IV Workshop on Coverings, Selections and Games in Topology

Caserta, Italy, June 25–30, 2012

Selection properties of uniform structures: symmetric and asymmetric cases

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1. Introduction

In 2002 (I Workshop on Coverings, Selections and games in Topology, Lecce, June 26–30), I introduced selection principles in uniform spaces (and proved that these selection principles are different from selection principles in topological spaces).

Goal: Selection properties in generalizations of metric spaces (quasi-metric spaces and partial metric spaces) and uniform spaces (quasi-uniform spaces).

Differences between selection properties of these generalizations and selection properties of metric and uniform spaces.

1.1. Quasi-metric

A mapping $q : X^2 \to [0, \infty)$ is called a *quasi-metric* on X if for all $x, y, z \in X$ the following hold:

(QM1) $x = y \Leftrightarrow q(x, y) = q(y, x) = 0$;

(QM2) $q(x, y) \leq q(x, y) + q(y, z)$.

(X, q) is a *quasi-metric space*.

If (QM1) is replaced by (QPM1): $q(x, x) = 0$ for each $x \in X$, then q is called a *quasi-pseudometric on* X.

If (X, q) is a quasi-(psedo)metric space, then the collection

$$S_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\},\ x \in X,\ \varepsilon > 0$$

is a base for the topology τ_q on X.

The collection

$$V_\varepsilon = \{(x, y) \in X^2 : d(x, y) < \varepsilon\}, \qquad \varepsilon > 0,$$

is a base for the quasi-uniformity $\mathbb{U}_q$ on X.

1.2. Quasi-uniformity

A *quasi-uniformity* on a set X is a filter $\mathbb{U}$ on $X \times X$ satisfying the following two conditions:

$(QU1)$ $\Delta_X \subset U$ for each $U \in \mathbb{U}$;

$(QU2)$ For each $U \in \mathbb{U}$ there is $V \in \mathbb{U}$ such that $V \circ V \subset U$,

where $\Delta_X = \{(x, x) : x \in X\}$ is the diagonal of X, and $V \circ V = \{(x, y) \in X \times X : \exists z \in X \text{ with } (x, z) \in V, (z, y) \in V\}$.

The pair $(X, \mathbb{U})$ is called a *quasi-uniform space*.

A quasi-uniformity $\mathbb{U}$ is a *uniformity* on X, and $(X, \mathbb{U})$ is a uniform space, if $\mathbb{U}$ satisfies also the condition implies $U^{-1} \in \mathbb{U}$, where $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$.

If $(X, \mathbb{U})$ is a quasi-uniform space, then $(X, \mathbb{U}^{-1})$ is also a quasi-uniform space. Here

$$\mathbb{U}^{-1} = \{U^{-1} : U \in \mathbb{U}\}$$

is called the *conjugate* of $\mathbb{U}$.

$\sup\{\mathbb{U}, \mathbb{U}^{-1}\} = \mathbb{U}^s$ is a uniformity (*symmetrization* of $\mathbb{U}$)

For $U \in \mathbb{U}$, $x \in X$ and $A \subset X$ we put

$$U[x] = \{y \in X : (x,y) \in U\},\ \ U[A] = \bigcup_{a \in A} U[a].$$

If $\mathbb{U}$ is a quasi-uniformity on X, then $\mathbb{U}$ generates a topology $\tau_{\mathbb{U}}$ on X such that for each $x \in X$ the family $\{U[x] : U \in \mathbb{U}\}$ is a local base at x.

For each of these three topologies we consider the following three classical (topological) selection properties.

A topological space Z has the *Menger property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of Z there are finite sets $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an open cover of Z.

Z has the *Rothberger property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of Z there are $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\}$ is an open cover of Z.

Z has the *Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of Z there are finite sets $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that each $z \in Z$ belongs to $\cup\mathcal{V}_n$ for all but finitely many n.

2. Q-U spaces and selections

A quasi-uniform space $(X, \mathbb{U})$ is:

(1) *precompact* (resp. *pre-Lindelöf*) if for each $U \in \mathbb{U}$ there is a finite (resp. countable) set $F \subset X$ such that $U[F] = X$;

(2) *totally bounded* if for each $U \in \mathbb{U}$ there is a finite cover $\mathcal{C}$ of X such that $C \times C \subset U$ for each $C \in \mathcal{C}$.

In uniform spaces these two notions coincide.

