Some selection properties related to bornologies

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We study some closure-type properties of the function spaces Y^X and $C(X,Y)$ endowed with two topologies: the topology $\tau_{\mathfrak{B}}$ of uniform convergence on a bornology $\mathfrak{B}$ on X and the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$.

The study of function spaces with the strong uniform topology on a bornology was initiated by G. Beer and S. Levi in 2009 and continued by A. Caserta, G. Di Maio, L'. Holá (2010), A. Caserta, G. Di Maio, Lj.D.R. Kočinac (2012).

Introduction

If (X,d) is a metric space, $x \in X$, $A \subset X$ and $\varepsilon > 0$ a real number, we write

$$S(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\},$$
$$A^{\varepsilon} := \bigcup_{a \in A} S(a,\varepsilon),$$

to denote the **open ε-ball** with center x and the **ε-enlargement** of A.

Bornology

A **bornology** on a metric space (X, d) is a family $\mathfrak{B}$ of nonempty subsets of X which is closed under finite unions, hereditary (i.e. closed under taking nonempty subsets) and forms a cover of X (Hu, 1948).

We suppose that X does not belong to a bornology $\mathfrak{B}$ on X.

A **base** for a bornology $\mathfrak{B}$ on (X, d) is a subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ which is cofinal in $\mathfrak{B}$ with respect to the inclusion, i.e. for each $B \in \mathfrak{B}$ there is $B_0 \in \mathfrak{B}_0$ such that $B \subset B_0$. A base is called *closed* (*compact*) if all its members are closed (compact) subsets of X.

Examples

Examples of bornologies on a metric space (X, d):

1. The family $\mathfrak{F}$ of all nonempty finite subsets of X (the smallest bornology on X and has a closed, in fact a compact, base);
2. The family of all nonempty subsets of X (the largest bornology on X);
3. The collection $\mathcal{K}_r$ of all nonempty relatively compact subsets (i.e. subsets with compact closure);
4. d-bounded subsets of X;
5. Totally d-bounded subsets of X.

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Strong uniform continuity

A mapping $f : X \to Y$ from a metric space (X, d) to a metric space (Y, ρ) is **strongly uniformly continuous** on a subset B of X if for each $\varepsilon > 0$ there is $\delta > 0$ such that $d(x_1, x_2) < \delta$ and $\{x_1, x_2\} \cap B \neq \emptyset$ imply $\rho(f(x_1), f(x_2)) < \varepsilon$. (Beer and Levi, 2009)

If $\mathfrak{B}$ is a bornology on X, then $f : X \to Y$ is called *strongly uniformly continuous on* $\mathfrak{B}$ if f is strongly uniformly continuous on B for each $B \in \mathfrak{B}$.

Beer and Levi also defined a new topology on the set Y^X of all functions from X into Y – the topology of strong uniform convergence. They initiated the study of function spaces Y^X and $C(X, Y)$ with this new topology.

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Function Spaces

Let (X, d) and (Y, ρ) be metric spaces and $\mathfrak{B}$ a bornology on X. By $\tau_{\mathfrak{B}}$ we denote the *topology of uniform convergence on* $\mathfrak{B}$ generated by a uniformity on Y^X having as a base the sets of the form

$$[B, \varepsilon] := \{(f, g) : \forall x \in B, \rho(f(x), g(x)) < \varepsilon\} \ (B \in \mathfrak{B}, \varepsilon > 0).$$

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Topology of strong uniform convergence

For given metric spaces (X, d) and (Y, ρ) and a bornology $\mathfrak{B}$ **with closed base** on X the *topology of strong uniform convergence on* $\mathfrak{B}$, denoted by $\tau^s_{\mathfrak{B}}$, is determined by a uniformity on Y^X having as a base the sets of the form

$$[B, \varepsilon]^s := \{(f, g) : \exists\delta > 0\, \forall x \in B^\delta, \rho(f(x), g(x)) < \varepsilon\}.$$

