A note on the translates of sets contained in the intersection ideal $\mathcal{M} \cap \mathcal{N}$

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Terminology.

Throughout this talk we deal with subsets of the Cantor space 2^ω with the standard topology, measure and modulo 2 coordinatewise addition denoted by $+$.

$\mathcal{M}$ — σ ideal of meager subsets of 2^ω.
$\mathcal{N}$ — σ ideal of measure zero subsets of 2^ω
$\mathcal{E}$ — σ ideal generated by F_σ measure zero sets in 2^ω.
$\mathcal{M} \cap \mathcal{N}$ — σ ideal of sets that are in $\mathcal{M}$ and in $\mathcal{N}$.

Suppose that I and J are σ-ideals of subsets in 2^ω with $I \subseteq J$.

Definition 1

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We shall say that X is I *additive* ($X \in I^*$) iff

$$X + A = \{x + a : x \in X,\ a \in A\} \in I,$$

for any set $A \in I$. $X \in (I, J)^*$ iff for every set $A \in I$, $X + A \in J$.

Theorem 2

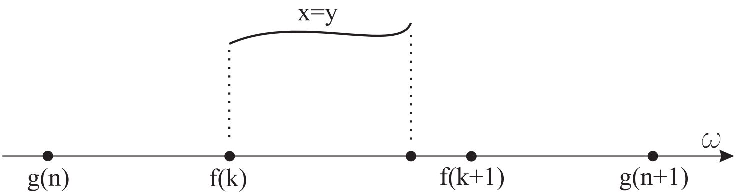
Theorem 2 (Bartoszyński–Judah–Shelah)

$X \in \mathcal{M}^*$ iff $\forall f \in \omega^{\omega^\uparrow}\ \exists g \in \omega^{\omega^\uparrow}$ and $y \in 2^\omega$ such that for every $x \in X$

$$\forall_n^\infty \exists k\ g(n) \leq f(k) < f(k+1) \leq g(n+1),$$

and

$$x \restriction [f(k), f(k+1)) = y \restriction [f(k), f(k+1)).$$



Theorems 3, 4 and Corollary 5

Theorem 3

$\mathcal{M}^* \subseteq \mathcal{E}^*$.

The following theorem is quite surprising.

Theorem 4 (Zindulka, 2010)

If $X \in \mathcal{E}^*$, then $X \in \mathcal{M}^*$.

Corollary 5 (Zindulka, 2010)

$\mathcal{E}^* = \mathcal{M}^*$.

This talk was inspired by the above results of Zindulka, and it consists of three parts.

1. We show that Theorem 4 with a slightly weaker assumption on X does not hold.
2. We consider relations between various ideals defined in terms of translations of sets that belong to the intersection ideal $\mathcal{M} \cap \mathcal{N}$.
3. We construct a "new" perfecty meager in the transitive sense subset of 2^ω.

Let $\mathbb{P}_{\aleph_2}$ be the $\aleph_2$ iteration of the Cohen forcing (with finite supports) over a model V of GCH. Suppose that G is a generic filter in $\mathbb{P}_{\aleph_2}$ over V.

Theorem 6

Theorem 6

There exists a set $X \in V[G]$, $|X| < \mathfrak{c}$, which is not meager additive, and such that for every $F \in \mathcal{E}$, $X + F \in \mathcal{M} \cap \mathcal{N}$.

Proof.

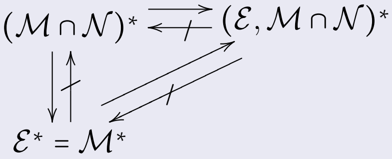
We construct in V, two sets X and Y which are strongly meager (in V), and such that $X + Y = 2^\omega \cap V$. Then we show that $X + F$ is meager for every $F \in \mathcal{E} \cap V[G]$. Notice that $X + F$ is measure zero, since every subset of 2^ω of small cardinality has strong measure zero. □

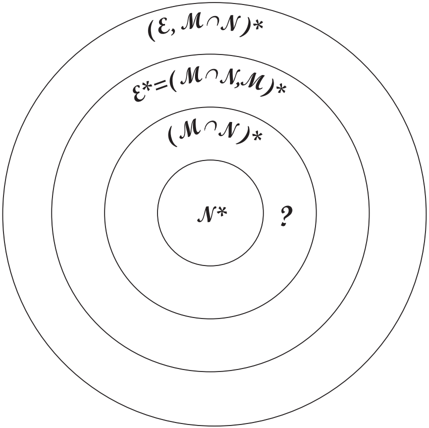
Theorem 7

Suppose now that $\to$ denotes the inclusion and $\not\leftarrow$ means that the reverse inclusion can not be proved in ZFC.

