# Cascades

- The *cascade* $(V, \leq)$ is a tree without infinite branches.
- $\emptyset_V$ - the root of $V$.
- $v^+$ - the set of immediate successors of $v$.
- $\max V$ - the set of maximal elements.
- A cascade $V$ is *sequential* if for every $v \in V \setminus \max V$ the set $v^+$ is countable and infinite.
- $\max V \cong \omega$ - the set of maximal elements ( for $V \neq \emptyset_V$) of sequential cascade.

# Rank of cascades

- We define a rank of elements of $V$ inductively:
- $rank(v) = 0$ for $v \in \max V$;
- if $v \notin \max V$ then

$$rank(v) = \min\{\alpha : \alpha > rank(w) \text{ for every } w \in v^+\}.$$

- $rank(V) := rank(\emptyset_V)$.

# Monotonic cascades

A cascade $V$ is *monotonic* iff
$v^+$ can be enumerate

$$w_1, w_2, w_3, \ldots$$

such that

$$rank(w_1) \leq rank(w_2) \leq rank(w_3), \ldots,$$

for every $v \in V \setminus \max V$

# Subcascades

- $V$ - the cascade.
- $w \in V$ - its element.
- $w^{\uparrow} = \{v \in V : w \leq v\}$ - the subcascade give by $w$.
- $\emptyset_{w^{\uparrow}} = w$
- $rank(w^{\uparrow}) = rank(w)$
- $\emptyset_{V}^{\uparrow} = V$

# Contours

- $V$ - the cascade and let $\max V = \omega$.
- $\int V$ - a *contour* i.e a special filter on $\max V$
- Contour is defined inductively for each subcascade $w^{\uparrow}$:
- $\int w^{\uparrow}$ is a trivial filter on $\{w\}$ for $w \in \max V$;
- otherwise
- $A \in \int w^{\uparrow}$ iff $(A \cap \max v) \in \int v^{\uparrow}$ for *almost every* $v \in w^{+}$;

# P-hierarchy

- We define $\mathcal{P}_\alpha$ for $1 \leq \alpha < \omega_1$ on $\beta\omega$ as follows:
- $u \in \mathcal{P}_\alpha$ iff
  1. there is no monotone sequential cascade $V_\alpha$ of rank $\alpha$ such that $\int V_\alpha \subset u$,
  2. for each $1 \leq \beta < \alpha$ there is a monotone sequential cascade $V_\beta$ of rank $\beta$ such that $\int V_\beta \subset u$.
- Moreover if for each $\alpha < \omega_1$ there is a monotone sequential cascade $V_\alpha$ of rank $\alpha$ such that $\int V_\alpha \subset u$ then we write $u \in \mathcal{P}_{\omega_1}$.

# P-hierarchy

- $\mathcal{P}_1$ - principal ultrafilters;
- $\mathcal{P}_2$ - $P$-points;
- How to find $\mathcal{P}_3$-ultrafilters?

# Know results

**Theorem (Starosolski, 2008)**

*The following conditions are equivalent*

- *P-points exist*
- $\mathcal{P}_\alpha \neq \emptyset$ *for each successor* $\alpha < \omega_1$;
- $\mathcal{P}_\alpha \neq \emptyset$ *for some successor* $1 < \alpha < \omega_1$;

**Theorem (Starosolski, 2010)**

*The following conditions are equivalent Assume CH.* $\mathcal{P}_\alpha \neq \emptyset$ *for each limit* $\alpha < \omega_1$;

# Flašková results

**Theorem (Flašková, 2006)**

*Assume CH. Let $\mathcal{I}$ be a tall $P-$ideal that contains all singletons than there exists $\mathcal{I}$-ultrafilter which is not a $P-$point.*

$u \in \beta\omega$ is is an $\mathcal{I}$-ultrafilter iff for any $f : \omega \longrightarrow A$ there is $U \in u$ such that $f(U) \in \mathcal{I}$.

**Theorem (Flašková, 2006)**

*Assume MA. Let $\mathcal{I}$ be an ideal of thin sets than there exists $\mathcal{I}$-ultrafilter which is not a $P-$point.*

# Our results

**Theorem (Machura & Starosolski, 2011)**

*Assume CH. Let $\mathcal{I}$ be tall $P-$ideal that contains all singletons and let $1 \leq \alpha \leq \omega_1$. Then there exists $\mathcal{I}$-ultrafilter which belongs to the class $\mathcal{P}_\alpha$.*

**Theorem (Machura & Starosolski, 2011)**

*Assume MA. Let $\mathcal{I}$ be an ideal of thin sets and let $1 \leq \alpha \leq \omega_1$ than there exists $\mathcal{I}$-ultrafilter which belongs to the class $\mathcal{P}_\alpha$.*

# Applications of the P-hierarchy

Baumgartner:

- $J_\alpha = \{B \subset \omega_1 : B \text{ has order type } < \alpha\}$
- $J_\alpha^*$ - a family of proper $I_\alpha$-ultrafilters i.e. $I_\alpha$-ultrafilters which are no $I_\beta$-ultrafilters for $\beta < \alpha$.
- If $J_\alpha^* \neq \emptyset$ , then $\alpha = \omega^\beta$ for some $\beta$

# Applications of the P-hierarchy

**Theorem (Starosolski, 2010)**

1. $u \in \mathcal{P}_2$ *iff* $u \in J^*_{\omega^2}$
2. *If* $u \in J^*_{\omega^\alpha}$ *then* $u \in \mathcal{P}_\beta$ *for some* $\beta \leq \alpha$

**Theorem (Starosolski, 2008)**

$J^*_{\omega^\omega} = \emptyset$ *(whilst* $\mathcal{P}_\omega \neq \emptyset$*).*