Stone-Čech remainders of ψ-spaces

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Almost disjoint family (of countable sets)

Definition

Let $\kappa \geq \omega$ be a cardinal number. An infinite family $\mathcal{A}$ of countably infinite subsets of κ ($\mathcal{A} \subset [\kappa]^\omega$) is an *almost disjoint family* provided $A \neq A'$ in $\mathcal{A} \Rightarrow A \cap A'$ is finite.

Maximal ADF

Definition

An almost disjoint family $\mathcal{A}$ is called *maximal* provided $\mathcal{A}$ is not properly contained in any other almost disjoint family.

Mrówka's ψ-space, generalized

Definition

Let $\psi(\kappa, \mathcal{A})$ denote the space with underlying set $\kappa \cup \mathcal{A}$ and with the topology having as a base all singletons $\{\alpha\}$ for $\alpha < \kappa$ and all sets of the form $\{A\} \cup (A \setminus F)$ where $A \in \mathcal{A}$ and F is finite.

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For the case $\kappa = \omega$, and $\mathcal{A}$ maximal, $\psi(\omega, \mathcal{A})$ is the well known space of S. Mrówka which he denoted $N \cup \mathcal{R}$, and denoted by Ψ in the book by Gillman and Jerison, "Rings of Continuous Functions."

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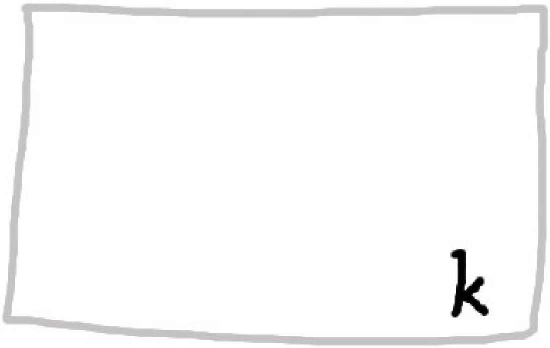
schematic



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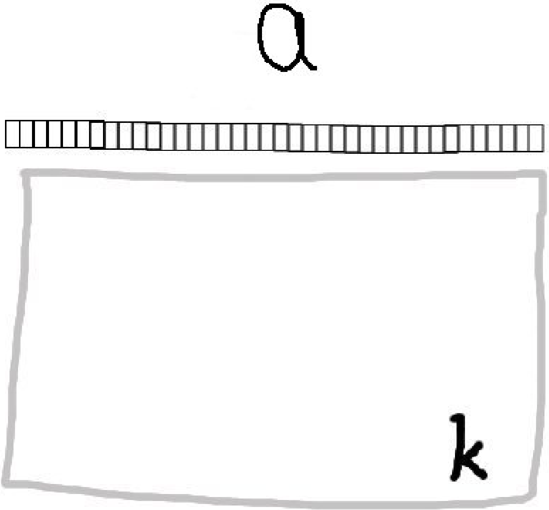

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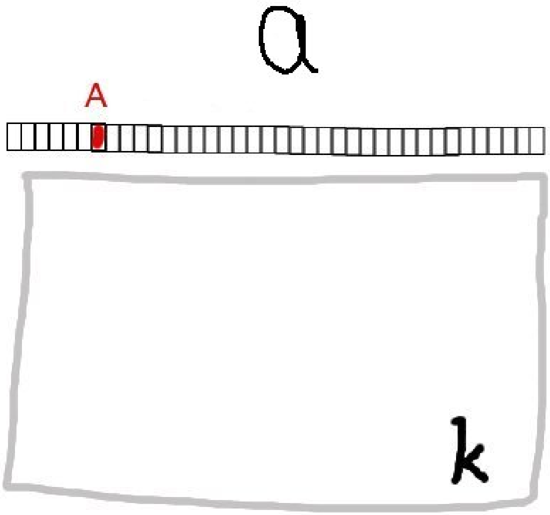

