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$_n$-Relator Menger and related properties

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1. Introduction and definitions

We define n-relator Menger spaces and investigate their properties and relations with other covering properties in relator spaces. Some our earlier results will be extended to n-relator Menger spaces considering subspaces, products and relator continuous mappings. We also give several examples which clarify relationships among considered properties and show that they are different.

Many authors investigated n-star compact and n-star Lindelöf properties in topological spaces. Van Douwen,Reed, Roscoe and Tree in [1] considered starcompact properties and star-Lindelöf properties and gave characterizations of countable compactness and pseudocompactness in the terms of n-star compactness. They also gave a great number of examples which show that n-star compactness (n-star Lindelöfness) and $(n+1)$-star compactness ($(n+1)$-star Lindelöfness) are not equivalent.

A. Szaz in several papers on relator spaces (see [8], [9], [10]) showed that many topological structures can be derived from relator spaces.

First we recall some basic facts on relations and relators.

If X and Y are nonempty sets, then every subset R of $X \times Y$ is called a relation on X to Y. If $X = Y$, then we say that R is a relation on X. For every $x \in X$, $R(x) = \{y \in X : (x, y) \in R\}$. If A is a subset of X, then $R(A) = \bigcup_{x \in A} R(x)$. The inverse R^{-1} can be defined in the following way: $R^{-1}(y) = \{x \in X : (x, y) \in R\}$ for every $y \in X$. $\Delta_X = \{(x, x) : x \in X\}$ is called the identity relation on X.

If R is a relation on X to Y and S is a relation on Y to Z, then the composition $S \circ R$ is defined such that $(S \circ R)(x) = S(R(x))$ for every $x \in X$. If R is a relation on X to Y and S is a relation on Z to W, then the product $R \times S$ of relations R and S is defined in the following way: $(R \times S)(x, y) = R(x) \times S(y)$ for every $x \in X$ and $y \in Z$. By $m(R)$ we mean $\underbrace{R \times R \times \ldots}_{m}$

A relation R on X is reflexive (symmetric, transitive) if $\Delta_X \subset R$ ($R \subset R^{-1}$, $R \circ R \subset R$). We can inductively define R^n for every $n \in \mathbb{N}$ in the following way: $R^1 = R$ and $R^{n+1} = R \circ R^n$. We can also define R^∞ by $R^\infty = \bigcup_{n=0}^{\infty} R^n$, where $R^0 = \Delta_X$.

Every nonvoid family $\mathcal{R}$ of relations on X is called a relator on X. The ordered pair $(X, \mathcal{R})$ is called a relator space.

We can derive topological structures from relator spaces in the following way: If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we write:

$$int_{\mathcal{R}}(A) = \{x \in X : \exists R \in \mathcal{R} : R(x) \subset A\};$$

$$cl_{\mathcal{R}}(A) = \{x \in X : \forall R \in \mathcal{R} : R(x) \cap A \neq \emptyset\};$$

$$\mathcal{T}_{\mathcal{R}} = \{A \subset X : A \subset int_{\mathcal{R}}(A)\};$$

$$\mathcal{F}_{\mathcal{R}} = \{A \subset X : cl_{\mathcal{R}}(A) \subset A\}.$$

Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be relator spaces. We say that a function $f : X \longrightarrow Y$ is relator continuous if for every $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $f(R(x)) \subset S(f(x))$ for every $x \in X$.

We say that a relator space $(Y, \mathcal{R}_Y)$ is the subspace of $(X, \mathcal{R})$ if $Y \subset X$ and for every $S \in \mathcal{R}_Y$ there exists $R \in \mathcal{R}$ such that $S = Y^2 \cap R$.

We assume that every relation from relator is reflexive. The notation and terminology are as in [2].

