq 1$ be a cardinal number not exceeding $\mathfrak{c}$. We say that a space X has *α types of countable dense sets* provided that α is the least cardinal for which there is a collection $\mathcal{A}$ of countable dense sets such that $|\mathcal{A}| \leq \alpha$ while moreover for any given countable dense set B of X there exist $A \in \mathcal{A}$ and a homeomorphism $f\colon X \to X$ such that $f(A) = B$.

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Theorem

If X is Borel and has fewer than $\mathfrak{c}$ types of countable dense sets, then X is Polish.

Corollary (Hrušák and Zamora Avilés)

If X is a CDH *and Borel, then X is Polish.*

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- Let $1 \leq n < \omega$. We call a space ${}^{1}/_{n}$-CDH if it has precisely n types of countable dense sets.
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- The topological sum of countably many copies of $[0, 1)$ has countably many types of countable dense sets.
- The pseudoarc P is an example of a homogeneous continuum that has $\mathfrak{c}$ types of countable dense sets, which is the maximum number possible.
- To see this, let A and B be disjoint composants of P. Moreover, let D and E be countable dense sets of A and B, respectively. There is a collection $\mathcal{F}$ consisting of $\mathfrak{c}$ pairwise nonhomeomorphic subsets of E, none of which is homeomorphic to the space of rational numbers $\mathbb{Q}$.

- Since every homeomorphism of P permutes its composants, it is easy to see that for distinct $F, F' \in \mathcal{F}$ we have that $D \cup F$ and $D \cup F'$ are of different type.

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- This does not seem to be a simple problem, since, as we will see, this is strongly related to the well-known *Topological Vaught Conjecture* which says that if G is any Polish group then any Polish G-space either has countably many orbits or has perfectly many orbits.

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- This does not seem to be a simple problem, since, as we will see, this is strongly related to the well-known *Topological Vaught Conjecture* which says that if G is any Polish group then any Polish G-space either has countably many orbits or has perfectly many orbits.
- The *Vaught conjecture* is a conjecture in model theory originally proposed by Robert Lawson Vaught in 1961. It states that the number of countable models of a first-order complete theory in a countable language is finite or ω or 2^ω.

- The topological Vaught conjecture is more general than the original Vaught conjecture: Given a countable language we can form the space of all structures on the natural numbers for that language. If we equip this with the topology generated by first order formulas, then it is known that the resulting space is Polish. There is a continuous action of the infinite symmetric Polish group S_∞ which gives rise to the equivalence relation of isomorphism. Given a complete first order theory T, the set of structures satisfying T is a minimal, closed invariant set, and hence Polish in its own right.

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- Let G be a topological group acting (continuously) on a space X, and let $\alpha \leq \mathfrak{c}$. We say that X has *α G-types of countable dense sets* if X contains a family $\mathcal{A}$ of countable dense sets such that for any countable dense set B of X there are $g \in G$ and $A \in \mathcal{A}$ such that $gB = A$. In other words, we restrict the homeomorphisms that we are interested in to those that we get from the action from G on X.

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Theorem

Let G be a closed subgroup of S_∞, and let α be the number of orbits for the canonical action $G \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Then there is an action of a Polish group H on $X = \mathbb{N} \times [0,1)$ such that X has α H-types of countable dense sets.

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Theorem

Let G be a Polish group for which there is a Polish G-space X with α G-types of countable dense sets. Then there is an action of a Polish group H on a Polish space Y having exactly α orbits.

- For a space X we let $\mathcal{H}(X)$ denote its group of homeomorphisms. It is well-known, and easy to prove, that for a locally compact space we can endow $\mathcal{H}(X)$ with a Polish group topology such that the natural action

$$\mathcal{H}(X) \times X \to X, \quad (g, x) \mapsto g(x) \quad (g \in \mathcal{H}(X), x \in X),$$

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Corollary

If there is a locally compact space X with α types of countable dense sets, then there is are Polish group G and a Polish G-space Y with α orbits.

