Topological games and perturbed optimization problems

Julian P. Revalski

Bulgarian Academy of Sciences

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Variational principles in optimization

Setting

X is a completely regular topological space;
$f : X \to \mathbf{R} \cup \{+\infty\}$ is a given proper, lsc and bounded below function;
Y is a (complete) metric space of (bounded) continuous functions in X (perturbations);

Question

How big is the set
$S(f) = \{g \in Y : f + g$ attains its minimum in X (or $f + g$ is well-posed)$\}$

- How big means: nonempty? dense in Y? residual in Y?
- $S(f)$ dense: *Variational principle for* f;
- $S(f)$ residual: *Generic variational principle for* f;
- $f + g$ is (Tykhonov) well-posed if it has unique minimum towards which converges every minimizing sequence for $f + g$.

Examples

Bishop-Phelps theorem-the Grandfather

X is a Banach space, $f \equiv i_{B_X}$, $Y = X^*$

Ekeland variational principle

X is a complete metric space, Y "translations" of the distance.

Stegall variational principle

X is a subset of a Banach space E with certain properties, $Y = E^*$.

Smooth principles of Borwein-Preiss and of Deville-Godefroy-Zizler

X is a Banach space and Y a family of smooth functions.

Continuous generic principles

f is continuous, Y is the family $C(X)$ of all bounded and continuous functions in X, endowed with the sup norm.
Lucchetti-Patrone, De Blasi-Myjak, Čoban, Kenderov, Revalski

Applications of variational principles

- Optimization (optimality conditions);
- Fixed point theory (existence);
- Optimal control;
- Hamilton-Jacobi equations (viscosity solutions);
- Geometry of Banach spaces;
- Differentiability properties of (convex) functions in Banach spaces;
- Critical point theory.

Sufficient (and necessary) conditions for validity

- Ekeland principle: completeness of the metric space X;
- Stegall principle: Radon-Nikodym property of the set X (X is a subset of a Banach space E and $Y = E^*$);
- Borwein-Preiss and Deville-Godefroy-Zizler smooth principles: enough rich family of smooth functions in the Banach space X (existence of bump functions);
- continuous principles: almost completeness properties (e.g. X must contain a dense completely metrizable subspace, or more general, topological games have to be involved)

A general variational principle (Kenderov-Revalski, 2010)

Lemma

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which is bounded from below. Let $x_0 \in \mathrm{dom}(f)$ and $\varepsilon > 0$ be so that $f(x_0) < \inf_X f + \varepsilon$. Then, there exists a continuous function $g \in C(X)$ such that $\|g\|_\infty \leq \varepsilon$ and the function $f + g$ attains its infimum at x_0.

Corollary

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which is bounded from below. Then the set $\{g \in C(X) : f + g$ attains its infimum on $X\}$ is dense in $C(X)$.

Corollary

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which is bounded from below. Then for every $x_0 \in \mathrm{dom}(f)$ there is $g \in C(X)$ such that x_0 is a minimizer for the perturbed function $f + g$.

Topological games: The Banach-Mazur game

Setting and rules of playing

Given a nonempty set S of a completely regular topological space X, two players, α and β, play a game, by choosing, alternating, nonempty open sets, U_n for β (who starts the game) and V_n for α so that $U_{n+1} \subset V_n \subset U_n$ for any $n \geq 1$. The so obtained sequence $\{U_n, V_n\}_{n\geq 1}$ is called a *play*.

Winning rule

α wins the play $\{U_n, V_n\}_{n\geq 1}$ if $\cap_n U_n = \cap_n V_n \subset S$. Otherwise β wins.

Notation: usually, $BM(X, S)$.
A well-known variant: denoted by $BM(X)$, is when the players are making their choices as above and the player α wins the corresponding play $\{U_n, V_n\}_{n\geq 1}$ if $\cap_n U_n = \cap_n V_n \neq \emptyset$. Otherwise β wins.

Appeared for the first time in the Scottish Book, 1936, as problem No. 43 posed by Mazur (unpublished answer by Banach).

The Banach-Mazur game: Winning strategies

A *strategy* for α in $BM(X,S)$ is a mapping s which assigns to every possible chain $(U_1, V_1, \ldots, U_n)$ corresponding to the first n legal moves of β and the first $n-1$ moves of α, $n \geq 1$, a non-empty open set $V_n \subset U_n$. The play $p = \{U_n, V_n\}_{n=1}^{\infty}$, where $V_n = s(U_1, V_1, \ldots, U_n)$ for every $n \geq 1$ is called an *s-play*.