2. Properties of n-relator Menger spaces

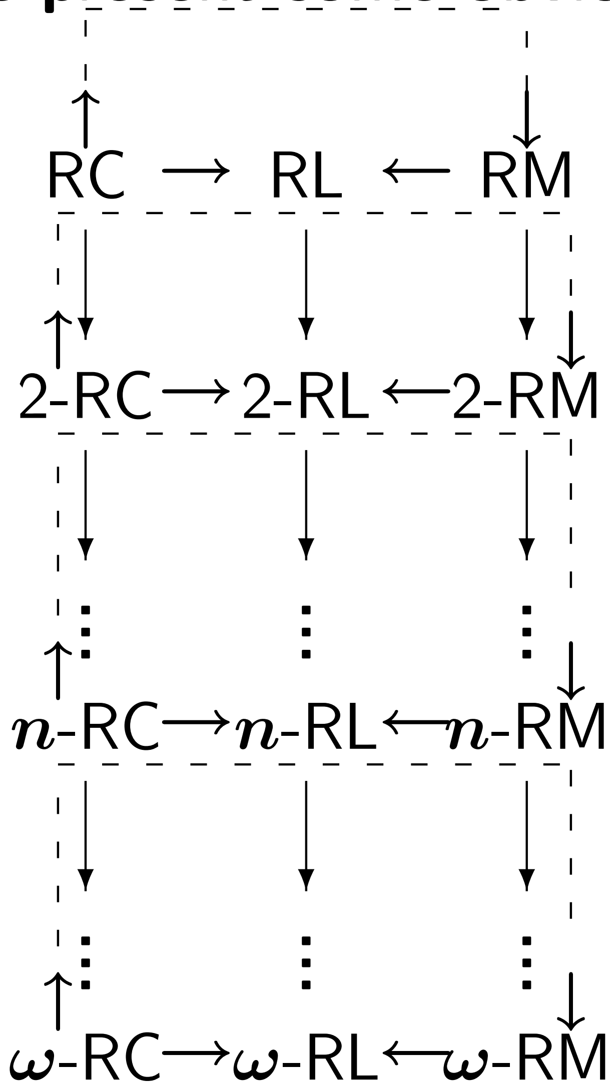
In [4] we introduced the notions of n-relator Menger property and ω-relator Menger property. In the same way we can define the notions of n-relator compact, ω-relator compact, n-relator Lindelöf and ω-relator Lindelöf properties.

Let us recall (see [4]) that a relator space $(X, \mathcal{R})$ is relator compact (relator Lindelöf) if for every relation R from $\mathcal{R}$ there exists a finite (resp. countable) subset A of X such that $R(F) = X$.

Definition 2.1 Let $(X,\mathcal{R})$ be a relator space. We say that:

- $(X,\mathcal{R})$ is n-relator Menger (ω-relator Menger), where $n \in \mathbb{N}$, if for every sequence $(R_k : k \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(F_k : k \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{k\in\mathbb{N}} R_k^n(F_k) = X$ (resp. $\bigcup_{k\in\mathbb{N}} R_k^\infty(F_k) = X$);
- $(X,\mathcal{R})$ is n-relator compact (n-relator Lindelöf) if for every $R \in \mathcal{R}$ there exists a finite (resp. countable) subset A of X such that $R^n(A) = X$;
- $(X,\mathcal{R})$ is ω-relator compact (ω-relator Lindelöf) if for every $R \in \mathcal{R}$ there exists a finite (resp. countable) subset A of X such that $R^\infty(A) = X$.

In the next diagram we present some obvious implications.



In the next section we will give examples which show that these implications can not be reversed. On the other hand, in [4] it was shown that if every relation from relator is transitive, then the notions of ω-relator Menger and n-relator Menger are equivalent for every $n \in \mathbb{N}$. Notice that if a relator has only one element, then the notions of n-relator Menger and n-relator Lindelöf properties are equivalent.

We present now some results concerning the operations of subsets, relator continuous functions and products.

The next three theorems extend statements of Theorem 2.2, Theorem 2.3 and Theorem 2.11 respectively from [4].

