# A connected, locally connected infinite metric space without separable sets of positive dimension

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Examples of non-separable metrizable connected spaces whose all nonempty connected separable subspaces are singletons were given by:

Roman Pol, 1975,
P.Simon, 2001,
M.Morayne and M.R.Wójcik, 2010,
T.Banakh, M.Vovk and M.R.Wójcik, 2011 (completely metrizable space)

Question(Banakh, Vovk, Wójcik): Does there exist such a space which is locally connected?

**EXAMPLE (E.Pol and R.Pol).**
**A connected, locally connected infinite metrizable space $E$ without separable sets of positive dimension.**

Note that by a classical result of Menger, completely metrizable connected and locally connected infinite spaces contain arcs.

Our construction is a modification of the example given by Roman Pol in 1975 which, in turn, combined the Knaster-Kuratowski Fan construction with basic properties of a certain non-separable set defined by A.H.Stone in 1963.

**Condensed Baire fan $C(\aleph_1)$.**

Put $\mathbb{N} = \{1, 2, \ldots\}$ and let $B(\aleph_1) = D(\aleph_1)^{\mathbb{N}}$ be the countable product of a discrete space $D(\aleph_1)$ of cardinality $\aleph_1$, i.e. the Baire space of weight $\aleph_1$. For $n \in \mathbb{N}$ let

$p_n : B(\aleph_1) \to D(\aleph_1)^n,\ p_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n)$, be the projections onto the first $n$ coordinates. Additionally, $p_0 : B(\aleph_1) \to \{0\}$ maps the Baire space to zero.

Let $\mathcal{V}_n = \{p_n^{-1}(z) : z \in D(\aleph_1)^n\}$, for $n \in \mathbb{N}$, and $\mathcal{V}_0 = \{B(\aleph_1)\} = \{p_0^{-1}(0)\}$.

Then $\mathcal{V}_n$ is a disjoint cover of $B(\aleph_1)$ consisting of open and closed sets and $\mathcal{V}_{n+1} \prec \mathcal{V}_n$.

Let $\mathbb{Q} = \{q_0 = 1, q_1, q_2, \ldots\}$ be the rational numbers from the interval $I$ and let
$B(\aleph_1) \times I/\sim\, = C(\aleph_1)$
be the set of equivalence classes with respect to the relation $\sim$ in the product $B(\aleph_1) \times I$, where $(x, t) \sim (y, s) \iff ((x, t) = (y, s))$ or ($t = s = q_n$ and $p_n(x) = p_n(y)$, for some $n$). Thus, $C(\aleph_1)$ is obtained from $B(\aleph_1) \times I$ by identifying to a point every set of the form $V \times \{q_n\}$, for $V \in \mathcal{V}_n$ $(n = 0, 1, 2, \ldots)$.

We denote by
$\pi : B(\aleph_1) \times I \to B(\aleph_1) \times I/\sim\, = C(\aleph_1)$
the map which associates to a point in $B(\aleph_1) \times I$ its equivalence class with respect to the relation $\sim$.

The quotient topology is non-metrizable, hence not suitable for our purpose, and we shall equip $C(\aleph_1)$ with the following coarser topology $\mathcal{T}$.

We shall say that an open rectangle $U \times V$ in $B(\aleph_1) \times I$ is saturated, if it is saturated with respect to $\sim$, and let

$\mathcal{B} = \{U \times V :\ U \times V = \pi^{-1}\pi(U \times V)$ is open in $B(\aleph_1) \times I\}$

be the collection of open saturated rectangles in $B(\aleph_1) \times I$.

The collection $\pi(\mathcal{B}) = \{\pi(U \times V) : U \times V \in \mathcal{B}\}$

is closed under finite intersections, and hence it is a basis for a topology $\mathcal{T}$ in $C(\aleph_1)$. The topological space

$(C(\aleph_1), \mathcal{T})$ **is the condensed Baire fan.**

In the sequel, we shall omit $\mathcal{T}$, denoting the condensed fan simply by $C(\aleph_1)$.
The map $\pi : B(\aleph_1) \times I \to C(\aleph_1)$ is continuous.

**The condensed Baire fan $C(\aleph_1)$ is metrizable.**

To check this, one can apply the Moore metrization theorem, i.e.,
it suffices to show that
$C(\aleph_1)$ is a $T_0$-space and
$C(\aleph_1)$ has a strong development $\mathcal{U}_1, \mathcal{U}_2, \ldots$
(i.e., there are open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of $C(\aleph_1)$ such that for every point $x \in C(\aleph_1)$ and every neighbourhood $U$ of $x$ there exist a neighbourhood $V$ of $x$ and a natural number $n$ such that $St(V, \mathcal{U}_n) \subset U$).

Indeed, it is easy to check that the sequence of covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$, where

$\mathcal{U}_n = \bigcup_{k=0}^{n-1} \{\, \pi(U \times V) : U \times V \in \mathcal{B}\,, q_k \in V,$
diam$V < \frac{1}{n}$ and $p_k(U)$ is a singleton $\} \cup$
$\{\pi(U \times V) : U \times V \in \mathcal{B}\,,$ diam$V < \frac{1}{n}$ and
$p_n(U)$ is a singleton $\}$,

is a strong development in the condensed Baire fan $C(\aleph_1)$.