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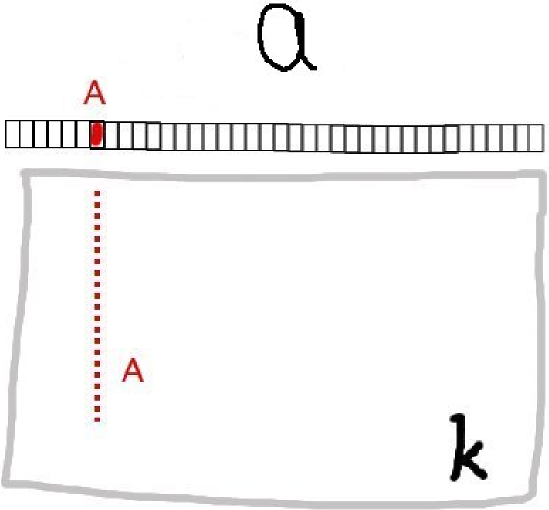
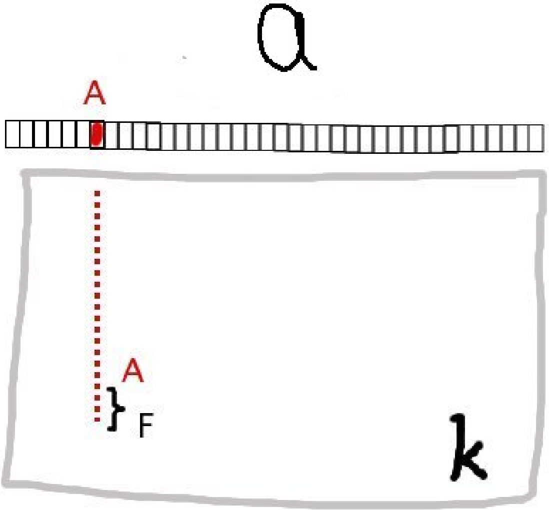
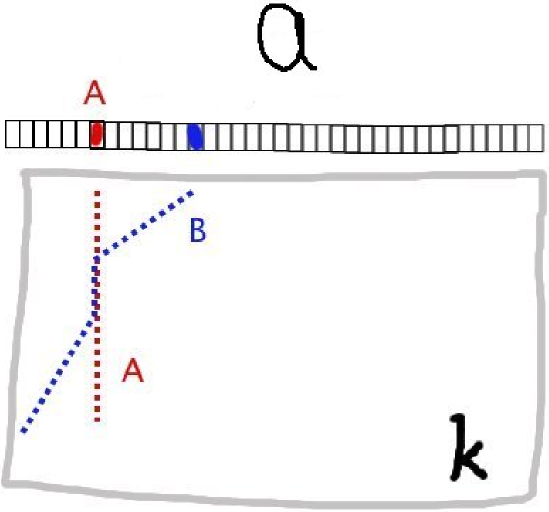


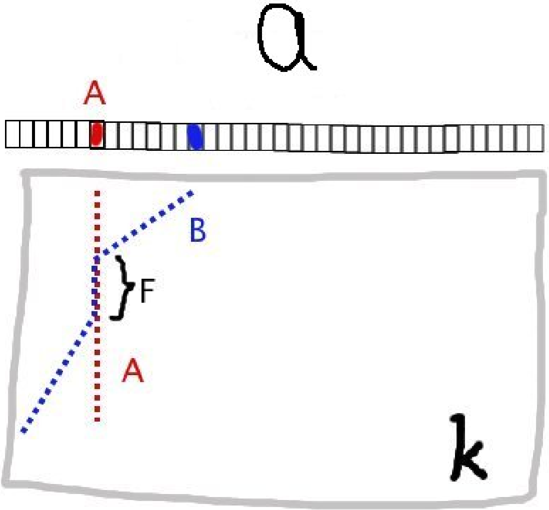
ψ is first countable, locally compact



ψ is Hausdorff



ψ is Hausdorff and zero-dimensional



Stone-Čech compactification

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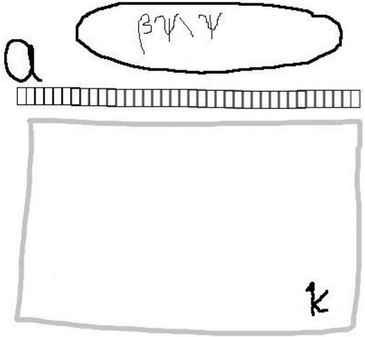
This is equivalent to the statement that every continuous $f : X \to \mathbb{R}$, has a continuous extension $\overline{f} : \beta X \to \mathbb{R}$.

