Indestructibility, Topological Games, Productive Lindelöfness, Selection Principles

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Definition 1

X satisfies $S_1^{\omega}(\mathcal{O},\mathcal{O})$, i.e., X is **Rothberger** if given any sequence of open covers $\{\mathcal{U}_n\}_{n<\omega}$, there are $\{U_n\}_{n<\omega}$, $U_n \in \mathcal{U}_n$, such that $\{U_n\}_{n<\omega}$ is a cover. $G_1^{\omega}(\mathcal{O},\mathcal{O})$ is the game in which in inning n, Player ONE chooses an open cover $\mathcal{U}_n$ and Player TWO picks $U_n \in \mathcal{U}_n$. TWO wins if $\{U_n\}_{n<\omega}$ is a cover; ONE wins otherwise. We write ONE $\not\uparrow G_1^{\omega}(\mathcal{O},\mathcal{O})$ if ONE does not have a winning strategy.

Theorem 2 (Pawlikowski94)

$S_1^{\omega}(\mathcal{O},\mathcal{O})$ *if and only if ONE* $\not\uparrow G_1^{\omega}(\mathcal{O},\mathcal{O})$.

Definition 3

$S_1^{\omega_1}(\mathcal{O},\mathcal{O})$, $G_1^{\omega_1}(\mathcal{O},\mathcal{O})$.

Definition 4

X is **indestructible** if it is Lindelöf in any countably closed forcing extension.

Theorem 5 (Scheepers-Tall10)

X is indestructible if and only if ONE $\not\uparrow G_1^{\omega_1}(\mathcal{O},\mathcal{O})$.

Problem 6 (Arhangel'skiĭ69)

Is it consistent that Lindelöf T_2 spaces with pseudocharacter $\leq \aleph_0$ (i.e., points G_δ) have size $\leq 2^{\aleph_0}$?

Theorem 7 (Tall95)

Con($\exists$ supercompact) $\rightarrow$ Con(indestructible Lindelöf spaces with pseudocharacter $\leq \aleph_0$ have size $\leq 2^{\aleph_0} = \aleph_1$).

Theorem 8 (Tall-Usuba12)

Con($\exists$ *measurable*) $\rightarrow$ *Con*(*indestructible Lindelöf spaces with pseudocharacter* $\leq \aleph_1$ *have size* $\leq 2^{\aleph_0} = \aleph_1$).

Problem 9 (Scheepers-Tall10)

Does $S_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ *imply* $ONE \not\uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.

Theorem 10 (Dias-Tall12)

For the lexicographic order topology on 2^{ω_1}, $ONE \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$, *but CH implies* $S_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.

$\aleph_1$-Borel Conjecture

Theorem 11 (Miller05)

Borel's Conjecture is equivalent to the statement that every Rothberger subset of $\mathbb{R}$ is countable.

Definition 12

A Lindelöf space is **projectively countable** (Arhangel'skiǐ : ω-simple) if whenever $f : X \to \mathbb{R}$, the range of f is countable. Similarly, **projectively** $\aleph_1$ if $f : X \to [0,1]^{\aleph_1}$ implies $|\text{range}\, f| \leq \aleph_1$.

Theorem 13 (Kočinac00; Bonanzinga, Cammaroto, Matveev 10)

Borel's Conjecture holds if and only if Rothberger = projectively countable.

Definition 14

The $\aleph_1$-**Borel Conjecture** is the statement that indestructible = projectively $\aleph_1$.

Theorem 15 (Tall-Usuba12, Usuba, Dias-Tall12)

Con($\exists$ an inaccessible) if and only if Con($\aleph_1$-Borel Conjecture).

Theorem 16

The $\aleph_1$ Borel Conjecture implies projectively countable Lindelöf spaces are projectively $\aleph_1$.

Example 17

The line formed from the cofinal branches of a Kurepa tree with no Aronszajn subtree is Lindelöf, projectively countable, not projectively $\aleph_1$. $V = L$ implies such trees exist.

Productive Lindelöfness

Definition 18

A space X is **productively Lindelöf** if $X \times Y$ is Lindelöf, for every Lindelöf Y. X is **powerfully Lindelöf** if X^ω is Lindelöf.

Theorem 19 (Michael74)

CH implies productively Lindelöf metrizable spaces are σ-compact and hence powerfully Lindelöf.

Problem 20 (Michael)

Is every productively Lindelöf space powerfully Lindelöf?

Problem 21 (Michael)

Is every productively Lindelöf metrizable space σ-compact?

Definition 22

$\mathcal{G}$ is a k-**cover** if every compact subset of X is included in a member of $\mathcal{G}$. X is **Alster** if every k-cover by G_δ's has a countable subcover.

Theorem 23 (Alster88)

a) *Alster implies productively Lindelöf and powerfully Lindelöf.*

b) *CH implies productively Lindelöf spaces of weight $\leq \aleph_1$ are Alster.*

Problem 24 (Alster88)

Is every productively Lindelöf space Alster?

Definition 25

A **Michael space** is a Lindelöf space X such that $X \times \mathbb{P}$ (the space of irrationals) is not Lindelöf.

Problem 26 (Michael63)

Is there a Michael space?

Recent Results

Theorem 27 (Burton-Tall12)
CH implies that if X is productively Lindelöf and $L(X^\omega) \leq \aleph_1$ then X is powerfully Lindelöf.

Corollary 28 (Burton-Tall12)
CH implies that if X is Lindelöf and the union of $\aleph_1$ compact sets, then X is powerfully Lindelöf.