Function Spaces

For a function $f \in (\mathsf{C}(X,Y), \tau_{\mathfrak{B}})$ the standard local base of f is the collection of sets

$$[B,\varepsilon](f) = \{g \in (\mathsf{C}(X,Y) : \rho(g(x), f(x)) < \varepsilon, \ \forall x \in B\}$$

while in $(\mathsf{C}(X,Y), \tau^s_{\mathfrak{B}})$ the standard local base of f is the collection of sets

$$[B,\varepsilon]^s(f) = \{g \in (\mathsf{C}(X,Y), \tau^s_{\mathfrak{B}}) : \exists \delta > 0,\ \rho(g(x), f(x)) < \varepsilon,\ \forall x \in B^{\delta}\}.$$

For each bornology $\mathfrak{B}$ with closed base on X the topology $\tau^s_{\mathfrak{B}}$ on Y^X is finer than the classical topology $\tau_{\mathfrak{B}}$ of uniform convergence on $\mathfrak{B}$, and if $\mathfrak{B}$ has a compact base, then $\tau^s_{\mathfrak{B}} = \tau_{\mathfrak{B}} \leq \tau_k$ on $\mathsf{C}(X,Y)$.In particular, $\tau_p \leq \tau^s_{\mathfrak{F}} \leq \tau^s_{\mathfrak{B}} \leq \tau^s_{\mathfrak{K}} = \tau_k$ on $\mathsf{C}(X)$

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$\mathfrak{B}$-cover

For a bornology $\mathfrak{B}$ on a space (X, d) an open cover $\mathcal{U}$ is called a $\mathfrak{B}$-cover if each element in $\mathfrak{B}$ is contained in a member of $\mathcal{U}$ and $X \notin \mathcal{U}$.

$\mathfrak{B}^s$-cover

An open cover $\mathcal{U}$ of a metric space (X, d) with a bornology $\mathfrak{B}$ is said to be a *strong* $\mathfrak{B}$*-cover* of X (or shortly a $\mathfrak{B}^s$*-cover* of X) if $X \notin \mathcal{U}$ and for each $B \in \mathfrak{B}$ there exist $U \in \mathcal{U}$ and $\delta > 0$ such that $B^\delta \subset U$.(Caserta, Di Maio, Holá)

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$\mathfrak{B}^s$ cover

Open covers

$\mathcal{O}_{\mathfrak{B}}$: The collection of all $\mathfrak{B}$-covers

$\mathcal{O}_{\mathfrak{B}^s}$: The collection of all strong $\mathfrak{B}$-covers of a space

$(\mathfrak{B}^s, \mathfrak{B})$-Lindelöf: Each $\mathfrak{B}^s$-cover contains a countable $\mathfrak{B}$-subcover.

$\Gamma_{\mathfrak{B}^s}$: The collection of all (countable) $\gamma_{\mathfrak{B}^s}$-covers of X.

A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be a **$\gamma_{\mathfrak{B}^s}$-cover** if it is infinite and for each $B \in \mathfrak{B}$ there are $n_0 \in \mathbb{N}$ and a sequence $(\delta_n : n \geq n_0)$ of positive real numbers such that $B^{\delta_n} \subset U_n$ for all $n \geq n_0$.

$\mathcal{U}$ is a $\gamma_{\mathfrak{B}}$-cover if each member B of $\mathfrak{B}$ belongs to U_n all but finitely many n.

(X, τ_1, τ_2), written simply X, will be a bitopological space (shortly bispace), i.e. the set X endowed with two topologies τ_1 and τ_2. For a subset A of X, $\mathrm{Cl}_i(A)$ will denote the closure of A in (X, τ_i), $i = 1, 2$.

A bispace (X, τ_1, τ_2) has countable **(τ_i, τ_j)-tightness** ($i \neq j$; $i, j = 1, 2$) if for each $A \subset X$ and each $x \in Cl_{\tau_i}(A)$ there is a countable $C \subset A$ such that $x \in Cl_j(C)$.

Theorem

Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X with closed base. The following are equivalent:

(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ has countable $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ tightness;

(2) X is a $(\mathfrak{B}^s, \mathfrak{B})$-Lindelöf space.