Stone-Čech compactification

We view the picture of $\beta\psi$ as three levels:



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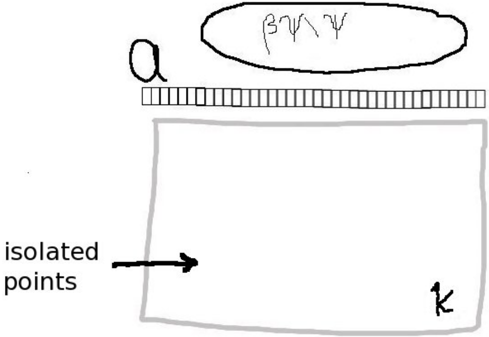

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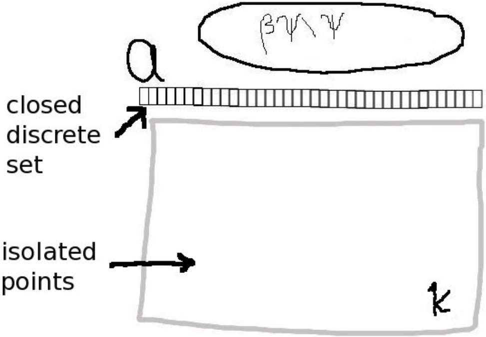

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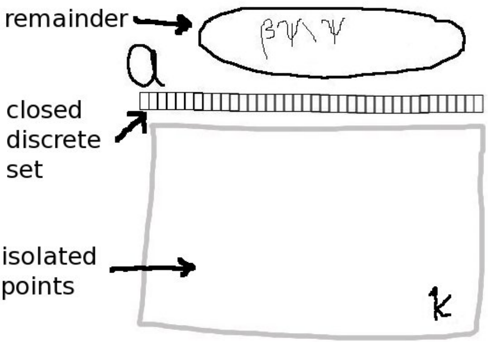
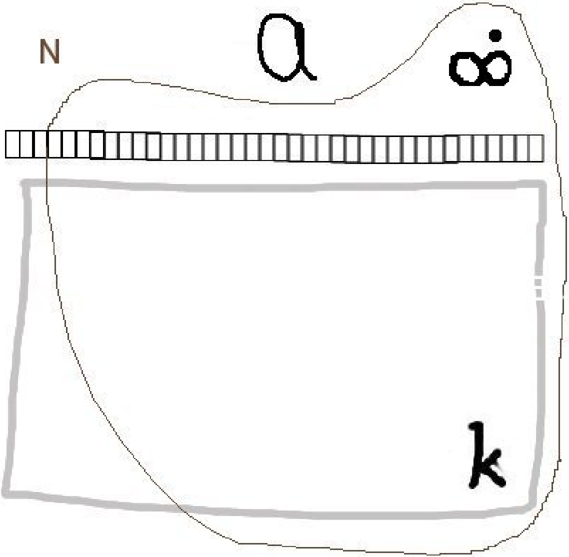


$\alpha\psi$: the one-point compactification



Mrówka's Theorem

Theorem (Mrówka, 1977)

There exists MADF $\mathcal{M} \subset \omega$ *satisfying* $|\beta\psi(\omega, \mathcal{M}) \setminus \psi(\omega, \mathcal{M})| = 1$

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Theorem (D-V)

If $\kappa \leq \mathfrak{c}$*, then there exists MADF* $\mathcal{M} \subset [\kappa]^\omega$ *satisfying* $|\beta\psi(\kappa, \mathcal{M}) \setminus \psi(\kappa, \mathcal{M})| = 1$

$\kappa > \mathfrak{c}$?

But "$|\beta\psi \setminus \psi| = 1$" does not generalize for $\kappa > \mathfrak{c}$:

Theorem (D-V)

For every $\kappa > \mathfrak{c}$, and for every $\mathcal{M} \subset [\kappa]^\omega$ MADF,
$|\beta\psi(\kappa, \mathcal{M})) \setminus \psi(\kappa, \mathcal{M})| \geq \sqrt[\omega]{\kappa} \geq \mathfrak{c}^+$.

Extending Mrówka's Theorem for $\kappa > \mathfrak{c}$

Definition of Mrówka MADF for all $\kappa \geq \omega$:

(1) For $\kappa \leq \mathfrak{c}$, $\mathcal{M}$ is called a *Mrówka MADF* if $|\beta\psi \setminus \psi| = 1$

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(2) For $\kappa > \mathfrak{c}$, $\mathcal{M}$ is called a *Mrówka MADF* if for every continuous $f : \psi \to \mathbb{R}$ there exists $r \in \mathbb{R}$ such that

$$|\psi(\kappa, \mathcal{M}) \setminus f^{-1}(r)| < |\mathcal{M}| = |\psi|.$$

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Indeed (1) implies (2): If $\mathcal{M}$ satisfies (1), let $p = \infty$ be the unique point in the reminder. Given continuous $f : \psi \to \mathbb{R}$ let $\overline{f} : \beta\psi \to \mathbb{R}$ be the continuous extension of f. Define $r = \overline{f}(p)$.