Selective versions:

Definition. A quasi-uniform space $(X, \mathbb{U})$ is:

(pre-M) *pre-Menger* if $\forall$ $(U_n : n \in \mathbb{N})$ of elements of $\mathbb{U}$ $\exists$ $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[F_n]$;

(pre-ωM) *pre-ω-Menger* if $\forall$ $(U_n : n \in \mathbb{N})$ in $\mathbb{U}$ $\exists$ $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each finite subset $A \subset X$ is contained in $U_n[F_n]$ for some $n \in \mathbb{N}$;

(pre-H) *pre-Hurewicz* if $\forall$ $(U_n : n \in \mathbb{N})$ of elements of $\mathbb{U}$ $\exists$ $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $U_n[F_n]$;

(pre-R) *pre-Rothberger* if $\forall$ sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathbb{U}$ $\exists$ $(x_n : n \in \mathbb{N})$ of elements of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[x_n]$;

(pre-GN) *pre-Gerlits-Nagy* if for each sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathbb{U}$ there is a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that each $x \in X$ belongs to all but finitely many $U_n[x_n]$.

Definition. Let $(X, \mathbb{U})$ be a quasi-uniform space and let $\mathcal{P} \in \{M, \omega M, H, R, GN\}$. X is said to be *$\mathcal{P}$-bounded* if the uniform space $(X, \mathbb{U}^s)$ is $\mathcal{P}$-bounded.

Remark. A quasi-uniform space $(X, \mathbb{U})$ is Menger-bounded if and only if for each sequence $(U_n : n \in \mathbb{N})$ there is a sequence $(\mathcal{C}_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ covers X and for each $n \in \mathbb{N}$, $C \times C \subset U_n$ for each $C \in \mathcal{C}_n$.

Proposition. Each Menger-bounded quasi-uniform space $(X, \mathbb{U})$ is (hereditarily) pre-Menger.

Evidently, we have

$$
\begin{array}{ccccccc}
\mathsf{comp} & \Rightarrow & \mathsf{Hur} & \Rightarrow & \mathsf{Menger} & \Leftarrow & \mathsf{Roth} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\mathsf{pre-com} & \Rightarrow & \mathsf{pre-Hur} & \Rightarrow & \mathsf{pre-Menger} & \Leftarrow & \mathsf{pre-Roth} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\mathsf{tot\,bound} & \Rightarrow & \mathsf{H-bound} & \Rightarrow & \mathsf{M-bound} & \Leftarrow & \mathsf{R-bound}
\end{array}
$$

Example. There is a (hereditarily) pre-Menger quasi-metric space which is not precompact.

Let $X = \mathbb{Q} \cup \{\infty\}$ with the usual order $\leq$, and let $q : X^2 \to [0, \infty)$ be the quasi-metric on X defined by

$$q(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{n} & \text{if } x = n, x < y, \\ 1 & \text{otherwise.} \end{cases}$$

Any subspace of X is pre-Menger. X is not precompact. Let $\varepsilon \in (0, 1)$, $F = \{x_1 < x_2 < \cdots < x_k\} \subset X$ finite. If F does not contain a natural number or contains only natural numbers n with $n < 1/\varepsilon$, then $\{B_q(x_i, \varepsilon) : i \leq k\}$ cannot cover X. So, let $F \cap \mathbb{N} \neq \emptyset$; suppose $x_1 \in F$ is the least natural number with $x_1 > 1/\varepsilon$. $\{B_q(x_i, \varepsilon) : i \leq k\}$ covers $[x_1, \infty] \neq X$.

Example. There is a hereditarily pre-Hurewicz quasi-pseudo-metric space which is not hereditarily precompact.

Let $X = \mathbb{N}$ and let $q : X^2 \to [0, \infty)$ be the quasi-pseudo-metric on X defined by

$$q(m,n) = \begin{cases} 0 & \text{if } m = n, \\ \frac{1}{n} & \text{if } m = 1, n > 1, \\ 1 & \text{otherwise.} \end{cases}$$

(X, q) is Hurewicz, so pre-Hurewicz. Also hereditarily pre-Hurewicz. Let $Y \subset X$, $(\varepsilon_n : n \in \mathbb{N})$ in in $(0, 1)$. If $1 \in Y$, then Y is pre-Hurewicz. If $1 \notin Y$ and $Y = \{m_1 < m_2 < \cdots < m_n < \cdots\}$, then for each $k \in Y$ and each $n \in \mathbb{N}$, the ball $B_q(k, \varepsilon_n) = \{k\}$. For each $n \in \mathbb{N}$ let $F_n = \{m_1, \cdots, m_n\}$. $(F_n : n \in \mathbb{N})$ witnesses:Y is pre-Hurewicz. Easy: Y is not precompact.