Theorem 7

The following diagram of inclusions holds.





Proof.

By Theorem 6 above we have that
$\mathcal{E}^* = \mathcal{M}^* \nleftarrow (\mathcal{E}, \mathcal{M} \cap \mathcal{N})^*$. □

Claim 8

Claim 8 (CH)

There exists a γ set X such that $X \notin (\mathcal{M} \cap \mathcal{N})^*$. As a consequence we obtain

$$\mathcal{E}^* \nrightarrow (\mathcal{M} \cap \mathcal{N})^*, \text{ and } (\mathcal{E}, \mathcal{M} \cap \mathcal{N})^* \nrightarrow (\mathcal{M} \cap \mathcal{N})^*.$$

Next fact contrasts with Theorem 6 above.

Claim 9

Claim 9

$(\mathcal{M} \cap \mathcal{N}, \mathcal{M})^* \to \mathcal{E}^* = \mathcal{M}^*$.

Proof.

For $f \in \omega^{\omega^{\uparrow}}$, we construct a large set $A \in \mathcal{M} \cap \mathcal{N}$ such that "$X + A \in \mathcal{M}$" implies that X satisfies the above Bartoszyński–Judah–Shelah characterization of a set in $\mathcal{M}^*$. □

This provides the proof of the inclusion $(\mathcal{M} \cap \mathcal{N})^* \to \mathcal{E}^* = \mathcal{M}^*$.

Since $\mathcal{N}^* \to \mathcal{M}^*$ (Shelah), it seems natural to ask the following question.

Problems 10, 11

Problem 10

Is it consistent with ZFC that the class $(\mathcal{M} \cap \mathcal{N})^*$ contains sets that are not in $\mathcal{N}^*$?

And if the answer is "yes".

Problem 11

Find characterization of sets in $(\mathcal{M} \cap \mathcal{N})^*$ in terms of infinite combinatorics on ω.

Also assuming that $(\mathcal{M} \cap \mathcal{N})^*$ is a wider class than $\mathcal{N}^*$, one might ask the following.

Problems 12 and Definition 13

Problem 12

Is there a property $\mathbb{P}$ (defined in terms of selections) which is weaker than "strongly γ", and such that all sets satisfying $\mathbb{P}$ belong to $(\mathcal{M} \cap \mathcal{N})^*$?

In relation to a certain problem of M. Scheepers the following notion of a subset of 2^ω was defined.

Definition 13

$X \subseteq 2^\omega$ is called a set *perfectly meager in the transitive sense* (an *AFC'* set) iff for every perfect set D, there exists an F_σ set F, with $X \subseteq F$, such that for every $t \in 2^\omega$, $(F + t) \cap D$ is meager in the relative topology of D.

Fact 14 and Remark 15

Fact 14

$AFC' \subsetneq PM$ (perfectly meager sets). Examples of AFC' sets: strongly meager sets, $\mathcal{M}^*$ sets, in particular strongly measure zero sets with the Hurewicz property, carefully constructed scales identified with subsets of 2^ω by characteristic functions.

Remark 15

A set X constructed in the proof of Theorem 6 is an AFC' set.

Notice that X is not strongly meager (Carlson's theorem), and it neither belongs to $\mathcal{M}^*$ nor it is equal to a scale (in the Cohen real model:
$\forall f \in V[G] \cap \omega^\omega \; \exists g \in V \cap \omega^\omega \; \exists_n^\infty f(n) < g(n)$).

Definition 16 and Problem 17

Definition 16 (Kraszewski)

$X \subseteq 2^\omega$ is called an *everywhere meager* set (an $\mathcal{EM}$ set) iff for any infinite $a \subseteq \omega$, the set $\{x \restriction a : x \in X\} \subseteq 2^a$ is a meager subset of 2^a.

$X \in \mathcal{EM}$ iff $X + A$ is meager for every set A of the form $\{x \in 2^\omega : x \restriction a \equiv \mathbb{O}\}$, where a is an infinite subset of ω.

It is easy to see that all AFC' sets mentioned above, including a "new" AFC' set X (see Remark 15), are in $\mathcal{EM}$. Thus the following question of Kraszewski remains open.

Problem 17

Is there an AFC' set which is not a member of the class $\mathcal{EM}$?

References

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