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$$|\psi \setminus f^{-1}(r)| \leq \omega < |\mathcal{M}| = |\psi|.$$

So (2) holds.

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But since (1) implies a stronger condition that (2), we state the definition of "Mrówka MADF" in two parts.

$\kappa > \mathfrak{c}$: Existence of a Mrówka MADF and a special p at infinity

For $\kappa > \mathfrak{c}$, $|\beta\psi \setminus \psi| \neq 1$, yet for Mrówka MADFs $\mathcal{M} \subset [\kappa]^\omega$, there is still a special point $p \in \beta\psi \setminus \psi$. Since $\kappa > \mathfrak{c}$ implies $|\beta\psi \setminus \psi| \neq 1$, p is not as special as $\{p\} = \beta\psi \setminus \psi$.

Special point in the remainder

Theorem (SCH for $\kappa > \mathfrak{c}$)

For every cardinal $\kappa \geq \omega$, and every Mrówka $\mathcal{M} \subset [\kappa]^\omega$ MADF, there is a point $p \in \beta\psi \setminus \psi$ having a clopen local base in $\beta\psi$.

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If κ is a regular cardinal (i.e., $\kappa = cf(\kappa)$), then p is a simple $P(\kappa)$-point in $\beta\psi$.

Otherwise (i.e.,$cf(\kappa) = \omega$ or $\kappa = \lambda^+$ and $cf(\lambda) = \omega$), then p is not even a weak P-point in $\beta\psi$ or $\beta\psi \setminus \psi$.

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Using a result of Jun Terasawa, such a space $\beta\psi$ need not be zero dimensional.

Mrówka's Theorem (attributed)

Theorem (attributed to S. Mrowka by J. Terasawa)

There exists a $\mathcal{M} \subset [\omega]^\omega$ *MADF such that* $\beta\psi \setminus \psi \cong (\omega_1 + 1)$ *with the order topology.*

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Question: What ordinals (other than 1 and $\omega_1 + 1$) can be reminders of a ψ-space?

Ordinal remainders of ψ-spaces

Which ordinals, with the order topology, can be Stone-Čech remainders of ψ-spaces? Our results:

Theorem (D-V)

If $\kappa \leq \mathfrak{c}$ then for every ordinal $\delta < \kappa^+$ there exists $\mathcal{M}_\delta \subset [\kappa]^\omega$, a MADF, such that $\beta\psi(\kappa, \mathcal{M}_\delta) \setminus \psi(\kappa, \mathcal{M}_\delta)$ is homeomorphic to $\delta + 1$.

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Theorem (D-V)

If $\kappa > \mathfrak{c}^+$, then no $\psi(\kappa, \mathcal{M})$ has any ordinal as a Stone-Čech remainder.

Ordinal remainders of ψ-spaces

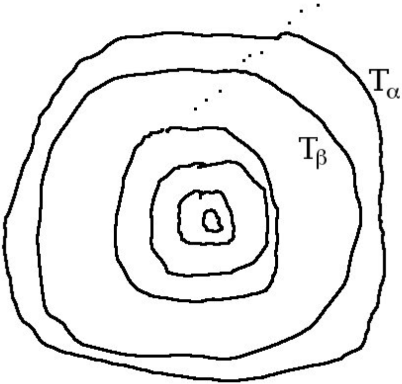
Summary:

For κ above $\mathfrak{c}$, no more one-point Stone-Čech remainders of ψ-spaces. Above $\mathfrak{c}^+$, no more ordinal remainders.

tower

Now let us ask about ordinal remainders in the classical case $\kappa = \omega$: Mrówka's original class of ψ-space. We need to define the tower number.

Mod-finite increasing chain



Mod-finite increasing chain: $\beta < \alpha$ implies $T_\beta <^* T_\alpha$

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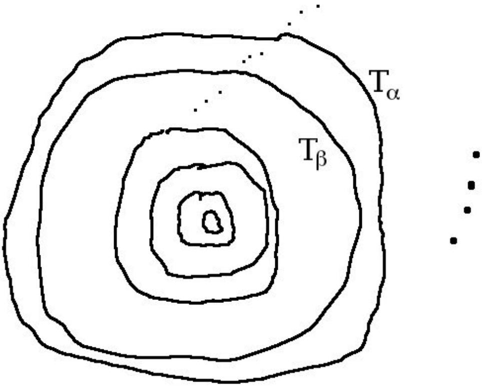