Example. Let (X, τ_{coc}) be an uncountable set with the cocountable topology, and let $\mathbb{U}$ be the Pervin quasi-uniformity compatible with τ_{coc}.

[(X, τ) topological space, $G \subset X$. $S_G = [(X \setminus G) \times [X \times G]$. The Pervin quasi-uniformity is generated by $\{S_G : G \in \tau\}$ (as a subbase)]

The spaces $(X, \mathbb{U})$ and $(X, \mathbb{U}^{-1})$ are both hereditarily precompact (since the space $(X, \mathbb{U})$ is totally bounded), hence hereditarily pre-Hurewicz. But $\tau_{\mathbb{U}^{-1}}$ is discrete, so $(X, \mathbb{U}^{-1})$ is not Hurewicz.

The supremum of two precompact quasi-uniformities need not be precompact (which is true for totally bounded case).

Theorem (KMRV, 1993; Künzi, 2009) The supremum of two hereditarily precompact quasi-uniformities is also hereditarily precompact.

However, the supremum of two pre-Hurewicz quasi-uniformities need not be pre-Hurewicz.

Example: Let $X = \mathbb{R}$. The collection of all $V_{a,b} = \{(x, y) : x = y$ or $a < x < b\}$, $a, b \in \mathbb{R}$, $a < b$, is a base for a pre-Hurewicz quasi-uniformity $\mathcal{U}_{a,b}$. But, $\mathcal{U}_{0,1} \vee \mathcal{U}_{1,2}$ is the discrete uniformity on X which cannot be pre-Hurewicz.

2.1. Subspaces

Known:

Theorem. ([K], 2004) Every subspace of a M-bounded uniform space $(X, \mathbb{U})$ is also M-bounded.

Theorem. A subspace of a Menger-bounded quasi-uniform space is also Menger-bounded.

Theorem. For a QU-space $(X, \mathbb{U})$ TFAE:

(1) X is hereditarily (relatively) pre-Menger;

(2) Each G_δ subset of $(X, \tau_{\mathbb{U}})$ is pre-Menger.

Theorem. If a quasi-uniform space $(X, \mathbb{U})$ is pre-Menger and Y is a dense subset of $(X, \mathbb{U}^{-1})$, then $(Y, \mathbb{U}_Y)$ is pre-Menger.

Theorem. If a quasi-uniform space $(X, \mathbb{U})$ is pre-Menger and Y is a dense subset of $(X, \mathbb{U}^{-1})$, then $(Y, \mathbb{U}_Y)$ is pre-Menger.

Theorem. ([K], 2004) If a uniform space $(X, \mathbb{U})$ contains a dense Hurewicz-bounded subspace Y, then X is also Hurewicz-bounded.

Example. There is a quasi-metric space (X, q) containing a dense pre-Menger subspace Y, but (X, q^s) is not pre-Menger.

$$X = \left\{ x = (x_k)_{k \in \mathbb{N}} \in \ell^\infty : \sum_{k=1}^{\infty} \frac{x_k}{2^k} = 0 \right\}$$

endowed with the quasi-metric q defined by

$$q(x, y) = \sup_{k \in \mathbb{N}} (y_k - x_k), \; x, y \in X.$$

Observe that $q^s(x, y) = \sup_{k \in \mathbb{N}} |y_k - x_k|$, and (X, q^s) is not separable. By a result of Künzi et al. (MV, 1994), (X, q^s) is not pre-Lindelöf and thus it cannot be pre-Menger.

On the other hand, the countable set

$$Y = \{x \in X : x_k \in \mathbb{Q}\,\forall k \in \mathbb{N},\ \text{and } \exists n_0 = n_0(x) \in \mathbb{N}$$

$$\text{with } x_{n_0} = x_m \forall\, m > n_0\}$$

is dense in X, and pre-Menger being countable.

2.2. Products

Theorem. ([K], 2004) The product of two Hurewicz-bounded uniform spaces is also Hurewicz-bounded.

Theorem. The product of two pre-Hurewicz quasi-uniform spaces is also pre-Hurewicz.