Problem 29
If X is productively Lindelöf, is $L(X^\omega) \leq 2^{\aleph_0}$?

Definition 30

X is a **Frolík space** if X is a closed subspace of a countable product of σ-compact spaces.

Theorem 31 (Burton-Tall12)

There is no Michael space if and only if Frolík spaces are productively Lindelöf.

Theorem 32 (Tall11)

Assume there are infinitely many Woodin cardinals and there exists a Michael space. Then productively Lindelöf projective subsets of $\mathbb{R}$ *are* σ*-compact.*

Connections with Selection Principles

Definition 33

X is **Hurewicz** if for each sequence $\{\mathcal{U}_n\}_{n<\omega}$ of open covers without finite subcovers there are finite $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\mathcal{F} = \{\bigcup \mathcal{F}_n : n < \omega\}$ is a cover and each point is in all but finitely many members of $\mathcal{F}$.

Theorem 34 (Tall11)

Alster implies Hurewicz.

Problem 35 (Aurichi-Tall11)

Is every productively Lindelöf (metrizable?) space Hurewicz?

Theorem 36 (Aurichi-Tall11)

$\mathfrak{d} = \aleph_1$ *implies every productively Lindelöf metrizable space is Hurewicz.*

Theorem 37 (Tall12)

If every productively Lindelöf metrizable space is Hurewicz (Menger), then so is every productively Lindelöf space.

Proof.

Lindelöf projectively Hurewicz (projectively Menger) spaces are Hurewicz (Menger) [Bonanzinga-Cammaroto-Matveev 10], [Arhangel'skiǐ00]. $\square$

Theorem 38 (Alas-Aurichi-Junqueira-Tall11)

$\mathfrak{b} = \aleph_1$ *implies every productively Lindelöf (metrizable) space is Menger.*

$$\begin{array}{ccc}
\mathrm{CH} & \rightarrow & (\text{productively Lindelöf metrizable spaces are } \sigma\text{-compact}) \\
\downarrow & & \downarrow \\
\mathfrak{d} = \aleph_1 & \rightarrow & (\text{productively Lindelöf spaces are Hurewicz}) \\
\downarrow & & \downarrow \\
\mathfrak{b} = \aleph_1 & \rightarrow & (\text{productively Lindelöf spaces are Menger})
\end{array}$$

Recent Improvements

Theorem 39 (Repovs-Zdomskyy12)

$\exists$ *Michael space (this follows from* $\mathfrak{b} = \aleph_1$*)* $\to$ (*productively Lindelöf spaces are Menger*).

Theorem 40 (Repovs-Zdomskyy)

$\mathrm{Add}(\mathcal{M}) = \mathfrak{d} \to$ (*productively Lindelöf spaces are Hurewicz*).

Theorem 41 (Zdomskyy)

$\mathfrak{u} = \aleph_1 \to$ (*productively Lindelöf spaces are Hurewicz*).

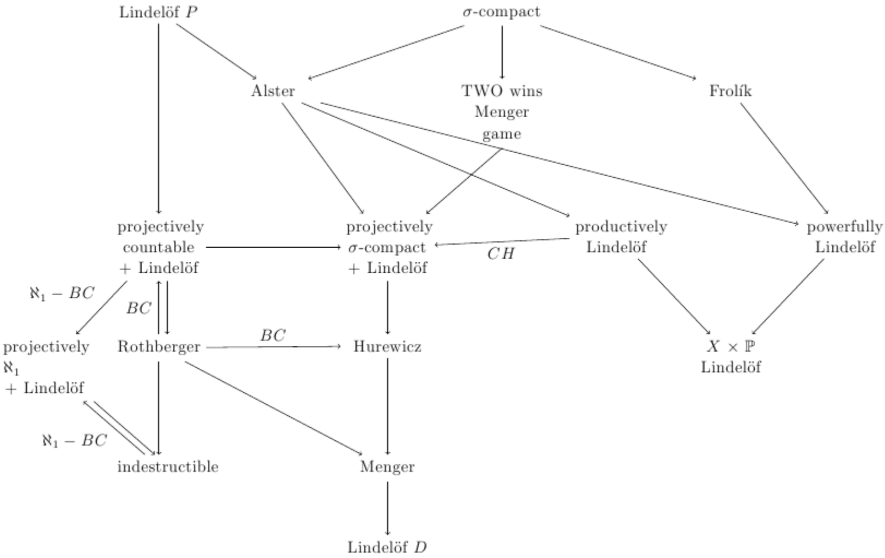
Theorem 42 (Brendle-Raghavan)

$\mathrm{Add}(\mathcal{M}) = \mathfrak{c} < \aleph_\omega \to$ (*productively Lindelöf metrizable spaces are* σ*-compact*).

Theorem 43 (Miller-Tsaban-Zdomskyy)

$\mathfrak{d} = \aleph_1 \to$ (*productively Lindelöf metrizable spaces are productively Hurewicz*).

MAYBE ALL CONCLUSIONS TRUE IN ZFC!


Lindelöf P
σ-compact
Alster
TWO wins Menger game
Frolík
projectively countable + Lindelöf
projectively σ-compact + Lindelöf
productively Lindelöf
powerfully Lindelöf
CH
$\aleph_1 - BC$
BC
projectively $\aleph_1$ + Lindelöf
Rothberger
BC
Hurewicz
$X \times \mathbb{P}$ Lindelöf
$\aleph_1 - BC$
indestructible
Menger
Lindelöf D

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