## THE EXAMPLE.

**A connected, locally connected infinite metrizable space $E$ without separable sets of positive dimension.**

A.H.Stone, 1963: There exists a set $S \subset B(\aleph_1)$ such that

$(*)$ each separable subspace of $S$ is countable, and
$(**)$ $S \cap U$ is non-Borel for each nonempty open $U$ in $B(\aleph_1)$,

Let $\mathbb{P} = I \setminus \mathbb{Q}$ be the irrationals. We fix a countable dense set $T \subset \mathbb{P}$, and let

$K = (S \times (\mathbb{P} \setminus T)) \cup ((B(\aleph_1) \setminus S) \times (\mathbb{Q} \cup T)) \subset B(\aleph_1) \times I$, and

$E = \pi(K) \subset C(\aleph_1)$ be a subspace of the condensed Baire fan.

**If $X$ is a separable subspace of $E$, then $X$ is 0-dimensional.**

We have $X = X_1 \cup X_2$, where
$X_1 = \pi(S \times (I \setminus T)) \cap X$ and
$X_2 = \pi((B(\aleph_1) \setminus S) \times (\mathbb{Q} \cup T)) \cap X$.

Since $X_1$ is separable, then there is a separable set $A \subset S$ such that
$X_1 \subset \pi(A \times (I \setminus T)) = \bigcup_{a \in A} \pi(\{a\} \times (I \setminus T))$.
But $A$ is countable by $(*)$ and every set $\pi(\{a\} \times (I \setminus T)) = \pi(\{a\} \times I)$ is 0-dimensional and closed in $X$. Thus $X_1$ is a countable union of 0-dimensional sets which are closed in $X$.

Moreover, $\pi((B(\aleph_1) \setminus S) \times (\mathbb{Q} \cup T))$ is a countable union of closed in $E$ zero-dimensional sets $\pi((B(\aleph_1) \setminus S) \times \{q\})$, $q \in \mathbb{Q} \cup T$, hence $X_2$ is a countable union 0-dimensional sets which are closed in $X$.

Thus $X = X_1 \cup X_2$ is 0-dimensional.

**The space $E$ is locally connected and connected.**

Let $\mathcal{P} = \{(a,b) \cap I \ : \ a,b \in \mathbb{R}, a < b\}$ be the set of intervals in $I$ and, for $i \in \mathbb{N} \cup \{0\}$, let $\mathcal{G}_i = \{ \ \pi(V \times J) \cap E \ : \ V \in \mathcal{V}_i, \ J \in \mathcal{P}$ and $J \cap \{q_0, \ldots, q_i\} = \{q_i\}\}$.

Then the family $\mathcal{G} = \bigcup_i \mathcal{G}_i$ is a base of open sets in $E$ and consists of connected sets.

It is easy to see that $\mathcal{G}$ is a base of $E$. The proof that every set $G \in \mathcal{G}$ is connected, is similar to the proof that the Knaster - Kuratowski Fan is connected, and uses the fact that by the property $(**)$ of the Stone's set $S$, $S \cap U$ is not an $F_\sigma$-set for each nonempty open $U$ in $B(\aleph_1)$.

**Definition.** A space $X$ is **economically metrizable** if the topology of $X$ can be generated by a metric $d$ such that for each infinite subset $A \subset X$ the set $d(A \times A) = \{d(a,b) : a, b \in A\}$ has cardinality $|\, d(A \times A) \,| \leq dens(A)$.

T.Banakh, M.Vovk and M.R.Wójcik (2011):
**If a space $X$ is economically metrizable then each subspace of density less than $2^{\aleph_0}$ is zero-dimensional.**

The following example shows that under CH this proposition cannot be reversed.

**There exists a metrizable space $M$ of weight $\aleph_1$ such that each separable subspace $A$ of $M$ is zero-dimensional, but $M$ is not economically metrizable.**

Indeed, let $M = K \cup (B(\aleph_1) \times \{1\}) \subset B(\aleph_1) \times I$, where $K = (S \times (\mathbb{P} \setminus T)) \cup ((B(\aleph_1) \setminus S) \times (\mathbb{Q} \cup T))$ is a space defined earlier.

Then the weight of $M$ is $\aleph_1$ and each separable subspace $A$ of $M$ is zero-dimensional.

It may be interesting to mention that the examples listed before answer some questions which arose naturally in mathematical economics and concern the notion of separably connected spaces.

A space $X$ is *separably connected*, if any two points of $X$ can be contained in a connected separable subset of $X$.

In their work on utility theory,
J.Candeal, C.Hervés and E.Induráin (Journal of Mathematical Economics 29 (1998))
asked if there exist connected metric spaces which are not separably connected.

All the examples mentioned at the beginning of my talk give the positive answer to this question.