Theorem 2.1 **Let $(Y, \mathcal{R}_Y)$ be a subspace of an n-relator Menger space $(X, \mathcal{R})$. If for every $T \in \mathcal{R}$ there exists $R \in \mathcal{R}$ such that $R^n \circ R^{n-1} \subset T^n$, then $(Y, \mathcal{R}_Y)$ is the n-relator Menger space.**

Theorem 2.2 **Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be relator spaces. If $(X, \mathcal{R})$ is n-relator Menger and $f : X \longrightarrow Y$ is a relator continuous surjection, then $(Y, \mathcal{S})$ is also n-relator Menger.**

Let us recall that a cover $\mathcal{U}$ of a space X is an ω-cover [3] if X does not belong to $\mathcal{U}$ and for every finite subset F of X there exists $U \in \mathcal{U}$ such that $F \subset U$.

Theorem 2.3 If $(X^m, \mathcal{R}^m)$ is n-relator Menger space for every $m \in \mathbb{N}$, then for every sequence $(R_k : k \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(F_k : k \in \mathbb{N})$ of finite subsets of X such that $\{{R_k}^n(F_k) : k \in \mathbb{N}\}$ is an ω-cover for X.

The next theorem is the generalization of the Theorem 2.10 from [4].

We say that a relator space $(X, \mathcal{R})$ is σ-n-relator compact if it is a countable union of n-relator compact subspaces.

Theorem 2.4 If $(X, \mathcal{R})$ is σ-n-relator compact and $(Y, \mathcal{S})$ is n-relator Menger, then the product $(X \times Y, \mathcal{R} \times \mathcal{S})$ is n-relator Menger.

In [5] we considered some properties weaker than the relator Menger property. We now study the relations between these properties and 2-relator Menger property.

Recall that a relator space $(X, \mathcal{R})$ is neighborhood relator Menger ([5]) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for every $O_n \in \mathcal{T}_{\mathcal{R}}$, $F_n \subset O_n$, $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} R_n(O_n) = X$.

Theorem 2.5 If a relator space $(X, \mathcal{R})$ is neighborhood relator Menger and $R(x) \in \mathcal{T}_{\mathcal{R}}$ for every $R \in \mathcal{R}$ and every $x \in X$, then $(X, \mathcal{R})$ is 2-relator Menger.

We say that a relator space $(X, \mathcal{R})$ is almost relator Menger ([5]) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(F_n : n \in \mathbb{N})$ such that $\bigcup_{n\in\mathbb{N}} cl_{\mathcal{R}}(R_n(F_n)) = X$.

Theorem 2.6 If a relator space $(X, \mathcal{R})$ is almost relator Menger and every relation $R \in \mathcal{R}$ is symmetric, then $(X, \mathcal{R})$ is 2-relator Menger.

In [4], we introduced the notion of k-relator Menger property in the following way: A relator space $(X, \mathcal{R})$ is k-relator Menger if for every sequence $(R_n : n \in \mathbb{N})$ of elements of $\mathcal{R}$, there exists a sequence $(K_n : n \in \mathbb{N})$ of relator compact subspaces of X such that $\bigcup_{n\in\mathbb{N}} R_n(K_n) = X$.

Theorem 2.7 If a relator space $(X, \mathcal{R})$ is k-relator Menger, then $(X, \mathcal{R})$ is 2-relator Menger.

Let $\mathbb{R}$ be the set of real numbers, and let $\mathcal{D}$ be a relator on $\mathbb{R}$ defined by $\mathcal{D} = \{\mathcal{D}_\varepsilon : \varepsilon \in \mathbb{R}, \varepsilon > 0\}$, where $D_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ for every $x \in \mathbb{R}$ (see [4], Example 1). The following theorem is analogous to the Theorem 2.1.6. from [1] in topological spaces.

Theorem 2.8 If a relator space $(X, \mathcal{R})$ is ω-relator compact, then every relator continuous function $f : X \longrightarrow (\mathbb{R}, \mathcal{D})$ is bounded.

3. Examples

In [1], Example 2.2.5, was given an example which shows that there exists a topological space which is strongly 2-starcompact and not 1-starcompact. Using this example, we show that there exists a relator space which is 2-relator compact, but not relator compact.

