Covering properties and products

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Selection principles (combinatorial properties of open covers)

Let $\mathcal{A}, \mathcal{B}$ be classes of open covers of a space X.

$\bigcup_{fin}(\mathcal{A}, \mathcal{B})$: *For each sequence* $\langle \mathcal{O}_m : m < \omega \rangle$ *of elements of* $\mathcal{A}$ *there is a sequence* $\langle \mathcal{T}_m : m < \omega \rangle$ *with each* $\mathcal{T}_m$ *a finite subset of* $\mathcal{O}_m$*, and* $\langle \bigcup \mathcal{T}_m : m < \omega \rangle \in \mathcal{B}$.

$S_1(\mathcal{A}, \mathcal{B})$: *For each sequence* $\langle \mathcal{O}_m : m < \omega \rangle$ *of elements of* $\mathcal{A}$ *there is a sequence* $\langle T_m : m < \omega \rangle$ *with each* $T_m \in \mathcal{O}_m$ *and* $\langle T_m : m < \omega \rangle \in \mathcal{B}$.

Similarly we define the property $S_{fin}(\mathcal{A}, \mathcal{B})$.
Various types of open covers are relevant to this discussion. A cover $\mathcal{U}$ of a topological space X is:

- An ω*-cover* if $X \notin \mathcal{U}$ and for each finite set $F \subset X$ there is a $U \in \mathcal{U}$ with $F \subset U$.
- A γ*-cover*, if it is an infinite cover and the set $\{U \in \mathcal{U} : x \notin U\}$ is finite for all $x \in X$.

Selection principles (combinatorial properties of open covers)

Collections of *open* covers of a space X (which are ω-, γ-covers) will be denoted by $\mathcal{O}(X)$ (resp. $\Omega(X)$, $\Gamma(X)$).

Collections of *countable Borel* covers of a space X (which are ω-, γ-covers) will be denoted by $\mathcal{B}(X)$ (resp. $\mathcal{B}_\Omega(X)$, $\mathcal{B}_\Gamma(X)$)

Properties we are going to consider:
$\bigcup_{fin}(\mathcal{O}, \mathcal{O})$ (Menger); $\bigcup_{fin}(\mathcal{O}, \Omega)$ (Scheepers);
$\bigcup_{fin}(\mathcal{O}, \Gamma)$ (Hurewicz); $S_1(\mathcal{O}, \mathcal{O})$ (Rothberger);
$S_1(\Omega, \Gamma)$ (γ-set); Borel versions of these properties.

Relations between selection principles

$$\begin{array}{ccccccccc}
|X| = \omega & \to & S_1(\Omega,\Gamma) & \to & S_1(\Gamma,\Gamma) & \to & \bigcup_{fin}(\mathcal{O},\Gamma) & \to & \bigcup_{fin}(\mathcal{O},\Omega) \\
 & & & & & \nearrow & & & \downarrow \\
 & & & & \sigma\text{-comp.} & & S_1(\mathcal{O},\mathcal{O}) & \to & \bigcup_{fin}(\mathcal{O},\mathcal{O})
\end{array}$$

because $\bigcup_{fin}(\mathcal{O},\mathcal{A}) = \bigcup_{fin}(\Omega,\mathcal{A}) = \bigcup_{fin}(\Gamma,\mathcal{A})$
for most of the families $\mathcal{A}$.

Theorem (Laver 1976)

It is consistent that every $S_1(\mathcal{O},\mathcal{O})$ set of reals is countable. □

Theorem (Z. 2005)

If $\mathfrak{u} < \mathfrak{g}$, then $\bigcup_{fin}(\mathcal{O},\mathcal{O}) = \bigcup_{fin}(\mathcal{O},\Omega)$ and $S_1(\mathcal{O},\mathcal{O}) \Rightarrow \bigcup_{fin}(\mathcal{O},\Gamma)$. □

Some examples

A subset X of $\mathbb{R}$ is called a *Luzin* (resp. *Sierpinski*) set, if $|X| > \omega$ and $|X \cap A| \leq \omega$ for every meager (resp. measure zero) set $A \subset \mathbb{R}$.

Theorem (Rothberger 38; Just-Miller-Scheepers-Szeptycki 96)

Every Luzin set has property $S_1(\mathcal{B}, \mathcal{B})$.
Every Sierpinski has property $\bigcup_{fin}(\mathcal{B}, \mathcal{B}_\Gamma)$. □

Theorem (Scheepers-Tall 2010)

Every ground model Lindelöf space is $S_1(O, O)$ after adding uncountably many Cohen reals. □

Some examples, continued

If $x, y \in [\omega]^\omega$, then $x \leq^* y$ means that the set $\{n : x(n) > y(n)\}$ is finite. $\mathfrak{b}$-scale is an unbounded set $\{b_\alpha : \alpha < \mathfrak{b}\} \subset [\omega]^\omega$ such that the enumeration is increasing with respect to $\leq^*$.