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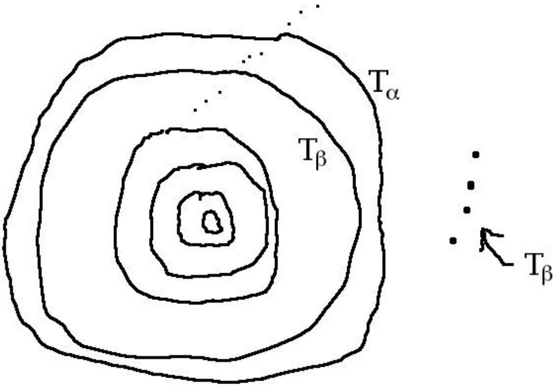

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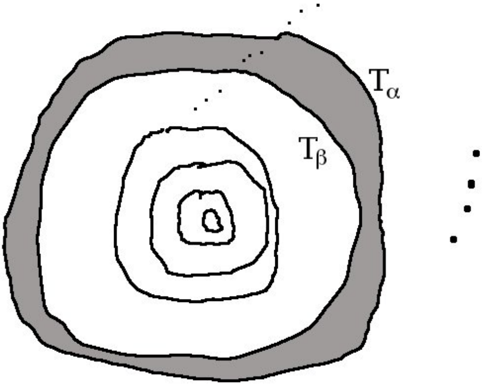

$$T_\beta <^* T_\alpha$$

tower

A mod-finite increasing chain in a set X is a *tower* in X if it is maximal in the sense that for all $H <^* X$ it is not the case that $T_\alpha \subset^* H$ for all α.

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Definition

The *tower number*, denoted $\mathfrak{t}$, is defined to the the smallest cardinality of a tower on ω.

$\mathfrak{t}$ is a "small uncountable cardinal," $\omega_1 \leq \mathfrak{t} \leq \mathfrak{c}$

Theorem

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For every $\lambda < \mathfrak{t}^+$, there exists $\mathcal{M} \subset [\omega]^\omega$ a MADF such that $\beta\psi \setminus \psi$ is homeomorphic to $\lambda + 1$.

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This implies the theorem attributed to Mówka by Terasawa because $\omega_1 \leq \mathfrak{t}$ hence $\omega_1 < \mathfrak{t}^+$.
Moreover, the theorem is best possible in ZFC in the sense that (it is consistent that) it is not true that the ordinal $\mathfrak{t}^+ + 1$ can be a remainder of a ψ space over ω.

Another result regarding the classical case $\kappa = \omega$

Recall, the cardinal $\mathfrak{a}$ is the smallest cardinality of a MADF on ω.

$$\omega_1 \leq \mathfrak{a} \leq \mathfrak{c}$$

$\mathfrak{a}$ is one of many "small uncountable cardinals."

Cardinality of $f^{-1}(r)$

Let $f : \psi \to \mathbb{R}$. A *fiber* of f is any set of the form $f^{-1}(r)$.
The cardinality of fibers is important in the definition of Mrówka MADF, where for every f there is $r \in \mathbb{R}$ such that $|\psi \setminus f^{-1}(r)| < |\mathcal{M}|$, which is stronger than saying $|f^{-1}(r)| = |\mathcal{M}|$.
A fiber $f^{-1}(r)$ is *full* means that $|f^{-1}(r)| = |\mathcal{M}| = |\psi|$.

Question: Does every continuous real valued function on a ψ-space (on ω) always have at least one "full" fiber.

Cardinality of $f^{-1}(r)$

Theorem (D-V)

The following are equivalent
(i) There exists a MADF $\mathcal{M} \subset [\omega]^\omega$, $|\mathcal{M}| = \mathfrak{c}$, *and there exists continuous* $f : \psi \to [0,1]$ *such that for every* $r \in [0,1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$ *(i.e., f does not have a full fiber).*

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(ii) $\mathfrak{a} < \mathfrak{c}$.

(ii) implies (i)

Assume $\mathfrak{a} < \mathfrak{c}$ and construct $\mathcal{M}$, and f.

Step 1: Show that every countable pairwise disjoint family $\mathcal{P} \subset [\omega]^\omega$ is a subset of some MADF $\mathcal{M}$ with $|\mathcal{M}| = \mathfrak{a}$.

