

The weak Hurewicz property of Pixley-Roy hyperspaces

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Pixley-Roy hyperspaces

All spaces are **regular**.

$\mathcal{F}[X]$: the space of all nonempty finite subsets of X with the **Pixley-Roy topology**(1969):

for $A \in \mathcal{F}[X]$ and an open set $U \subset X$, let

$$[A, U] = \{B \in \mathcal{F}[X] : A \subset B \subset U\};$$

the family $\{[A, U] : A \in \mathcal{F}[X], U \text{ open in } X\}$ is a base for the Pixley-Roy topology.

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- ① $\mathcal{F}[X]$ is zero-dimensional, completely regular and hereditarily metacompact.
- ② $\mathcal{F}[X]$ is Moore iff $\mathcal{F}[X]$ is first-countable iff X is first-countable.

Three covering properties and their weak versions

Definition 3.1 (Hurewicz 1925, Daniels 1988)

A space X is **Menger** (resp., **weakly Menger**) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\} = X$ (resp., $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\}$ is dense in X).

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Definition 3.2 (Rothberger 1938, Daniels 1988)

A space X is **Rothberger** (resp., **weakly Rothberger**) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are $U_n \in \mathcal{U}_n$ such that $\bigcup\{U_n : n \in \omega\} = X$ (resp., $\bigcup\{U_n : n \in \omega\}$ is dense in X).

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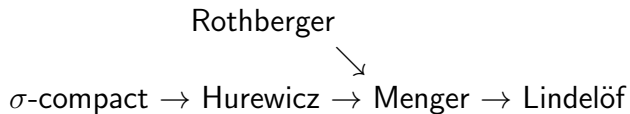
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Definition 3.3 (Hurewicz 1925)

A space X is **Hurewicz** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

Rothberger

σ -compact \rightarrow Hurewicz \rightarrow Menger \rightarrow Lindelöf

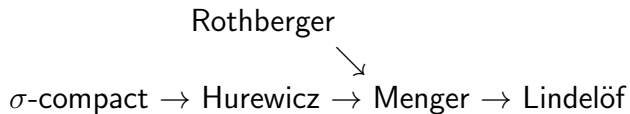


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- 2 \mathbb{C} is not Rothberger.

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- ① \mathbb{P} is not Menger.
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- ③ A Lusin set is Rothberger, but not Hurewicz.
A Sierpiński set is Hurewicz, but not $\sigma\text{-compact}$.

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- ④ In ZFC, $\exists X \subset \mathbb{R}$: Menger, not Hurewicz
(Fremlin and Miller 1984, Chaber and R. Pol 2002).

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- ⑤ In ZFC, $\exists Y \subset \mathbb{R}$: Hurewicz, not σ -compact (Just, Miller, Scheepers and Szeptycki 1996).

van Douwen's problem and Daniels' results

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Are $\mathcal{F}[\mathbb{R}]$ and $\mathcal{F}[\mathbb{P}]$ homeomorphic?

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Theorem 4.2 (Daniels, 1988)

- ① If $\mathcal{F}[X]$ is weakly Menger (resp., weakly Rothberger), then every finite power of X is Menger (resp., Rothberger).
- ② If X is metrizable and every finite power of X is Menger (resp., Rothberger), then $\mathcal{F}[X]^\kappa$ is weakly Menger (resp., weakly Rothberger) for any cardinal κ .

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Corollary 4.3

$\mathcal{F}[\mathbb{P}]$ is not a continuous image of $\mathcal{F}[\mathbb{R}]$.

The weak Hurewicz property in the sense of Kočinac

Recall:

Definition 5.1

A space X is **Hurewicz** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

Definition 5.2 (Kočinac, 2001)

A space X is **weakly Hurewicz in the sense of Kočinac** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are a dense subset $Y \subset X$ and finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every point of Y is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

Kočinac observed (2001): If $\mathcal{F}[X]$ is weakly Hurewicz in the sense of Kočinac, then every finite power of X is Hurewicz.

Question 5.3

Is the converse true for a metrizable space?

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Proposition 5.4

For a space X , the following are equivalent:

- 1 $\mathcal{F}[X]$ is weakly Hurewicz in the sense of Kočinac,
- 2 X is countable.

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- 2 X is countable.

Example 5.5

$\mathcal{F}[\mathbb{C}]$ is not weakly Hurewicz in the sense of Kočinac.

The weak Hurewicz property

Definition 6.1

A space X is **weakly Hurewicz** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that for every nonempty open set $U \subset X$, $U \cap (\bigcup \mathcal{V}_n) \neq \emptyset$ for all but finitely many $n \in \omega$.

weakly Hurewicz in the sense of Kočinac \rightarrow weakly Hurewicz

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$\prod\{X_\alpha : \alpha < \kappa\}$ is weakly Hurewicz, if every finite subproduct of $\{X_\alpha : \alpha < \kappa\}$ is weakly Hurewicz.

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Lemma 6.4 (Kočinac and Scheepers, 2003)

For a space X , the following are equivalent:

- ① *every finite power of X is Hurewicz,*
- ② *for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open ω -covers of X , there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every finite set $F \subset X$ is contained in some member of \mathcal{V}_n for all but finitely many $n \in \omega$.*

Lemma 6.5 (Creede, 1970)

A space (X, τ) is **semi-stratifiable** if and only if there is a function $g : \omega \times X \rightarrow \tau$ such that (i) $\{x\} = \bigcap \{g(n, x) : n \in \omega\}$ for all $x \in X$, (ii) $x \in \bigcap \{g(n, x_n) : n \in \omega\}$ implies $x_n \rightarrow x$.

metrizable \rightarrow stratifiable = monotonically normal + semi-stratifiable

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Theorem 6.6

- ① If $\mathcal{F}[X]$ is weakly Hurewicz, then every finite power of X is Hurewicz.
- ② if X is semi-stratifiable and every finite power of X is Hurewicz, then $\mathcal{F}[X]^\kappa$ is weakly Hurewicz for any cardinal κ .