Theorem (Bartoszynski-Tsaban 2006, Bartoszynski-Shelah 2001 for $n = 1$.)

Let B be a $\mathfrak{b}$-scale and $n \in \omega$. Then $(B \cup [\omega]^{<\omega})^n$ is $\bigcup_{fin}(\mathcal{O}, \Gamma)$. □

A set $B = \{b_\alpha : \alpha < \kappa\} \subset [\omega]^\omega$ is a *tower* if $|b_\alpha \setminus b_\beta| < \omega$ for all $\alpha > \beta$ and B has no pseudointersection.

Theorem (A. Miller-Tsaban 2010; Scheepers 1998 under $\mathfrak{t} = \mathfrak{b}$.)

If $B = \{b_\alpha : \alpha < \mathfrak{b}\}$ is an unbounded tower (and $\mathfrak{h} = \mathfrak{b}$), then $B \cup [\omega]^\omega$ has property $S_1(\Gamma, \Gamma)$ (in all finite powers). □

Unbounded towers exist in models of $\mathfrak{b} < \mathfrak{d}$, and hence $\mathfrak{t} = \mathfrak{b}$ is not equivalent to the existence of an unbounded tower.

Theorem (Orenstein-Tsaban 2011)

If $\mathfrak{p} = \mathfrak{b}$ and $B = \{b_\alpha : \alpha < \mathfrak{b}\}$ is an unbounded tower, then $B \cup [\omega]^\omega$ has property $S_1(\Omega, \Gamma)$. □

Some limitations

Theorem (Sierpinski 192?; Just-Miller-Scheepers-Szeptycki 1996)

Luzin sets do not have property $\bigcup_{fin}(\mathcal{O},\Gamma)$.
Sierpinski sets do not have property $S_1(\mathcal{O},\mathcal{O})$. □

Theorem (Repovš-Tsaban-Zdomskyy 2008)

If $\mathfrak{b}=\mathfrak{c}$, *then there exists a* $\mathfrak{b}$*-scale* B *such that* $B\cup[\omega]^{<\omega}$ *does not have property* $S_1(\Gamma,\Gamma)$. □

Question (Scheepers)

Is there a ZFC example of a set X *of reals without copies of* 2^ω *inside which is* $\bigcup_{fin}(\mathcal{O},\Gamma)$ *but does not have property* $S_1(\Gamma,\Gamma)$*?*

Proposition (Scheepers-Tall 2011)

The old reals do not have property $\bigcup_{fin}(\mathcal{O},\Gamma)$ *after adding* ω_1*-many Cohen reals.* □

Theorem (Just-Miller-Scheepers-Szeptycki 1996)

Assume CH. Then

- *There exists a Luzin set L such that L^2 does not have property $\bigcup_{fin}(\mathcal{O}, \mathcal{O})$;*
- *There exists a Sierpinski set S such that S^2 does not have property $\bigcup_{fin}(\mathcal{O}, \mathcal{O})$.* □

Theorem (A. Miller-Tsaban-Z. 201?)

(CH) There exist subsets X, Y of 2^ω with property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ such that $X \times Y$ is not $\bigcup_{fin}(\mathcal{O}, \mathcal{O})$, i.e., it is not Menger. □

Surprisingly, but

Theorem (Barman-Dow 201?)

In any model obtained by adding $> \omega_1$ many Cohen reals over a model of CH the product of finitely many sets of reals with property $S_1(\Omega, \Gamma)$ (i.e., γ-sets) is $\bigcup_{fin}(\mathcal{O}, \mathcal{O})$.

The same conclusion follows from PFA. □

Productive spaces

Let $\mathcal{P}$ be a property. A space X is said to be *productively* $\mathcal{P}$, if $X \times Y$ has property $\mathcal{P}$ provided so does the space Y.

Theorem (A. Miller-Tsaban-Z. 201?)

Let B be a $\mathfrak{b}$-scale. Then

- $B \cup [\omega]^{<\omega}$ *is productively* $\bigcup_{fin}(\mathcal{O}, \Gamma)$ *for sets of reals;*
- *If B is dominating (this implies $\mathfrak{b} = \mathfrak{d}$), then* $B \cup [\omega]^{<\omega}$ *is productively* $\bigcup_{fin}(\mathcal{O}, \Omega)$ *for sets of reals;*
- *If B is dominating or $\mathfrak{b} < \mathfrak{g}$, then* $B \cup [\omega]^{<\omega}$ *is productively* $\bigcup_{fin}(\mathcal{O}, \mathcal{O})$ *for sets of reals;*
- *In Miller's model ,* $B \cup [\omega]^{<\omega}$ *is productively* $S_1(\mathcal{O}, \mathcal{O})$ *for sets of reals.*
- *If $\mathfrak{b} = \omega_1$ and B is an unbounded tower, then* $B \cup [\omega]^{<\omega}$ *is productively** $S_1(\Omega, \Gamma)$.
- *If B is an unbounded tower and X is* $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$*, then* $(B \cup [\omega]^{<\omega}) \times X$ *is* $S_1(\Gamma, \Gamma)$. □

Proof of the Hurewicz case

Base of the standard topology on $\mathcal{P}(\omega)$ consists of the sets $O(s,l) = \{b \subset \omega : b \cap [0,l) = s\}$, where $l \in \omega$ and $s \subset [0,l)$.