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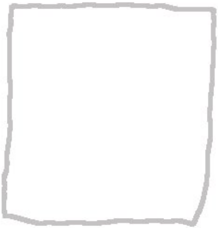

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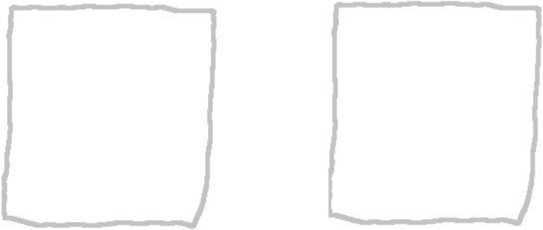

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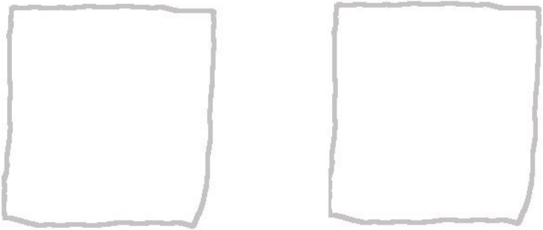
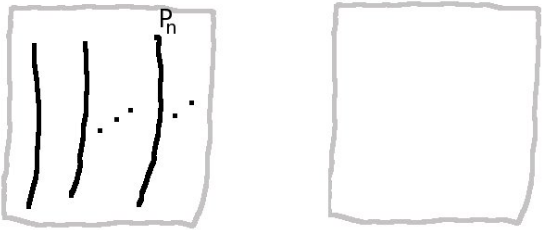


Two copies of ω

(ii) implies (i)



(ii) implies (i)

We now use the MADF $\mathcal{A}$ which has cardinality $\mathfrak{a}$

$$|\mathcal{A}| = \mathfrak{a}$$

(ii) implies (i)

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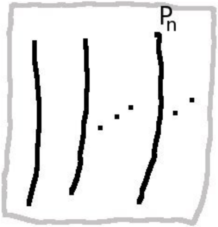
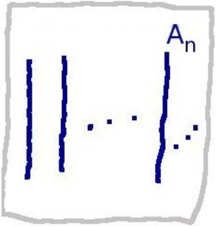
$$|\mathcal{A}| = \mathfrak{a}$$

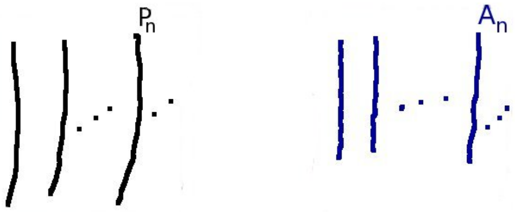
Pick distinct $A_n \in \mathcal{A}$ for all $n \in \omega$. By removing a finite set F_n from each A_n, We get that the sets $A_n \setminus F_n$ are pairwise disjoint. Then $\mathcal{A} \setminus \{A_n : n \in \omega\} \bigcup \{A_n \setminus F_n : n \in \omega\}$ is also a MADF with $|\mathcal{A}| = \mathfrak{a}$. Denote this slightly revised version of $\mathcal{A}$ by the same name $\mathcal{A}$.

(ii) implies (i)

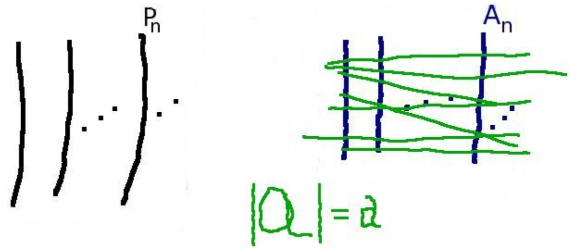




(ii) implies (i)



(ii) implies (i)



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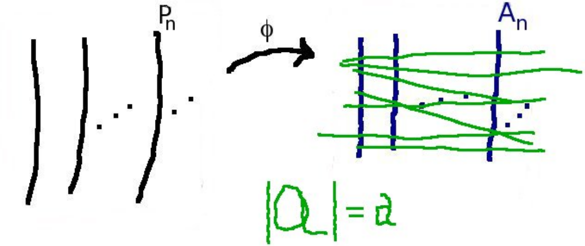

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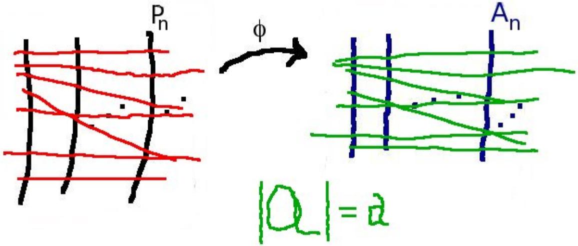

