The weak Hurewicz property of Pixley-Roy hyperspaces

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Pixley-Roy hyperspaces

All spaces are **regular**.
$\mathcal{F}[X]$: the space of all nonempty finite subsets of X with the **Pixley-Roy topology**(1969):
for $A \in \mathcal{F}[X]$ and an open set $U \subset X$, let

$$[A, U] = \{B \in \mathcal{F}[X] : A \subset B \subset U\};$$

the family $\{[A, U] : A \in \mathcal{F}[X], U \text{ open in } X\}$ is a base for the Pixley-Roy topology.

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Fact 2.1

1. *$\mathcal{F}[X]$ is zero-dimensional, completely regular and hereditarily metacompact.*
2. *$\mathcal{F}[X]$ is Moore iff $\mathcal{F}[X]$ is first-countable iff X is first-countable.*

Three covering properties and their weak versions

Definition 3.1 (Hurewicz 1925, Daniels 1988)

A space X is **Menger** (resp., **weakly Menger**) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\} = X$ (resp., $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\}$ is dense in X).

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Definition 3.2 (Rothberger 1938, Daniels 1988)

A space X is **Rothberger** (resp., **weakly Rothberger**) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are $U_n \in \mathcal{U}_n$ such that $\bigcup\{U_n : n \in \omega\} = X$ (resp., $\bigcup\{U_n : n \in \omega\}$ is dense in X).

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Definition 3.3 (Hurewicz 1925)

A space X is **Hurewicz** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

$$\begin{array}{ccccccc} & & \text{Rothberger} & & & & \\ & & & \searrow & & & \\ \sigma\text{-compact} & \to & \text{Hurewicz} & \to & \text{Menger} & \to & \text{Lindelöf} \end{array}$$

Example 3.4

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5. In ZFC, $\exists Y \subset \mathbb{R}$: Hurewicz, not σ-compact (Just, Miller, Scheepers and Szeptycki 1996).

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van Douwen's problem and Daniels' results

Problem 4.1 (van Douwen, 1977)

Are $\mathcal{F}[\mathbb{R}]$ *and* $\mathcal{F}[\mathbb{P}]$ *homeomorphic?*

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Theorem 4.2 (Daniels, 1988)

1. *If $\mathcal{F}[X]$ is weakly Menger (resp., weakly Rothberger), then every finite power of X is Menger (resp., Rothberger).*
2. *If X is metrizable and every finite power of X is Menger (resp., Rothberger), then $\mathcal{F}[X]^\kappa$ is weakly Menger (resp., weakly Rothberger) for any cardinal κ.*

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Corollary 4.3

$\mathcal{F}[\mathbb{P}]$ is not a continuous image of $\mathcal{F}[\mathbb{R}]$.

The weak Hurewicz property in the sense of Kočinac

Recall:

Definition 5.1

A space X is **Hurewicz** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

Definition 5.2 (Kočinac, 2001)

A space X is **weakly Hurewicz in the sense of Kočinac** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are a dense subset $Y \subset X$ and finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every point of Y is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

Kočinac observed (2001): If $\mathcal{F}[X]$ is weakly Hurewicz in the sense of Kočinac, then every finite power of X is Hurewicz.

Question 5.3

Is the converse true for a metrizable space?

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Proposition 5.4

For a space X, the following are equivalent:

1. *$\mathcal{F}[X]$ is weakly Hurewicz in the sense of Kočinac,*
2. *X is countable.*

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For a space X, the following are equivalent:

1. *$\mathcal{F}[X]$ is weakly Hurewicz in the sense of Kočinac,*
2. *X is countable.*

Example 5.5

$\mathcal{F}[\mathbb{C}]$ is not weakly Hurewicz in the sense of Kočinac.

The weak Hurewicz property

Definition 6.1

A space X is **weakly Hurewicz** if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that for every nonempty open set $U \subset X$, $U \cap (\bigcup \mathcal{V}_n) \neq \emptyset$ for all but finitely many $n \in \omega$.

weakly Hurewicz in the sense of Kočinac $\rightarrow$ weakly Hurewicz

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$\prod\{X_\alpha : \alpha < \kappa\}$ *is weakly Hurewicz, if every finite subproduct of* $\{X_\alpha : \alpha < \kappa\}$ *is weakly Hurewicz.*

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Lemma 6.4 (Kočinac and Scheepers, 2003)

For a space X, the following are equivalent:

1. *every finite power of X is Hurewicz,*
2. *for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open ω-covers of X, there are finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that every finite set $F \subset X$ is contained in some member of $\mathcal{V}_n$ for all but finitely many $n \in \omega$.*

Lemma 6.5 (Creede, 1970)

A space (X, τ) *is* **semi-stratifiable** *if and only if there is a function* $g : \omega \times X \to \tau$ *such that (i)* $\{x\} = \bigcap\{g(n, x) : n \in \omega\}$ *for all* $x \in X$*, (ii)* $x \in \bigcap\{g(n, x_n) : n \in \omega\}$ *implies* $x_n \to x$*.*

metrizable $\to$ stratifiable=monotonically normal $+$ semi-stratifiable

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metrizable $\to$ stratifiable=monotonically normal + semi-stratifiable

Theorem 6.6

1. *If* $\mathcal{F}[X]$ *is weakly Hurewicz, then every finite power of* X *is Hurewicz.*
2. *if* X *is semi-stratifiable and every finite power of* X *is Hurewicz, then* $\mathcal{F}[X]^\kappa$ *is weakly Hurewicz for any cardinal* κ*.*