Base of the standard topology on $\mathcal{P}(\omega)$ at $a \subset \omega$ consist of the sets $O(a \cap [0,l), l)$, where $l \in \omega$.

Recall from the Boaz' talk:

Let $\mathcal{U} = \langle U_n : n \in \omega \rangle$ be an ω-cover of $[\omega]^{<\omega}$. Then there are increasing sequences $\langle a(k) : n \in \omega \rangle$ and $\langle n(k) : k \in \omega \rangle$ such that

if $a \subset \omega$ *has empty intersection with* $[a(k), a(k+1))$*, then* $a \in U_{n(k)}$.

These sequences can be constructed by induction: Given $a(k)$, let $n(k)$ be such that $U_{n(k)} \supset \mathcal{P}([0, a(k)))$.
For every $s \subset [0, a(k))$ find $l_s \in \omega$ such that $O(s, l_s) \subset U_{n(k)}$. Let $a(k+1)$ be any upper bound of $\{l_s : s \subset [0, a(k))\}$.

Let $\mathcal{U} = \langle U_n : n \in \omega \rangle$ be an ω-cover of $[\omega]^{<\omega}$. Set $a_{\mathcal{U}}(0) = 0$. Recursively construct increasing sequences $\langle a_{\mathcal{U}}(k) : n \in \omega \rangle$ and $\langle n_U(k) : k \in \omega \rangle$ such that

1. $n_{\mathcal{U}}(k)$ is the minimal natural number m such that $\mathcal{P}([0, a_{\mathcal{U}}(k)]) \subset U_m$; and
2. $a_{\mathcal{U}}(k+1)$ is the minimal natural number l such that if $a \subset \omega$ has empty intersection with $[a_{\mathcal{U}}(k), l)$, then $a \in U_{n_{\mathcal{U}}(k)}$.

Suppose that Y has the property $\bigcup_{\text{fin}}(O, \Gamma)$ and $\langle \mathcal{U}_m : m \in \omega \rangle$ is a sequence of ω-covers of $(B \cup Q) \times Y$ by clopen subsets of $2^\omega \times 2^\omega$, where $\mathcal{U}_m = \langle U_{m,n} : n \in \omega \rangle$ and $U_{m,n} = V_{m,n} \times W_{m,n}$.

For every subset A of $2^\omega \times 2^\omega$ and $t \in 2^\omega$ set $A^t = \{x : (x,t) \in A\}$. Set also $\mathcal{U}_m^y = \langle U_{m,n}^y : n \in \omega \rangle$ for every $y \in Y$ and note that $U_{m,n}^y$ is either $V_{m,n}$ or $\emptyset$ depending on whether $y \in W_{m,n}$ or $y \notin W_{m,n}$, respectively.

Consider the map $\varphi : Y \to \omega^\omega$, $\varphi(y)(k) = n_{\mathcal{U}_k^y}(k+1)$. It is continuous.
Since Y is Hurewicz,there exists $b \in \omega^\omega$ such that $\varphi(y) \leq^* b$ for all $y \in Y$.
Let $\beta < \mathfrak{b}$ be such that $K := \{k : b(k) < b_\beta(k)\}$ is infinite. Set $O_k = \bigcup_{i \leq b(k)} U_{k,i}$. We claim that $\{O_k : k \in K\}$ is a γ-cover of $\{b_\alpha : \alpha \geq \beta\} \times Y$.

Proof of the Hurewicz case, continued

Indeed, let us fix $\alpha \geq \beta$ and $y \in Y$. It is enough to show that if $b_\alpha(k) > b(k)$ and $b(k) > \varphi(y)(k) = n_{\mathcal{U}_k^y}(k+1)$, then $(b_\alpha, y) \in O_k$.

Given such a $k \in K$, observe that there exists $j \leq k$ such that $[n_{\mathcal{U}_k^y}(j), n_{\mathcal{U}_k^y}(j+1)) \cap b_\alpha = \emptyset$, and hence $b_\alpha \in U^y_{k, n_{\mathcal{U}_k^y}(j)}$. The latter means that $(b_\alpha, y) \in U_{k, n_{\mathcal{U}_k^y}(j)}$.

Since $b(k) > n_{\mathcal{U}_k^y}(k+1) > n_{\mathcal{U}_k^y}(j)$, we conclude that $(b_\alpha, y) \in O_k$, which finishes our proof of the fact that $\{O_k : k \in K\}$ is a γ-cover of $\{b_\alpha : \alpha \geq \beta\} \times Y$.

The last slide

Thank you for your attention.