Example 3.1 Let $\mathcal{A} = \{N_s : s \in S\}$ be an infinite family of infinite subsets of $\mathbb{N}$ such that for every distinct s and s' from S, $N_s \cap N_{s'}$ is finite, every $n \in \mathbb{N}$ is an element of only finitely many members of family $\mathcal{A}$ and $\mathcal{A}$ is a maximal family with this property. Suppose $\mathbb{N} \cap S = \emptyset$ and let $X = \mathbb{N} \cup S$. Fix a finite subset F of X and define a relation R_F on X in the following way: for every $n \in \mathbb{N}$, $R_F(n) = \{n\} \cup \{s : \{n\} \in N_s \backslash F\}$, and for every $s \in S$, $R_F(s) = \{s\} \cup (N_s \backslash F)$. Let $\mathcal{R} = \{R_F\}$ be a relator on X.

The following example shows that there exists a relator space which has ω-relator compact property and has not n-relator compact property for any $n \in \mathbb{N}$.

Example 3.2 **Let $\mathbb{R}$ be the set of real numbers and let $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. We define a relation D_ε on $\mathbb{R}$ in the following way: $D_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ for every $x \in \mathbb{R}$. Let $\mathcal{D} = D_\varepsilon$ be a relator on $\mathbb{R}$. A relator space $(\mathbb{R}, \mathcal{D})$ is ω-relator compact since for every $x, y \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $x \in (y - n\varepsilon, y + n\varepsilon)$, so for every finite $F \subset \mathbb{R}$ we have ${D_\varepsilon}^\infty(F) = \mathbb{R}$.**

Now we show that $(\mathbb{R}, \mathcal{D})$ is not n-relator compact for any $n \in \mathbb{N}$. Let $F \subset \mathbb{R}$ be any finite subset of $\mathbb{R}$. It is obvious that $\bigcup_{x \in F}(x - n\varepsilon, x + n\varepsilon) \neq \mathbb{R}$. $\square$

The next example shows that there exists 2-relator Menger (2-relator Lindelöf) space which is not relator Menger (relator Lindelöf).

Example 3.3 Let S be an uncountable set such that $S \cap \omega_1 = \emptyset$ and let $\mathcal{A} = \{N_s : s \in S\}$ be a maximal family of uncountable subsets of ω_1 such that $N_s \cap N_{s'}$ is countable for every distinct $s, s' \in S$ and every $\alpha \in \omega_1$ belongs to N_s for only countably many $s \in S$. Put $X = \omega_1 \cup S$ and fix countable subset A of ω_1. We define a relation D_A on X in the following way: $D_A(\alpha) = \{\alpha\} \cup \{s \in S : \alpha \in N_s\}$ for every $\alpha \in \omega_1$ and $D_A(s) = \{s\} \cup \{N_s \backslash A\}$ for every $s \in S$. Let $\mathcal{D} = \{D_A\}$ be a relator on X. The relator space $(X, \mathcal{D})$ is 2-relator Menger (2-relator Lindelöf) but not relator Menger (relator Lindelöf) (since the relator $\mathcal{D}$ has only one element, the notions of relator Menger and relator Lindelöf property are equivalent, so it is enough to prove only one of these claims).

The following example shows that there exists an ω-relator Menger space which is not n-relator Menger space for any $n \in \mathbb{N}$.

Example 3.4 **Let $\mathbb{R}^\omega$ be the set of all sequences of real numbers and let d be a usual metric on $\mathbb{R}$. Denote by $D_{\varepsilon,k} = \underbrace{B_\varepsilon \times B_\varepsilon \times \ldots \times B_\varepsilon}_{k} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \ldots,$, where $\varepsilon \in \mathbb{R}$, $k \in \mathbb{N}$, and for every $x \in \mathbb{R}$, $B_\varepsilon(x) = \{y \in \mathbb{R} : d(x,y) < \varepsilon\}$. If we put $\mathcal{D} = \{D_{\varepsilon,k} : \varepsilon \in \mathbb{R}, k \in \mathbb{N}\}$, then $\mathcal{D}$ is a relator on $\mathbb{R}^\omega$ (see [4]).**

Problem 3.1. **Does for every $n \in \mathbb{N}$ exist a relator space which has the $(n+1)$-relator Menger property and does not have the n-relator Menger property?**

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