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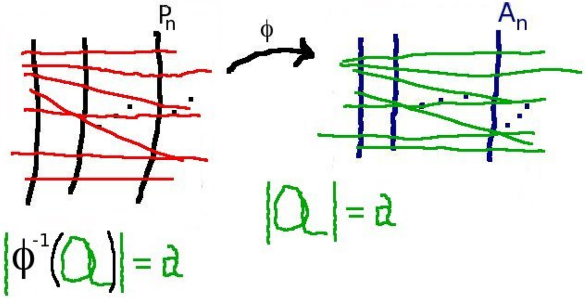
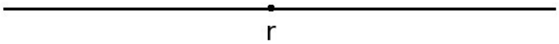


(ii) implies (i)

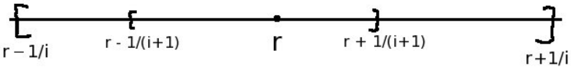
Step 2: Think of the countable set ω as being $\mathbb{Q} \cap [0,1]$, the set of rational numbers in $[0,1]$. Then for each $r \in [0,1]$ there is an ADF $\mathcal{M}_r \subset [\mathbb{Q} \cap [0,1]]^\omega$ such that $|\mathcal{M}_r| = \mathfrak{a}$, every $M \in \mathcal{M}_r$ converges to r (in the usual sense in $[0,1]$) and $\mathcal{M}_r$ satisfies the following maximal condition:

(*) For every $C \in [\mathbb{Q} \cap [0,1]]^\omega$, if C converges to r, then there exists $M \in \mathcal{M}_r$ such that $M \cap C$ is infinite.

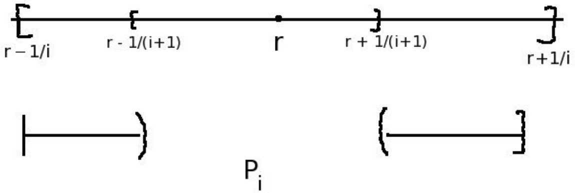
(ii) implies (i)



(ii) implies (i)



(ii) implies (i)



(ii) implies (i)

Let $\mathcal{P}$ denote the set of all infinite members of $\{P_i : i \geq 1\}$. By Step 1, there is an MADF $\mathcal{M} \subset \mathbb{Q} \cap [0,1]$ such that $\mathcal{M} \supset \mathcal{P}$, and $|\mathcal{M}| = \mathfrak{a}$. Put

$$\mathcal{M}_r = \mathcal{M} \setminus \mathcal{P}$$

Check that $\mathcal{M}_r$ has the required properties: First, $M \in \mathcal{M}_r$ converges to r because $M \cap P_i$ is finite for all $i \in \omega$. Second, to see that $\mathcal{M}_r$ has the required maximal property, let C be a sequence of rationals converging to r. There exists $M \in \mathcal{M}$ such that $M \cap C$ is infinite because $\mathcal{M}$ is a MADF. But why is $M \in \mathcal{M} \setminus P$?

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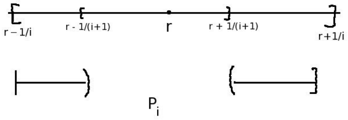

$M \in \mathcal{M} \setminus \mathcal{P} = \mathcal{M}_r$.

(ii) implies (i)

Step 3: Then $\mathcal{M} = \bigcup_{r\in[0,1]} \mathcal{M}_r$ has cardinality $\mathfrak{c}$. Next we construct the desired function f. Define

$$f : \psi(\mathbb{Q} \cap [0,1], \mathcal{M}) \to [0,1]$$

by taking f to be the identity on $\mathbb{Q} \cap [0,1]$, and for $M \in \mathcal{M}$, define $f(M) = r$ where $M \in \mathcal{M}_r$.

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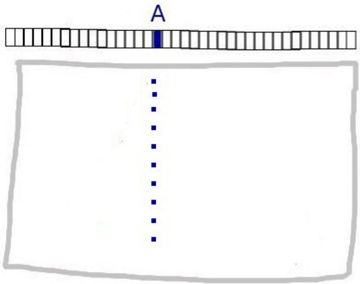
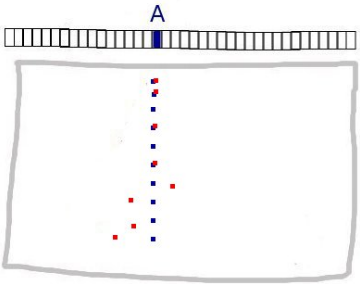
To check continuity of f it suffices to check that whenever a sequence in $\mathbb{Q} \cap [0,1]$ converges to a non-isolated point $A \in \mathcal{M}$, then the image of the sequence converges to the image of the point A in $[0,1]$ (sequentially continuous).