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- This suggests the following problem, i.e., the problem whether 'the Topological Vaught Conjecture' is true for locally compact spaces.

Question

Let X be a locally compact space. Does X either has at most ω or exactly $\mathfrak{c}$ types of countable dense sets?

- In this section we are interested in the structure of spaces having fewer than $\mathfrak{c}$ countable dense sets. Our results are valid for locally compact spaces, but since our proofs only require 'good' actions by Polish groups G, we formulate our results in the language of G-spaces.

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- Let X be a G-space, for some topological group G. By the G-*type* of the countable dense set $D \subseteq X$ we mean the collection $\{gD : g \in G\}$. We are interested in G-spaces having α G-types of countable dense sets, where α is some cardinal number less than $\mathfrak{c}$.

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- Let X be a G-space, for some topological group G. By the G-*type* of the countable dense set $D \subseteq X$ we mean the collection $\{gD : g \in G\}$. We are interested in G-spaces having α G-types of countable dense sets, where α is some cardinal number less than $\mathfrak{c}$.
- Different groups may yield different cardinal numbers of course. If X is any dense in itself space, then the trivial group $G = \{e\}$ has $\mathfrak{c}$ G-types of countable dense sets. Moreover, if X is locally compact, then the number of $\mathcal{H}(X)$-types of countable dense sets is equal to the number of types of countable dense sets.

- Here is our first structure theorem.

Theorem

Let G be a Polish group, and let X be a G-space that is dense in itself and Baire. Assume that X has fewer than $\mathfrak{c}$ G-types of countable dense sets. Then

$$S = \bigcup \{Gx : (x \in X) \,\&\, (Gx \text{ is of the first category in } X)\}$$

is closed in X and scattered (hence countable). Moreover, S has finite Cantor-Bendixson rank, X is Polish, S is invariant under the action of G, and G makes $X \setminus S$ homogeneous provided it is connected.

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- For locally compact spaces, the result can be stated without referring to group actions.

Corollary

Let X be a locally compact dense in itself space. Assume that X has fewer than $\mathfrak{c}$ types of countable dense sets. Then there is a closed and scattered subset S of X of finite Cantor-Bendixson rank which is invariant under all homeomorphisms of X while moreover $X \setminus S$ is homogeneous provided it is connected.

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- The question is natural whether 'homogeneous' can be improved to 'CDH-ness'.
- The Structure Theorem can indeed be improved if we additionally assume that the number of types of countable dense subsets of X is countable. The case where the number of types of countable dense sets is uncountable, but fewer than $\mathfrak{c}$, remains an open problem.

Theorem

Let G be a Polish group, and let X be a G-space that is dense in itself and Baire. Assume that X has at most countably many G-types of countable dense sets. Then

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is closed in X and scattered (hence countable). Moreover, S has finite Cantor-Bendixson rank, X is Polish, S is invariant under the action of G, and G makes $X \setminus S$ CDH. *Moreover, $|S| \leq n-1$ if X has at most n G-types of countable dense sets.*

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- So for locally compact spaces we get:

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- Hence the locally compact spaces that are ${}^{1}/_{n+1}$-CDH, look exactly like the topological sum of n copies of $[0, 1)$.

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Corollary

Let X be a homogeneous locally compact space. If X is not CDH*, then X has uncountably many types of countable dense sets.*

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- From now on, all topological spaces under discussion are Tychonoff.

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Theorem (Arhangel'skii and vM)

If X is CDH, *then $|X| \leq \mathfrak{c}$.*

Proof.

Fix a countable dense subspace M of X, and put $Y = X \setminus M$. For each $y \in Y$, put $Z_y = M \cup \{y\}$. The subspaces M and Z_y of X are homeomorphic, by our assumption. Therefore, for each $y \in Y$, we can fix a homeomorphism f_y of M onto Z_y.

Let x_y be the preimage of y under f_y. Put $M_y = M \setminus \{x_y\}$, and denote by g_y the restriction of f_y to M_y. Observe that x_y is not isolated in M. Hence, M_y is dense in M.