The strategy s is *winning* for α if α wins every s-play $p = \{U_n, V_n\}_{n=1}^{\infty}$ in the game $BM(X,S)$ (i.e. $\cap_n U_n \subset S$).

The concept of a (winning) strategy for β in $BM(X,S)$ as well as the corresponding notions in the variant $BM(X)$ are defined analogously.

The Banach-Mazur game: some basic facts

Oxtoby (Answer to Mazur's original question)

• α has a winning strategy $BM(X,S)$ if and only if S is residual in X;
• if X is a complete metric space, β has a winning strategy in $BM(X,S)$ if and only if S is of the first Biare category in some open set of X.

Baire spaces and Banach-Mazur game

β does not have a winning strategy in $BM(X)$ exactly when X is a Baire space (Krom, Oxtoby).

Determiness

These games are not necessarily *determined*, that is, the absence of a winning strategy for one of the players does not imply the existence of a winning one for the opponent player.

α-favorable spaces

complete metric spaces, (locally) compact spaces, ...

The Banach-Mazur game $BM(X)$ and continuous variational principles

Existence of solutions (Kenderov and Revalski)

If $f : X \to \mathbf{R}$ is a continuous bounded function, then the set $\{g \in C(X) : f + g \text{ attains its minimum in } X\}$ contains a dense G_δ-subset of $C(X)$ if and only if the player α has a winning strategy in the Banach-Mazur game $BM(X)$.

Well-posedness (Čoban, Kenderov and Revalski)

A special kind of winning strategies for the player α in the Banach-Mazur game $BM(X)$ are needed if we want also that the perturbations $f + g$ are well-posed.

Another topological game

The game

Two players, Σ and Ω, play in X by choosing alternatively, nonempty sets (Σ starts) A_n for Σ and B_n for Ω so that for any $n \geq 1$, $A_{n+1} \subset B_n \subset A_n$ and B_n must be relatively open subset of A_n.
Ω wins the play $\{A_n, B_n\}_{n \geq 1}$ if $\cap_n \overline{A}_n = \cap \overline{B}_n \neq \emptyset$. Otherwise Σ wins. We will denote this game by $G(X)$.
The game $G(X)$ (together with other, stronger winning conditions) was used by Michael to characterize metric spaces with completeness properties.

Strategies

The notions of (winning) strategy for Ω or Σ are defined as in the case of the Banach-Mazur game.

Relation with the Banach-Mazur game $BM(X)$

The game $G(X)$ can be considered as a generalization of the Banach-Mazur game $BM(X)$:

Proposition

Let X be a completely regular topological space which admits a winning strategy for the player Ω in the game $G(X)$. Then X admits also a winning strategy for the player α in the Banach-Mazur game $BM(X)$.

Ω-favorable spaces

complete metric spaces, (locally) compact spaces, countably compact spaces, Čech complete spaces, ... ; stable under open continuous mappings.

The game $G(X)$ and variational principles

Existence of minima

Theorem

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ *be a proper lower semicontinuous function which is bounded from below in* X. *Let the player* Ω *have a winning strategy* ω *in the game* $G(X)$. *Then the set* $\{g \in C(X) : f + g$ *attains its infimum in* $X\}$ *contains a dense* G_δ*-subset of* $C(X)$.

Question

Is the condition of existence of a winning strategy also necessary?—Open.

The game $G(X)$ and variational principles

Well-posedness of the perturbations

Theorem

Let X be a completely regular topological space for which there exists a winning strategy ω for the player Ω in the game $G(X)$ with the additional property that for any ω-play $\{A_n, B_n\}_{n\geq 1}$ the intersection $\cap_n \overline{A}_n = \cap_n \overline{B}_n$ is a singleton, say $x \in X$, and $\{B_n\}_{n\geq 1}$ behaves as a base for x. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function in X which is bounded from below. Then the set $\{g \in C(X) : f + g \text{ is well-posed }\}$ contains a dense G_δ-subset of $C(X)$.

Question

Is the condition of existence of such a winning strategy also necessary?—Again open.

Fragmentability

Fragmentable spaces

A topological space X is *fragmentable* (Jayne and Rogers) if there is a metric d in X such that for any nonempty set A of X and any $\varepsilon > 0$ there exists a nonempty relatively open subset B of A such that $\mathrm{diam}(B) := \sup\{d(x, y) : x, y \in B\} < \varepsilon$.