Theorem. ([BKS], 2007) G is ω-Menger bounded $\Leftrightarrow$ G^n is M-bounded $\forall\ n \in \mathbb{N}$.

Theorem. ([BKS]) G^n, $n \in \mathbb{N}$, is Menger-bounded $\Leftrightarrow$ $\forall$ $(U_i : i \in \mathbb{N})$ in $\mathcal{N}_e$ $e \in G$ $\exists$ $(A_i : i \in \mathbb{N})$ finite in G s.t.: $\forall$ $F \subset G$ with $|F| = n$, $\exists$ i such that $F \subset A_i \cdot U_i$.

Theorem. (Machura+Shelah+Tsaban, 2010) Under CH there is a Menger-bounded group $G \leq \mathbb{Z}^{\mathbb{N}}$ whose square is not Menger-bounded.

Theorem. (Machura+Shelah+Tsaban, 2010) Under some additional assumptions (weaker than CH) there is for each $k \in \mathbb{N}$ a metrizable group G such that G^k is Menger-bounded but G^{k+1} is not.

Note. [Banakh+Zdomskyy (2006) and (independently) Mildenberger+Shelah] Consistently, every topological group G such that G^2 is Menger-bounded has Menger-bounded all finite powers.

3. Partial metric spaces

Let X be a nonempty set. $p : X \times X \to [0, \infty)$ is a *partial metric* on X if $\forall\ x, y, z \in X$ p satisfies:

(PM1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(PM2) $p(x,x) \leq p(x,y)$;

(PM3) $p(x,y) = p(y,x)$;

(PM4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

(X, p) is a *partial metric space*.

In difference of metric spaces in partial metric spaces the distance $p(x, x)$ need not be 0.

An open ball in a partial metric space X with center $x \in X$ and radius $\varepsilon > 0$ is the set

$$B_p(x, \varepsilon) := \{y \in X : p(x, y) < \varepsilon + p(x, x)\}.$$

The collection $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is the base of a T_0 topology τ_p on X.

Each partial metric p on X generates the quasi-metric q_p and the metric d_p on X:

$$q_p(x, y) = p(x, y) - p(x, x), \ \ (x, y \in X),$$

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \ \ (x, y \in X).$$

Associated topologies τ_{q_p}, τ_{d_p}. [Matthews, 1994] $\tau_{q_p} = \tau_p$ and $\tau_p \leq \tau_{d_p}$.

Similar to (quasi-)uniform selection principles.

Definition. A partial metric space (X, p) is:

(1) p-*Menger-bounded* (or M_p-*bounded*) if for each sequence $(\varepsilon_n : n \in \mathbb{N})$ there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} \bigcup_{x\in F_n} B_p(x, \varepsilon_n)$;

(2) p-*Rothberger-bounded* (or R_p-*bounded*) if for each sequence $(\varepsilon_n : n \in \mathbb{N})$ there is a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $X = \bigcup_{n\in\mathbb{N}} B_p(x_n, \varepsilon_n)$;

(3) p-*Hurewicz-bounded* (or H_p-*bounded*) if for each sequence $(\varepsilon_n : n \in \mathbb{N})$ there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to $\bigcup_{a\in\ F_n} B_p(a, \varepsilon_n)$ for all but finitely many n.

Example. A partial metric space which is R_p-bounded, but not R_{d_p}-bounded.

Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$, $x, y \in X$. Then $d_p(x, y) = |x - y|$. (X, p) is R_p-bounded. Indeed: $(\varepsilon_n : n \in \mathbb{N})$ is a sequence of positive reals, then for $(x_n : n \in \mathbb{N})$, $x_n = n-1$, we have $X = \bigcup_{n\in\mathbb{N}} B_p(x_n, \varepsilon_n)$ since $B_p(x_n, \varepsilon_n) = [0, n - 1 + \varepsilon_n)$. On the other hand, X is not R_{d_p}-bounded.

Let (X, p_X) and (Y, p_Y) be partial metric spaces and let $Z = X \times Y$. For $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in Z$ define

$$p_Z(z_1, z_2) = p_X(x_1, x_2) + p_Y(y_1, y_2).$$

It is not hard to check that p_Z is a partial metric on Z. The pair (Z, p_Z) is called the *product partial metric space* of X and Y.

Theorem. The product (Z, p_Z) of two H_p-bounded partial metric spaces (X, p_X) and (Y, p_Y) is also H_p-bounded.

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THANK YOU!