On a class of Namioka spaces determined by a topological game

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The Problem

$$f : X \times Y \to Z$$

$f_x(y) = f(x, y) : Y \to Z$ is continuous for every $x \in X$;
$f_y(x) = f(x, y) : X \to Z$ is continuous for every $y \in Y$;

What one can say about continuity of $f : X \times Y \to Z$?

Joint continuity versus separate continuity

Example

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

defined by

$$f(x,y) = \frac{x \cdot y}{x^2 + y^2} \text{ if } (x,y) \neq (0,0)$$

$$f(x,y) = 0 \text{ if } (x,y) = (0,0)$$

Some historical remarks on research of the problem

We recall some historical remarks about the works related to this problem. Here we mainly follow the work by Z. Piotrowski
Z. Piotrowski, Separate continuity and joint continuity, Real Analysis Exchange, **11** (1985-1986), pp. 515–531.

Joint continuity versus separate continuity

Example [Genocchi, 1884]

A function continuous along every straight line but discontinuous at $(0,0)$:

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

defined by

$$f(x,y) = \frac{x^2 \cdot y}{x^4 + y^2} \text{ if } (x,y) \neq (0,0)$$

$$f(x,y) = 0 \text{ if } (x,y) = (0,0)$$

Joint continuity versus separate continuity

Theorem [R. Baire, 1899]

Given separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ there is a residual set of lines parallel to each axis consiting of points of continuity.

Theorem [R. Baire, 1899]

Given separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$, then for open disc $U(x_0, y_0)$ and for every $\varepsilon > 0$ there exists a disc $V \subseteq U$ such that $\mid f(x, y) - f(x_0, y_0) \mid < \varepsilon$ for every $(x, y) \in V$.

Joint continuity versus separate continuity

Densifying the set $D(f)$ of points of discontinuity: G.C. Young, W.H. Young (1910)

There exists a function

$$f : [0, 1] \times [0, 1] \to \mathbb{R}$$

which is continuous with respect to every straight line and which has uncountably many points of discontinuity in every rectangle.

Existence problem [Z. Piotrowski, 1985]

Find the essential properties of the set $C(f)$ of points of continuity of a separately continuous function $f : X \times Y \to Z$.

Hypothesis

If X and Y are "nice' then $C(f)$ is "usually" a dense G_δ subset of $X \times Y$

Existence problem

Theorem [F. Topsoe, J. Hoffman-Jorgenson, 1980]

Let X be Hausdorff and Y, Z be metric. If $f : X \times Y \to Z$ is separatedly continuous, then $C(f)$ is residual subset of $X \times Y$ such that all its Y-sections $\{x : (x, y) \in C(f)\}$, $y \in Y$ are residual in X.

Uniformization problem [Z. Piotrowski, 1985]

Uniformization problem

Given a separately continuous function $f : X \times Y \to Z$ find a "uniform" or "thick" subset A of X such that $A \times Y$ is contained in $C(f)$.

Special case of uniformization problem

$(*)$ Given any function $f : X \times Y \to Z$, then there is a dense G_δ subset A of X such that $A \times Y \subset C(f)$

Namioka's theorems

Namioka's theorems [1974]

Let X be a regular complete, Y be locally compact σ-compact, and let Z be pseido-metric, then $(*)$ holds

Namioka's problem

Does (*) holds for every Baire space X, every compact space Y and every metric space Z?

Namioka spaces

Definition [J.P.R. Christensen [1981]]

A space X is called Namioka if $(*)$ holds for any compact space Y and any metric space Z.

Theorem: J. Saint Raymond (1983)

(1) Separable Baire spaces are Namioka.

(1) Tychonoff Namioka spaces are Baire

(2) In the class of metric spaces
X is Namioka if and only if X is Baire.

Christiansen-type topological game 1981–1984

Description of Christensen's game 1 [cf Banah-Mazur game]

Two players A and B in a topological space X. In what follows U, V are open nonempty sets:
B starts by choosing set $U_1 \subset X$. Then A chooses $V_1 \subset U_1$. Then B chooses $U_2 \subset V_1$. Next A chooses $V_2 \subset U_2$, etc. A wins if

$$\bigcap_i V_i = \bigcap_i U_i \neq \emptyset$$

Description of Christensen's game 2

Two players A and B in a topological space X. In what follows U, V are open nonempty sets:
B starts by choosing set $U_1 \subset X$. Then A chooses $V_1 \subset U_1$ and $x_1 \in U_1$. Then B chooses $U_2 \subset V_1, x_1 \notin U_2$. Next A chooses $V_2 \subset U_2, x_2 \notin U_3$, etc. A wins if any subsequence of $x_n : n \in N$ accumulates to at least one point in the set

$$\bigcap_i V_i = \bigcap_i U_i.$$

A-favorable well A-favorable spaces

Definition

A space X is called A-favorable if A has a winning strategy in game 1.

Definition

A space X is called well A-favorable if A has a winning strategy in game 2.

Theorem (J.P.R. Christensen)

Well A-favorable spaces are Namioka.

Example [Talagran, 1985]

There exists an A-favorable space (hence, Baire space) which is not Namioka.

Szymanski-A.Š topological game

Description of the game

Two players A and B in a topological space X; in what follows U, V are open nonempty sets:
B starts by choosing set $U_0 \subset X$. Then A chooses for each $x \in U_0$ a neighborhood $V_1(x) \in U_0$. B responses by choosing $U_1(x)$ in each $V_1(x)$.
Generally $U_1(x)$ *need not contain* x*!*.
Next A chooses one of the sets U_1 constructed by B and for each point $x \in U_1$ constructs a neighborhood $V_2(x) \subset U_1$, etc.,

Description of the game

Gain of A

Player A wins if there is a sequence x_n such that

1. $x_1 \in U_0$ and $x_{n+1} \in U_n(x_n)$ for each n = 1,2,...
2. there exists an accumulation point of the sequence x_n

A. Šostak, A. Szymanski, *On a class of special Namioka spaces*, Topology Proc., **22** 1997, 485–499.

Weakly well A-fabourable spaces are Namioka

Definition

A space X is called weakly well A-favorable if A has a winning strategy.

Theorem A.Szyman'ski – A.Š

Weakly well A-favorable spaces are Namioka.

The class of weakly A-favorable spaces studied by A.Sz. and A.Š

Theorem

A well A-favorable space is weakly well A-favorable.

Theorem

A weaklyl well A-favorable space is Baire.

Theorem

A Baire strict p-space is weakly well A-favorable