W_δ sets and the strong Choquet game

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Strong Choquet game $Ch(X)$

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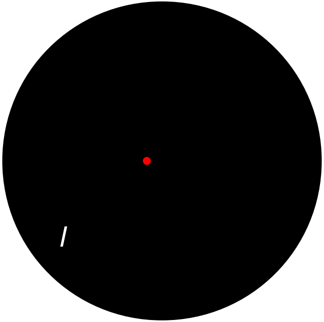

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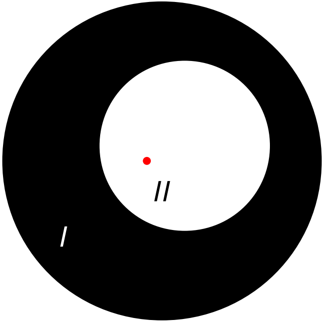

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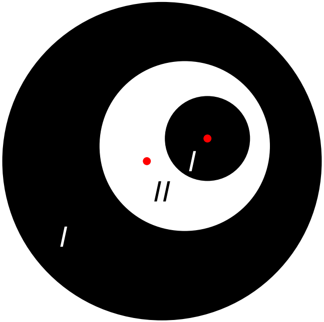

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Results about $Ch(X)$ for a metrizable X

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   (i.e. X is not hereditarily Baire).

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2. Y is a *W_δ-set* in X, if Y has a sieve (G, T) in X such that $\bigcap_n G(t_n) \subseteq Y$ for each branch (t_n) of T.

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- $G_\delta \Rightarrow W_\delta$

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X 1st countable, T_1, and hereditarily Baire

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X 1st countable, T_1, and hereditarily Baire $\Rightarrow$ Player I has no w.s. in $Ch(X)$

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Player II has a w.s. in $Ch(X)$ + X *normal* $\Rightarrow$
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Let $X = (\omega_1 + 1)^2 \setminus \{(\omega_1, \omega_1)\}$ be the Tychonoff square.

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- Player *I* has a w.s. in $Ch(CL(X))$,
- every nonempty W_δ subspace of $CL(X)$ is of the 2nd category in itself.

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Moors: if Y is a metric hereditarily Baire space

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nonempty W_δ-subsets of Y are of the 2nd category in themselves

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Let X be a Baire space, and Y be a 1st countable space in which the nonempty W_δ-subsets are of the 2nd category in themselves.

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Example

(CH) There exists a space X in which the nonempty W_δ-subsets are Baire, but X^2 is not a Baire space.

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Corrolary

Let X be a Baire space, and Y be a 1st countable space such that Player I has no w.s. in $Ch(Y)$. Then $X \times Y$ is a Baire space.