Assume now that the cardinality of Y exceeds $\mathfrak{c}$. Then there are two distinct points p and q in Y such that $x_p = x_q$, $g_p = g_q$, and $M_p = M_q$ (since the cardinality of the set of all possible mappings of arbitrary subsets of M to M doesn't exceed $\mathfrak{c}$).

However, this is impossible, since it says that the same continuous mapping $h = g_p = g_q$ of the dense subspace $H = M_p = M_q$ of the space M to the Hausdorff space X can be extended to a continuous mapping of the space M into X in two different ways (f_p and f_q are the two distinct extensions). □

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Since X is a regular T_1-space, it follows that for every dense subspace S of X and $p \in S$ we have $\chi(p, S) = \chi(p, X)$. Hence we may assume that E consists of all points of first countability of X. We will show that $E = X$.

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Claim

E is open in X.

Fix a countable subset D of X dense in X, and put $C = D \setminus E$, $S = D \cap E$, $D' = (S \setminus \overline{C}) \cup C$. It is straightforward to check that D' is a countable subset of D dense in X.

Fix an arbitrary $t \in E$, and put $D'' = D' \cup \{t\}$. Since X is a CDH-space, there exists a homeomorphism h of D'' onto D'. Since $t \in E$, the space D' is first countable at $h(t)$. Hence $h(t) \in E$, since D' is dense in X. Therefore, $h(t) \notin \overline{C}$. Since, obviously, $h(C) \subseteq C$, it follows that t is not in the closure of C. Hence E is open in X.
It follows from the Claim, that the set $D \cap E$ is dense in X. Thus, S is a countable first countable dense subspace of X. Now take any $t \in X$. Then $S \cup \{t\}$ is homeomorphic to S. It easily follows that X is first countable at t. Hence, $X = E$.

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Corollary ($2^\omega < 2^{\omega_1}$)

Every CDH-*compactum is first-countable.*

Proof.

Let X be a CDH-compactum. Then $|X| \leq \mathfrak{c}$ by what we proved. It consequently follows from $\mathfrak{c} < 2^{\omega_1}$ that X is first-countable at a dense set of points. And so we are done by the previous result. □

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- This is not a result in ZFC, since by the result of Steprāns and Zhou, the Cantor cube $\{0,1\}^{\omega_1}$ can be CDH.
- Hence for an example of a CDH-compactum that is not metrizable in ZFC, one should look for a first-countable space.
- The first obvious candidate is the Alexandroff double arrow space. But:

Theorem

The Alexandroff double arrow space is not CDH.

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Is the countable infinite product of copies of the Alexandroff double arrow space CDH*?*

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- In fact, X is both hereditarily separable and hereditarily Lindelöf.

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- Other (consistent) examples can be found by considering certain filters of subsets of ω.
- Let $\mathcal{X}$ be a collection of subsets of ω. By identifying every element $X \in \mathcal{X}$ with its characteristic function, we can study $\mathcal{X}$ as subspace of 2^ω.

Theorem (Medini and Milovich)

Assume MA*(countable). Then there exists a non-principal ultrafilter* $\mathcal{U} \subseteq \mathcal{P}(\omega)$ *with any of the following properties: (a)* $\mathcal{U}$ *is* CDH *and a* P*-point, (b)* $\mathcal{U}$ *is* CDH *and not a* P*-point, (c)* $\mathcal{U}$ *is not* CDH *and not a* P*-point, (d)* $\mathcal{U}^\omega$ *is* CDH.

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- Unfortunately, the existence of non-meager P-filters is still open in ZFC.
- Many interesting open problems remain:

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Is there a CDH *filter (ultrafilter) in* ZFC*? Is there a non-*CDH *and non-meagre filter (ultrafilter) in* ZFC*?*

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Can one prove from ZFC the existence of a CDH Bernstein subspace of $\mathbb{R}$?

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Does there exist in ZFC *a compact* CDH*-space that is not metrizable?*