Examples

Metric spaces, Eberlein compacta, bounded subsets of some dual Banach spaces (the latter with the weak and weak start topologies).

Applications

Useful in the study of generic single-valuedness of set-valued mappings, in the study of differentiability of convex functions and in the geometry of Banach spaces

Fragmentability and topological games

The fragmenting game $FG(X)$

The same setting as the game $G(X)$: two players, Σ and Ω play exactly as in $G(X)$ (i.e. $A_{n+1} \subset B_n \subset A_n$ and B_n is a nonempty relatively open subset of A_n for any $n \geq 1$). Ω wins the play $\{A_n, B_n\}$ if the intersection $\cap_n A_n$ is no more than a singleton (may be empty!). Otherwise Σ wins.

Characterization: Kenderov and Moors

The space X is fragmentable if and only if the player Ω has a winning strategy in the fragmenting game $FG(X)$.

Fragmentability and variational principles

Corollary

Let X be a completely regular topological space which is fragmented by a metric d whose topology contains the initial topology in X and d is conditionally complete (i.e. each d-Cauchy sequence converges in the initial topology). Then for any proper lower semicontinuous function $f : X \to \mathbf{R} \cup \{+\infty\}$ which is bounded from below the set $\{g \in C(X) : f + g \text{ is well-posed } \}$ contains a dense G_δ-subset of $C(X)$.

Nonexistence of winning strategies

Theorem

Let X be a completely regular topological space which does not admit a winning strategy for the player Σ in the game $G(X)$. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinous function which is bounded from below. Then, the set $\{g \in C(X) : f + g$ attains its infimum in $X\}$ is everywhere of the second Baire category in $C(X)$.

Let $G'(X)$ be the game played as $G(X)$ in which Ω wins the corresponding play $\{A_n, B_n\}_{n\geq 1}$ if the intersection $\cap_n \overline{A}_n = \cap_n \overline{B}_n$ is a singleton, say $x \in X$, and $\{B_n\}_{n\geq 1}$ behaves as a base for x. Otherwise Σ wins the play.

Theorem

Let X do not admit a winning strategy for the player Σ in the game $G'(X)$. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinous function which is bounded from below. Then, the set $\{g \in C(X) : f + g$ is well-posed$\}$ is everywhere of the second Baire category in $C(X)$.

Nondeterminess: Examples

Example

Let X be a Bernstein subset of the unit interval with the inherited topology. It is known that no one of the players in the Banach-Mazur game $BM(X)$ has a winning strategy (X is Baire and the results on generic continuous principles). It can be seen that X does not admit a winning strategy for none of the players in the game $G(X)$ either.

Example

There is a compact space X such that the sets
$D_1 = \{g \in C(X) : g$ *attains its minimum at exactly one point*$\}$ *and*
$D_2 = \{g \in C(X) : g$ *attains its minimum at exactly two points* $\}$ *are everywhere of second Baire category in* $(C(X), \|\cdot\|_\infty)$*. In particular, none of the players in the fragmenting game* $FG(X)$ *possesses a winning strategy.*

Another tool: set-valued analysis

The solution mapping

Let X be a completely regular topological space and let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous proper extended real-valued function. Then the solution mapping $M_f : C(X) \rightrightarrows X$ is defined as:

$$M_f(g) := \{x \in X : (f+g)(x) \leq (f+g)(y) \quad \forall y \in X\}, \; g \in C(X).$$

The properties of the solution mapping play fundamental role in the proofs of the results presented so far.

Properties of M_f:

(a) $\mathrm{Gr}(M_f)$ is closed in the product topology in $C(X) \times X$;

(b) $\mathrm{Dom}(M_f)$ is dense in $C(X)$;

(c) M_f maps $C(X)$ onto $\mathrm{dom}(f)$;

(d) for any two opens sets U of $C(X)$ and V of X such that $M_f(U) \cap V \neq \emptyset$ there is a nonempty open set $U' \subset U$ such that $M_f(U') \subset V$ (important property which is called *minimality* or *quasi-continuity*);

(e) if $(U_n)_{n\geq 1}$ is a base of neighborhoods of $g_0 \in C(X)$ then $M_f(g_0) = \cap_n \overline{M(U_n)}$;

(f) if the restriction of f on $\mathrm{dom}(f)$ is continuous then M_f is relatively open: for any open set U of $C(X)$ the set $M_f(U)$ is a (nonempty) relatively open subset of $\mathrm{dom}(f)$.