Theorem on universal spaces for dimension function (m,n)-dim

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June 27, 2012

Preliminaries

All spaces are assumed to be normal T_1. The set of all open covers with $\leq m$ elemets of a space X is denoted by $\mathrm{cov}_m(X)$.
By the *order* of a family u we mean the largest n such that u contains n sets with a non-empty intersection. The *order* of u is denoted by $\mathrm{ord}\, u$. So,

$$\mathrm{ord}\, u \leq 1 \iff u \text{ is a disjoint family.}$$

Definitions

Def 1

Let $u = (U_1, \dots U_m) \in \mathrm{cov}_m(X)$ and let $\phi = (F_1, \dots, F_m)$ be a family of closed subsets of X such that
$F_j \subset U_j, j = 1, \dots, m$;
ord $\phi \leq 1$.
Then (ϕ, u) is called an *m-pair* in X. The set of all *m-pairs* in X is denoted by $m(X)$.

Def 2

Let (ϕ, u) be an *m-pair* in X and let $v = (V_1, \dots, V_m)$ be a family of open subsets of X such that
$F_j \subset V_j \subset U_j, j = 1, \dots, m$;
ord $v \leq n$.
Then (ϕ, v, u) is called (m, n)-*triple* in X.

Def 3

Let $(\phi, u) \in m(X)$. A closed set $P \subset X$ is called an *n-partition* of (ϕ, u) (notation: $P \in \mathrm{Part}(\phi, u, n)$), if there exists a family v of open subsets of X such that (ϕ, v, u) is an (m, n)-*triple* in X and $P = X \setminus \bigcup v$.

Def 4

Let $(\phi_i, u_i) \in m(X), i = 1, \ldots, r$.
The sequence $((\phi_1, u_1), \ldots, (\phi_r, u_r))$ is said to be an *n-inessential* in X, if there exists partitions $P_i \in \mathrm{Part}(\phi_i, u_i, n)$ such that

$$P_1 \cap \ldots \cap P_r = \varnothing.$$

Definitions

Def 5

Let τ be an infinite cardinal number. An inverse system $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ with surjective projections π^α_β is said to be a *τ-spectrum* if

1) all X_α are compact spaces;
2) for every chain B in A with $|B| \leq \tau$ there is a $\sup B$ in A;
3) for every chain B in A with $|B| \leq \tau$ and $\sup B = \beta$, the diagonal product $\triangle\{\pi^\beta_\alpha \colon \alpha \in B\}$ is a homeomorphism between X_β and $\lim(S|B)$.

Main definition

Def 6 (V. Fedorchuk)

Let $m, n \in \mathbb{N}, n \leq m$. To every space X one assigns the dimension (m, n)-dim$X$, wich is an integer ≥ -1 or ∞. The dimensional function (m, n)-dim is defined in the following way:
(1) (m, n)-dim$X = -1$ iff X$= \varnothing$;
(2) (m, n)-dim$X \leq k$, where $k = 0, 1, \ldots,$ if every sequence $((\phi_1, u_1), \ldots, (\phi_{k+1}, u_{k+1}))$, $(\phi_i, u_i) \in m(X)$ is n-inessential in X;
(3) (m, n)-dim$X = \infty$, if (m, n)-dim$X > k$ for all $k = -1, 0, 1, \ldots$.

Remark

(m, n)-dim is a generalization of classic Lebesgue covering dimension. V. Fedorchuk had proved that $(2, 1)$-dim$X =$ dim$X$ for every space X. For more details see [a].

[a] *V.V. Fedorchuk* Finite dimensions defined by means of m-coverings, to be published in Mathem. Vesnik.

Required theorems

Factorization theorem (N. Martynchuk)

Let $f\colon X \to Y$ be a surjective mapping of compact spaces. Then there exist a compact space Z and mappings $g\colon X \to Z$ and $h\colon Z \to Y$ such that $f = h \circ g$, $(m,n)\text{-dim}Z \leq (m,n)\text{-dim}X$, and $wZ = wY$.

Theorem on compactification (N. Martynchuk)

For every space X $(m,n)\text{-dim}X = (m,n)\text{-dim}\beta X$.

Following results

Corollary 1

Let X be a compact space with $(m, n)\text{-dim}X \leq r$. If $X = \lim S$, where $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ is a τ-spectrum, then for each $\beta \in A$ there exists $\alpha \in A$, $\beta \leq \alpha$, such that $(m, n)\text{-dim}X_\alpha \leq r$.

Corollary 2

For every space X there exists a compactification bX such that $wX = w(bX)$ and $(m, n)\text{-dim}bX \leq (m, n)\text{-dim}X$.

Lemma

Let A be an indexing set and $X = \oplus_{\alpha \in A} X_\alpha$ be a discrete sum of X_α. If $(m,n)\text{-}\dim X_\alpha \le r$ for every α, then $(m,n)\text{-}\dim X \le r$.

Theorem

For any integers $m \ge 1, n \ge 1, r \ge 0$, any infinite cardinal number τ there exists a compact space $B^\tau_{(m,n,r)}$ such that $w(B^\tau_{(m,n,r)}) = \tau, (m,n)\text{-}\dim(B^\tau_{(m,n,r)}) = r$, and $B^\tau_{(m,n,r)}$ contains up to homeomorphism every space X with $wX \le \tau$ and $(m,n)\text{-}\dim X \le r$.

Thank you