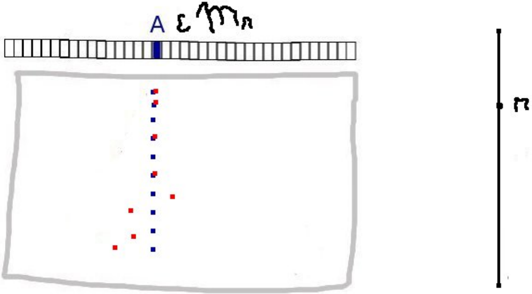
(ii) implies (i)



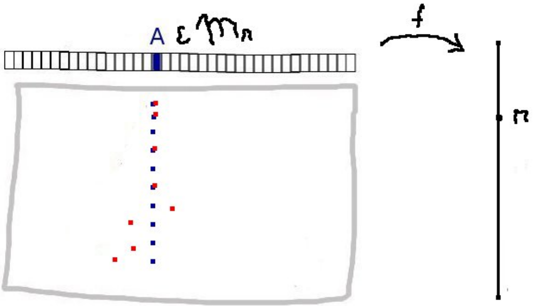
(ii) implies (i)



(ii) implies (i)



(ii) implies (i)



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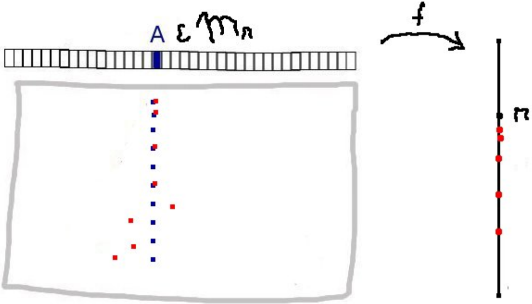

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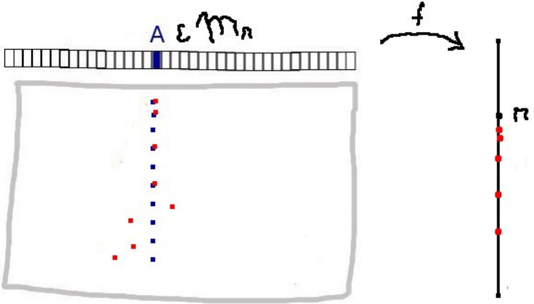

Thus f is sequentially continuous, hence continuous.

(ii) implies (i)

Finally we show that every $r \in [0,1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$.

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Because f is one-to-one on $\mathbb{Q} \cap [0,1]$, the set of isolated points in ψ, and for r irrational, $f^{-1}(r) = \mathcal{M}_r$ and $|\mathcal{M}_r| = \mathfrak{a} < \mathfrak{c}$ (the inequality is the hypothesis of (ii)). Thus $|f^{-1}(r)| \leq \mathfrak{a} < \mathfrak{c}$ for all $r \in [0,1]$. $\square$

Mrówka's papers on $\psi(\omega, \mathcal{M})$ (i.e., $N \cup \mathcal{R}$)

The theorem we call Mrówka's Theorem is contained in his 1977 paper which refers to two of his earlier papers:

- S. Mrówka, On completely regular spaces, Fund. Math. 41 (1954) 105-106.
- S. Mrówka, Some comments on the author's example of an non-R-compact space, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 18 (1970) 443-448.
- S. Mrówka, Some set-theoretic constructions in topology, Fund. Math. 94 (1977) 83-92.

D-V papers on ψ plus Terasawa

- Alan Dow and Jerry E. Vaughan, Mrówka maximal almost disjoint families for uncountable cardinals, Topology and Appl. 157 (2010), 1379-1394.
- Alan Dow and Jerry E. Vaughan, Ordinal Remainders of ψ-spaces, Topology and Appl. 158, Issue 14 (1 September 2011) 1852-1857.
- Alan Dow and Jerry E. Vaughan, Ordinal remainders of classical ψ-spaces, Fund. Math. (to appear).
- Jun Terasawa, Spaces $\mathbf{N} \cup \mathcal{R}$ and their dimensions, Topology and its Applications 11 (1980) 93-102.

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