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Let $\mathcal{A}$ and $\mathcal{B}$ be sets consist of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$:
For each sequence $(A_n : n \in \mathbb{N})$ *of elements of* $\mathcal{A}$ *there is a sequence* $(b_n : n \in \mathbb{N})$ *such that for each* n, $b_n \in A_n$, *and* $\{b_n : n \in \mathbb{N}\}$ *is an element of* $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$:
For each sequence $(A_n : n \in \mathbb{N})$ *of elements of* $\mathcal{A}$ *there is a sequence* $(B_n : n \in \mathbb{N})$ *of finite sets such that for each* n, $B_n \subset A_n$, *and* $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

For a space X and a point $x \in X$ the symbol Ω_x denotes the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$, and Σ_x is the set of sequences converging to x.

Classical selection principles

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consist of families of subsets of an infinite set X. Then:

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Countable fan tightness

A space X has **countable fan tightness** if for each $x \in X$ we have that $\mathsf{S}_{fin}(\Omega_x, \Omega_x)$ holds.

X has *countable strong fan tightness* if for each $x \in X$ the selection principle $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds.

Lemma

Let $\mathcal{U}$ be a $\mathfrak{B}^s$-cover of X. Set $A = \{f \in \mathsf{C}(X) : \exists U \in \mathcal{U}, f(x) = 1$ for all $x \in X \setminus U\}$. Then $\underline{0} \in \mathrm{Cl}_{\tau_{\mathfrak{B}^s}}(A)$.

Let (X, τ_1, τ_2) be a bispace.

(τ_i, τ_j)-fan tightness

X has countable (τ_i, τ_j)-*fan tightness* $(i \neq j;\ i, j = 1, 2)$ if for each $x \in X$ and each sequence $< A_n : n \in \mathbb{N} >$ of subsets of X such that $x \in Cl_i(A_n)$ for each $n \in \mathbb{N}$, there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, with $x \in Cl_j\left(\bigcup_{n \in \mathbb{N}} F_n\right)$, i.e. if for each $x \in X$, $\mathsf{S}_{fin}(\Omega_x^{\tau_i}, \Omega_x^{\tau_j})$ is satisfied.

Theorem

Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X. The following are equivalent:

(1) $(\mathrm{C}(X), \tau_{\mathfrak{B}}^s, \tau_{\mathfrak{B}})$ has countable $(\tau_{\mathfrak{B}}^s, \tau_{\mathfrak{B}})$-fan tightness;

(2) X satisfies $\mathsf{S}_{fin}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}})$.

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(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ has countable strong fan tightness;

(2) X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}})$.

(τ_i, τ_j)-Fréchet-Urysohn

A bispace (X, τ_1, τ_2) is said to be **(τ_i, τ_j)-Fréchet-Urysohn** if for each A subset of X and each $x \in Cl_{\tau_i}(A)$ there is a sequence in A τ_j-converging to x. X is *strictly* (τ_i, τ_j)-*Fréchet-Urysohn* if it fulfills the selection property $\mathsf{S}_1(\Omega_x^{\tau_1}, \Sigma_x^{\tau_2})$.

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Let (X, d) be a metric space and $\mathfrak{B}$ be a bornology on X. The following are equivalent:

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A bispace (X, τ_1, τ_2) is said to be **(τ_i, τ_j)-Fréchet-Urysohn** if for each A subset of X and each $x \in Cl_{\tau_i}(A)$ there is a sequence in A τ_j-converging to x. X is *strictly (τ_i, τ_j)-Fréchet-Urysohn* if it fulfills the selection property $\mathsf{S}_1(\Omega_x^{\tau_1}, \Sigma_x^{\tau_2})$.

Theorem

Let (X, d) be a metric space and $\mathfrak{B}$ be a bornology on X. The following are equivalent:

(1) $(\mathsf{C}(X), \tau_{\mathfrak{B}}^s, \tau_{\mathfrak{B}}))$ is a strictly $(\tau_{\mathfrak{B}}^s, \tau_{\mathfrak{B}})$-Fréchet-Urysohn space;

(2) X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}}^s, \Gamma_{\mathfrak{B}})$.

Theorem

If (X, d) is a metric space and $\mathfrak{B}$ a bornology on X, then the following assertions are equivalent:

(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ is a $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ Fréchet-Urysohn space;

(2) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}}))$ is a strictly $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$-Fréchet-Urysohn space;

(3) Each $\mathfrak{B}^s$-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_{\mathfrak{B}}$-cover of X;

(4) X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}})$.

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